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Nonlinearity management in higher dimensions

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Abstract. In the present short communication, we revisit nonlinearity management of the time-periodic nonlinear Schrödinger equation and the related averaging procedure. We prove that the averaged nonlinear Schrödinger equation does not support the blow-up of solutions in higher dimensions, independently of the strength in the nonlinearity coefficient variance. This conclusion agrees with earlier works in the case of strong nonlinearity management but contradicts those in the case of weak nonlinearity management. The apparent discrepancy is explained by the divergence of the averaging procedure in the limit of weak nonlinearity management.

1. Introduction

In the past few years, there has been a large volume of literature regarding the applications of the nonlinear Schrödinger equation (NLS) in the presence of the so-called *nonlinearity management* (often referred to also as Feshbach resonance management). The NLS is a prototypical dispersive nonlinear wave equation of the form:

$$iu_t = -\Delta u + \Gamma(t)|u|^2 u + V(x)u, \tag{1}$$

where $u(x,t)$ is a complex envelope field, $V(x) \geq 0$ is an external potential, $\Gamma(t)$ is a time-periodic nonlinearity coefficient, and Δ is the Laplacian operator with $x \in \mathbb{R}^d$, $d \geq 1$.

Nonlinearity management arises in applications in optics for transverse beam propagation in layered optical media [1], as well as in atomic physics for the Feshbach resonance of the scattering length of inter-atomic interactions in Bose-Einstein condensates (BECs) [2]. In the latter case, the periodic variation of $\Gamma(t)$ through an external magnetic field has been used as a means of producing robust matter-wave breathers in quasi-one-dimensional BECs [3]. It has also been suggested that the nonlinearity management may prevent collapse-type phenomena in higher dimensions [4]. Theoretical studies of the nonlinearity management were performed with a reduction of the time-periodic PDE problem (1) to a time-periodic ODE problem using a variational method [5] and a moment method [6].

The physical relevance of the time-periodic NLS equation, as evidenced by the above works, led to further developments in analysis of the PDE problem (1). As an example, the success of the averaging theory [7] for optical solitons in the presence of *strong dispersion management*, led to an analogous development for *strong nonlinearity management* produced originally in [8] and systematized in [9]. The time-periodic NLS equation (1) is averaged in the limit $\epsilon \to 0$, where ϵ measures a short period and the large variation of $\Gamma(t)$ in the decomposition

$$\Gamma = \gamma_0 + \frac{1}{\epsilon}\gamma\left(\frac{t}{\epsilon}\right), \tag{2}$$

where $\gamma(\tau)$, $\tau = t/\epsilon$ has a unit period and zero mean. After the averaging procedure, the time-periodic PDE problem (1) is governed by the averaged NLS equation,

$$iw_t = -\Delta w + \gamma_0|w|^2 w + V(x)w - \sigma^2\left(\left|\nabla|w|^2\right|^2 + 2|w|^2\Delta|w|^2\right)w, \tag{3}$$

where

$$\sigma^2 = \int_0^1 \gamma_{-1}^2(\tau)d\tau, \tag{4}$$

and $\gamma_{-1}(\tau)$ is the mean-zero anti-derivative of $\gamma(\tau)$. Derivation and local well-posedness of solutions of the averaged NLS equation (3) in function space $H^\infty(\mathbb{R})$ are reviewed for $d = 1$ in [9].

In the present paper, we consider global well-posedness of solutions of the averaged NLS equation (3) in the energy space $H^1(\mathbb{R}^d)$ for $d \geq 1$. The use of H^1 and $d \geq 1$ seems more appropriate for physical applications of the averaged model (3). In particular, we address the question whether the averaged NLS equation with a nonlinearity management ($\sigma^2 > 0$) arrests the blowup of solutions of the NLS equation in two and three dimensions that would occur if no nonlinearity management was applied ($\sigma^2 = 0$). We show that the averaged NLS equation is globally well-posed and no blowup of solutions occurs for $\sigma^2 > 0$. This is demonstrated initially, in Section 2, from the point of view of scaling arguments. The statement is subsequently proved using rigorous estimates in Section 3. In Section 4 we compare the above conclusion and earlier works where possibilities of blowup of solutions of the full time-periodic NLS equation (1) have been reported. Section 5 summarizes our findings.

