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# Equivariant homology and K-theory of affine Grassmannians and Toda lattices

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# EQUIVARIANT ( $K$ -)HOMOLOGY OF AFFINE GRASSMANNIAN AND TODA LATTICE

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## 1. INTRODUCTION

1.1. Let  $G$  be an almost simple complex algebraic group, and let  $\mathrm{Gr}_G$  be its affine Grassmannian. Recall that if we set  $\mathbf{O} = \mathbb{C}[[t]]$ ,  $\mathbf{F} = \mathbb{C}((t))$ , then  $\mathrm{Gr}_G = G(\mathbf{F})/G(\mathbf{O})$ .

It is well-known that the subgroup  $\Omega K$  of polynomial loops into a maximal compact subgroup  $K \subset G$  projects isomorphically to  $\mathrm{Gr}_G$ ; thus  $\mathrm{Gr}_G$  acquires the structure of a topological group. An algebro-geometric counterpart of this structure is provided by the *convolution diagram*  $G(\mathbf{F}) \times_{G(