2. Formal Scaling Arguments

The averaged NLS equation (3) has a standard Hamiltonian form (see [9]) with the Hamiltonian functional:

$$H(w) = \int_{\mathbb{R}^d} \left(|\nabla w|^2 + \frac{\gamma_0}{2}|w|^4 + V(x)|w|^2 + \sigma^2 |w|^2 \left| \nabla |w|^2 \right|^2 \right) dx. \tag{5}$$

Due to the gauge invariance, the averaged NLS equation (3) also conserves the squared L^2 norm:

$$P(w) = \int_{\mathbb{R}^d} |w|^2 dx. \tag{6}$$

Solitary wave solutions of the averaged NLS equation (3) are critical points of $H(w)$ at the level set of fixed values of $P(w)$.

Using formal scaling arguments [10] (see also the review of Ref. [11]), we consider a two-parameter family of dilatations:

$$w = bW(ax), \tag{7}$$

where (a, b) are parameters and $W(\xi)$ is a suitable function of $\xi = ax$. The squared L^2 norm (6) is preserved by the dilatations (7) whenever $b = a^{d/2}$. The Hamiltonian (5) at the dilatations (7) is scaled as a function of parameter $a > 0$:

$$H(a) = I_0(a) + a^2 I_1 + \gamma_0 a^d I_2 + \sigma^2 a^{2d+2} I_3, \tag{8}$$

where

$$I_1 = \int_{\mathbb{R}^d} |\nabla W|^2 d\xi, \quad I_2 = \frac{1}{2}\int_{\mathbb{R}^d} |W|^4 d\xi, \quad I_3 = \int_{\mathbb{R}^d} |W|^2 \left| \nabla |W|^2 \right|^2 d\xi.$$

and

$$I_0(a) = \int_{\mathbb{R}^d} V\left(\frac{\xi}{a}\right) |W|^2 d\xi.$$

Let us consider the case of no nonlinearity management and no external potential, when $\sigma^2 = 0$ and $V(x) = 0$. It follows from (8) that the Hamiltonian function $H(a)$ is positive definite in the defocusing case, when $\gamma_0 > 0$. In the focusing case, when $\gamma_0 < 0$, the Hamiltonian function $H(a)$ is bounded from below for $d = 1$ and $d = 2$, $\gamma_{\rm cr} < \gamma_0 < 0$ and is unbounded from below for $d = 2$, $\gamma_0 < \gamma_{\rm cr}$ and $d = 3$, where

$$\gamma_{\rm cr} = -\frac{I_1}{I_2}.$$

When $H(a)$ is unbounded from below as $a \to \infty$, the critical points of $H(w)$ at a fixed value of $P(w)$ (i.e., solitary wave solutions) can not be stable for small width a^{-1} and instability of solitary waves implies a blowup of localized initial data in the time evolution of the cubic NLS equation (see [11] for details).

When the nonlinearity management is applied, the last term in the decomposition (8) always dominates and it preserves the boundness of $H(a)$ from below for any $\sigma^2 > 0$. This indicates on the level of formal scaling arguments that the blowup of solutions is arrested by the nonlinearity management term in the averaged NLS equation (3). We shall prove this conjecture with rigorous analysis of well-posedness of solutions. We also note that the first term in the decomposition (8) does not change the conclusions above if $V(x)$ is a smooth non-negative potential, such that $I_0(a) \geq 0$. Typical examples of $V(x)$ are parabolic magnetic traps, when $V \sim x^2$, and periodic optical lattices, when $V \sim \sin^2(k_0 x)$.

3. Rigorous Analysis of Well-Posedness

The rigorous analysis of the local well-posedness of the averaged NLS equation (3) is not a trivial task. In fact, to the best of our knowledge, one cannot verify even the local existence and uniqueness of solutions to this problem. The problem has been considered in one dimension $d = 1$ by Poppenberg where a local well-posedness result with data in H^∞ was established [12]. In higher dimensions $d \geq 2$, one needs to require that the initial data be in $H^{s,m}$ for sufficiently large s,m, that is $\int_{\mathbf{R}^n}(1+|x|)^{2m}(|w_0|^2+|\partial^s w_0|)^2 dx < \infty$ (see [13]). In addition, one needs to assume a "non-trapping" condition on the symbol of the second order operator. This is a geometric condition, which depends on the profile of the initial data (see [13] for details).

To summarize, we cannot state a precise condition under which the averaged NLS equation (3) has a (local) solution that preserves values of P and H constant in time. We will however show, that whenever such a local solution exists for a short time $0 < t < t_0 < \infty$, it can be extended globally for all $t > 0$.

To that end, we represent the Hamiltonian $H(w)$ in the form:

$$H(w) = H_1(w) + \gamma_0 H_2(w),$$

where

$$H_1(w) = \int_{\mathbb{R}^d} \left(|\nabla w|^2 + V(x)|w|^2 + \sigma^2 |w|^2 |\nabla |w|^2|^2\right) dx \geq 0$$

and

$$H_2(w) = \frac{1}{2} \int_{\mathbb{R}^d} |w|^4 dx \geq 0.$$

We consider the focusing case $\gamma_0 < 0$ and prove that $H_1(w)$ and $H_2(w)$ are bounded by the two conserved quantities $H(w)$ and $P(w)$.

First, we quote a variant of the Gagliardo-Nirenberg inequality (see (1.1.16) on p. 15 in [14]).

Lemma 1. *For all $1 \leq p, q, r \leq \infty$, $\theta \in (0,1)$ and $r^{-1} = \theta p^{-1} + (1-\theta)q^{-1}$, it is true for every function $f(x)$ on $x \in \mathbb{R}^d$ that*

$$\|f\|_{L^r} \leq \|f\|_{L^p}^{\theta} \|f\|_{L^q}^{1-\theta}. \tag{9}$$

Next, we define and use the Fourier transform and its inverse for a function $f(x)$ on $x \in \mathbb{R}^d$:

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle x, \xi \rangle} dx, \quad f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{2\pi i \langle x, \xi \rangle} d\xi. \tag{10}$$

The Plancherel's formula gives $\|f\|_{L^2} = \|\hat{f}\|_{L^2}$ and the inequality $\|f\|_{L^\infty} \leq \|\hat{f}\|_{L^1}$ is immediate from the definition.

Let $\chi(s)$ be a $C_0^\infty(\mathbb{R}_+)$ function with a compact support on $s \in [0,2]$, such that $\chi(s) = 1$ on $s \in [0,1]$. Let $\chi_d(x)$ be a $C_0^\infty(\mathbb{R}^d)$ function, such that $\chi_d(x) = \chi(|x|)$. The (smooth) Fourier multiplier $P_{<N}$ is defined for every positive N by $\widehat{P_{<N} f}(\xi) = \chi_d(\xi/N)\hat{f}(\xi)$. Equivalently,

$$P_{<N} f(x) = N^d \int_{\mathbb{R}^d} \widehat{\chi_d}(N(x-y)) f(y) dy. \tag{11}$$

Let $P_{>N} = Id - P_{<N}$. Then both $P_{>N}, P_{<N}$ are self-adjoint, bounded on L^2 operators. The following statement is a modification of the Sobolev embedding theorem.

Lemma 2. *There exists a constant $C_d > 0$, which depends only on the dimension $d \geq 1$, so that it is true for every function $f(x)$ on $x \in \mathbb{R}^d$ that*

$$\|f\|_{L^2} \leq C_d \left(\|\nabla f\|_{L^2} + \|f\|_{L^1}\right). \tag{12}$$

Proof. Since $f = P_{<1}f + P_{>1}f$, it will suffice to show that

$$\|P_{<1}f\|_{L^2} \le C_d\|f\|_{L^1} \tag{13}$$

$$\|P_{>1}f\|_{L^2} \le C\|\nabla f\|_{L^2} \tag{14}$$

The bound (14) follows simply by the Plancherel's theorem and $\widehat{\partial_j f}(\xi) = 2\pi i\xi_j\hat{f}(\xi)$. Indeed,

$$\|P_{>1}f\|_{L^2}^2 \le \int\limits_{|\xi|>1} |\hat{f}(\xi)|^2 d\xi \le \int\limits_{|\xi|>1} |\xi|^2|\hat{f}(\xi)|^2 d\xi = \frac{1}{4\pi^2}\int\limits_{|\xi|>1} |\widehat{\nabla f}(\xi)|^2 d\xi \le$$

$$\le \frac{1}{4\pi^2}\int\limits_{\mathbb{R}^d} |\widehat{\nabla f}(\xi)|^2 d\xi = \frac{1}{4\pi^2}\|\nabla f\|_{L^2}^2.$$

By duality, the bound (13) follows from the estimate $\|P_{<1}f\|_{L^\infty} \le C_d\|f\|_{L^2}$. That is trivial as well, since

$$\|P_{<1}f\|_{L^\infty} \le \left\|\widehat{P_{<1}f}\right\|_{L^1} \le \int\limits_{|\xi|\le 2} |\hat{f}(\xi)|d\xi \le C_d(\int\limits_{|\xi|\le 2} |\hat{f}(\xi)|^2 d\xi)^{1/2} \le C_d\|f\|_{L^2}.$$

The positive constant C_d can be taken to be the square root of the volume of the ball in $\mathbb{R}^d$ with radius 2. □

The central result of our analysis is the following theorem.

Theorem 1. *There exist $\varepsilon(d) > 0$ and $C(\varepsilon, d, \sigma) > 0$, so that for every $0 < \varepsilon < \varepsilon(d)$, it is true for every function $\phi(x)$ on $x \in \mathbb{R}^d$ that*

$$\|\phi\|_{L^4}^4 \le \varepsilon H_1(\phi) + C(\varepsilon, d, \sigma)\left(\|\phi\|_{L^2}^2 + \|\phi\|_{L^2}^4\right). \tag{15}$$

Proof. Let $f = h^{3/2}$ with $h(x) > 0$ in (12) and obtain

$$\int\limits_{\mathbb{R}^d} h^3 dx \le C_d^2\left(\frac{3}{2}\left(\int\limits_{\mathbb{R}^d} h|\nabla h|^2 dx\right)^{1/2} + \int\limits_{\mathbb{R}^d} h^{3/2}dx\right)^2.$$

Next, we set $h = |\phi|^2$ and obtain

$$\begin{aligned}\int\limits_{\mathbb{R}^d} |\phi|^6 dx &\le C_d^2\left(\frac{3}{2}\left(\int\limits_{\mathbb{R}^d} |\phi|^2|\nabla|\phi|^2|^2 dx\right)^{1/2} + \int\limits_{\mathbb{R}^d} |\phi|^3 dx\right)^2 \\ &\le C_d^2\left(\frac{9}{2}\int\limits_{\mathbb{R}^d} |\phi|^2|\nabla|\phi|^2|^2 dx + 2\left(\int\limits_{\mathbb{R}^d} |\phi|^3 dx\right)^2\right) \\ &\le C_{d,\sigma}(H_1(\phi) + \|\phi\|_{L^3}^6),\end{aligned}$$

for some positive constant $C_{d,\sigma}$. We have used here that

$$\left(\sqrt{a} + b\right)^2 \le 2\left(a + b^2\right).$$

By the Gagliardo-Nirenberg inequality (9), we have $\|\phi\|_{L^3} \le \|\phi\|_{L^2}^{1/2}\|\phi\|_{L^6}^{1/2}$, such that the last inequality is rewritten in the form:

$$\|\phi\|_{L^6}^6 \le C_{d,\sigma}H_1(\phi) + C_d^2\|\phi\|_{L^6}^3\|\phi\|_{L^2}^3 \le C_{d,\sigma}H_1(\phi) + C_d^2\left(\varepsilon\|\phi\|_{L^6}^6 + \frac{1}{4\varepsilon}\|\phi\|_{L^2}^6\right), \tag{16}$$

where in the last line we have used the Cauchy-Schwartz inequality:

$$\forall \varepsilon > 0: \quad ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}. \tag{17}$$

Let $\varepsilon < \varepsilon(d)$, where $2C_d^2 \varepsilon(d) = 1$. Then, the term $\|\phi\|_{L^6}^6$ can be estimated from the bound (16) as follows:

$$\|\phi\|_{L^6}^6 \leq \tilde{C}_{d,\sigma} H_1(\phi) + \tilde{C}_d \|\phi\|_{L^2}^6 \tag{18}$$

for some constants $\tilde{C}_{d,\sigma} > 0$ and $\tilde{C}_d > 0$. By the Gagliardo-Nirenberg inequality (9), we have $\|\phi\|_{L^4} \leq \|\phi\|_{L^6}^{3/4} \|\phi\|_{L^2}^{1/4}$, such that the upper bound for $\|\phi\|_{L^4}$ follows from (17) and (18):

$$\|\phi\|_{L^4}^4 \leq \|\phi\|_{L^2} (\hat{C}_{d,\sigma} \sqrt{H_1(\phi)} + \hat{C}_d \|\phi\|_{L^2}^3) \leq \frac{\hat{C}_{d,\sigma}}{4\varepsilon} \|\phi\|_{L^2}^2 + \hat{C}_d \|\phi\|_{L^2}^4 + \varepsilon H_1(\phi),$$

which is the desired upper bound (15). □

As a corollary of the main theorem, we pick $\varepsilon = \varepsilon(d)/2$ and immediately obtain the following upper bounds.

Corollary 1. *There exists constants $C_1 > 0$ and $C_2 > 0$ that depend on d,γ_0,σ, so that*

$$H_1(w) \leq C_1 \left(H(w) + P(w) + P^2(w) \right) \tag{19}$$

and

$$H_2(w) \leq C_2 \left(H(w) + P(w) + P^2(w) \right). \tag{20}$$

Since $H(w)$ and $P(w)$ are conserved in the time evolution, the Cauchy problem for the averaged NLS equation (3) has global solutions in the energy space $H^1(\mathbb{R}^d)$, if the initial data $w(x,0)$ gives rise to local solutions in the same energy space. Therefore, the blowup of solutions of the NLS equation with $\sigma^2 = 0$ in $d \geq 2$ is arrested by the nonlinearity management for any $\sigma^2 > 0$.

4. Averaged Equation versus Full Dynamics

We have proven that the blowup does not occur in the averaged NLS equation (3) in higher dimensions $d \geq 2$ for $\sigma^2 > 0$. This result raises the question whether the blowup of solutions is arrested in the full NLS equation (1) for any non-zero variance of the time-periodic nonlinearity coefficient $\Gamma(t)$. We address this question within the ODE reduction of the time-periodic problem, which was considered recently with a variational method [4, 5] and a moment method [6]. In both cases, the time evolution of the radially symmetric localized solutions of the full NLS equaiton (1) is approximated by a time-dependent, generalized Ermakov-Pinney [15] equation:

$$\ddot{R}(t) = \frac{Q_1}{R^3} + \Gamma(t) \frac{Q_2}{R^{d+1}}, \tag{21}$$

where $R(t) \geq 0$ is an effective width of a localized solution, while (Q_1, Q_2) are constants found from an initial data, such that $Q_2 > 0$ (see [6] for details). We shall consider the critical case $d = 2$ and rewrite the ODE (21) with the nonlinearity coefficient $\Gamma(t)$ in (2) in the explicit form:

$$\ddot{R}(t) = \frac{\alpha + \beta \gamma(t/\epsilon)}{R^3}, \tag{22}$$

where $\alpha = Q_1 + \gamma_0 Q_2$, $\beta = Q_2/\epsilon > 0$, and γ is a mean-zero ϵ-periodic function of t. Conditions for blowup and existence of bounded oscillations in solutions of the ODE (22) were recently reviewed in [6] (see also references therein). The sufficient condition for the blowup (when $R(t) \to 0$ in a finite time $t \to t_0$) is

$$\alpha + \beta \max_{0 \leq t \leq \epsilon} (\gamma) < 0. \tag{23}$$

The sufficient condition for the unbounded growth of the solution width (when $R(t) \to \infty$ as $t \to \infty$) is

$$\alpha + \beta \min_{0 \le t \le \epsilon}(\gamma) > 0. \tag{24}$$

The necessary condition for the bounded oscillations of $R(t) > 0$ for any $t \ge 0$ is

$$\alpha < 0, \quad \alpha + \beta \max_{0 \le t \le \epsilon}(\gamma) > 0. \tag{25}$$

Numerical simulations in [6] showed that the condition (25) was also sufficient in the case $\alpha < 0$ for the blowup arrest for any $t \ge 0$.

It is obvious from the explicit scaling that $\beta \gg |\alpha|$ in the asymptotic limit $\epsilon \to 0$. Therefore, the second condition (25) is satisfied for $\alpha < 0$ and solutions of the time-periodic ODE (22) do not collapse, in agreement with our results derived for the averaged NLS equation (3).

Previous work [4] (see also the review in [9]) has also addressed the averaged NLS equation (3) in the limit of *weak* nonlinearity management, when $\gamma(\tau)$ is rescaled as $\gamma = \epsilon\tilde{\gamma}(\tau)$ and the parameter σ^2 is small in the limit $\epsilon \to 0$ as $\sigma^2 = \epsilon^2\tilde{\sigma}^2$. Solutions of the averaged NLS equation (3) at the leading order $\epsilon = 0$ blow up in a finite time, but the small ϵ^2-terms formally stabilize the blow-up for any $\tilde{\sigma}^2 > 0$. Although this approximation of the critical NLS equation has been considered in many applications of nonlinear optics (see Sections 4-5 in [16] and references therein), it is clearly insufficient for a correct identification of the domain, where the blowup of solutions occurs. Indeed, while the averaged NLS equation (3) with small σ^2 predicts no blowup of solutions, it is clear that the weak nonlinearity management corresponds to the case $\beta \approx |\alpha|$ and solutions of the ODE problem (22) (and those of the full PDE problem (1)) collapse in the domain (23).

Following the work [5], we address the failure of the averaging procedure for weak nonlinearity management in a simple time-periodic ODE problem:

$$\ddot{R}(t) = \frac{\alpha + \beta\sin(2\pi\tau)}{R^3}, \qquad \tau = \frac{t}{\epsilon}, \tag{26}$$

where $\alpha < 0$, $\beta > 0$ and (α, β) are order of $O(1)$ in the limit $\epsilon \to 0$. By using the formal asymptotic multi-scale expansion method (see [9] for details), we construct an asymptotic solution to the problem (26):

$$R = r(t) + \epsilon^2 R_2(\tau, r) + \epsilon^4 R_4(\tau, r) + \mathrm{O}(\epsilon^6), \qquad \tau = \frac{t}{\epsilon}, \tag{27}$$

where R_2 and R_4 are recursively found from the set of linear inhomogeneous problems,

$$R_2 = -\frac{\beta}{(2\pi)^2 r^3}\sin(2\pi\tau),$$

$$R_4 = -\frac{3\alpha\beta}{(2\pi)^4 r^7}\sin(2\pi\tau) + \frac{3\beta^2}{8(2\pi)^4 r^7}\cos(4\pi\tau).$$

The mean-value term $r(t)$ satisfies an extended dynamical equation that excludes secular growth of the correction terms of the series (27) in τ:

$$\ddot{r} = \frac{\alpha}{r^3} + \tilde{\epsilon}^2\frac{3\beta^2}{2r^7} + \tilde{\epsilon}^4\frac{15\alpha\beta^2}{2r^{11}} + \mathrm{O}(\tilde{\epsilon}^6), \qquad \tilde{\epsilon} = \frac{\epsilon}{2\pi}. \tag{28}$$

The averaged ODE problem (28) is an equation of motion for an effective particle with a coordinate $r(t)$ in the potential field with an effective potential energy:

$$U(r) = \frac{\alpha}{2r^2} + \tilde{\epsilon}^2\frac{\beta^2}{4r^6} + \tilde{\epsilon}^4\frac{3\alpha\beta^2}{4r^{10}} + \mathrm{O}(\tilde{\epsilon}^6). \tag{29}$$

When $\alpha < 0$ and $\tilde{\epsilon} = 0$, the particle with $r(0) > 0$ reaches $r = 0$ at a finite time $t = t_0 < \infty$, that indicates the blowup of a localized solution. When the next $\tilde{\epsilon}^2$-term is taken into account

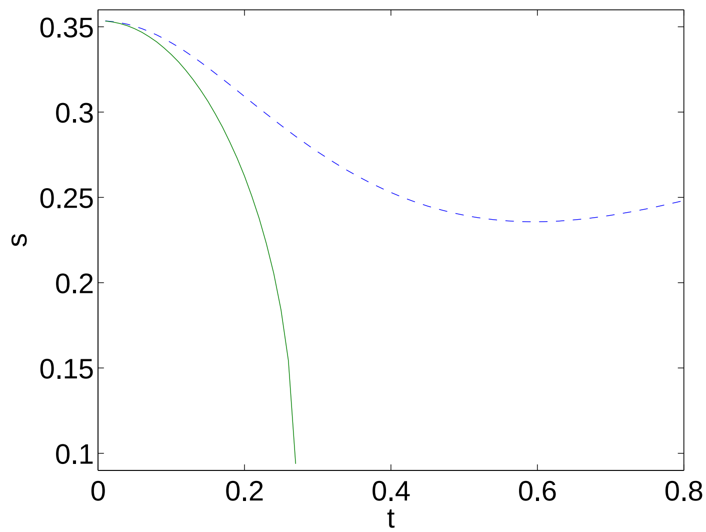

Figure 1. Numerical simulations of the full NLS equation (1) (solid curve) and the averaged NLS equation (3) with a fourth order in time scheme, where spacings are $dx = 0.075$ and $dt = 10^{-5}$. The half-width s of the wavefunction is shown as a function of time t.

(as in the approximation of weak nonlinearity management [4, 5]), the blow-up is arrested and the mean-value term $r(t)$ oscillates in an effective minimum of the potential energy $U(r)$, truncated at $\tilde{\epsilon}^2$-terms. When the next $\tilde{\epsilon}^6$-term is taken into account (beyond the approximation of weak nonlinearity management), the potential energy $U(r)$ with $\alpha < 0$ does not prevent the blowup of the localized solution depending on the initial data $r(0)$. Existence versus non-existence of blowup depends on the ratio of parameters (α, β) but the difference can only be detected in the averaging method if convergence of the power series (29) is established in a closed analytical form.

Similarly, the averaged NLS equation (3) can not be used in the limit of weak nonlinearity management for an accurate prediction of existence versus non-existence of blowup of solutions. In order to illustrate this point, we have performed numerical simulations of the full NLS equation (1) in $d = 2$ with

$$\Gamma(t) = -20.76 + 8\sin(2\pi t).$$

We have observed that collapse of localized initial data does occur (see solid curve on Fig. 1) by monitoring the half-width (of one dimensional slices along $y = 0$) of the wavefunction (for radially symmetric Gaussian initial data),

$$s = \frac{1}{2}\left(\frac{\int x^2 |u(x,0,t)|^2 dx}{\int |u(x,0,t)|^2 dx}\right)^{1/2},$$

until it becomes comparable to the lattice grid spacing used (at that scale collapse is arrested, since the numerical scheme cannot resolve scales below the grid spacing). On the other hand, numerical simulations of the averaged NLS equation (3) with the same parameters show that the half-width s never decreased below $s < 0.23$ (see dashed curve on Fig. 1) indicating the absence of collapse in accordance with the rigorous results presented above. Therefore, the averaged NLS equation (3) can only be used for modeling of the blowup arrest in the limit of strong nonlinearity management of the full NLS equation (1) when $\max(\gamma) \gg \epsilon|\gamma_0|$ and ϵ is small.

5. Conclusion

In conclusion, we have studied the global well-posedness of solutions of the averaged NLS equation that describe strong nonlinearity management of the time-periodic NLS equation. We have showed with formal scaling arguments and rigorous analysis that the blowup of solutions in higher dimensions is arrested within the averaged NLS equation. We have also discussed the non-applicability of the averaged NLS equation to the weak nonlinearity management, where the blowup of solutions can occur beyond the weak management limit.

It is an open problem to study well-posedness of the full time-periodic NLS equation, depending on parameters of the nonlinearity management and profile of initial data. Rigorous results on the latter problem are only available within the ODE approximation (21), when the PDE model is reduced to a dynamical system with one degree of freedom. It would be particularly interesting to study mathematically and to examine numerically whether the theoretical prediction from the method of moments provides an optimal bound for the full PDE model with arbitrary initial data.

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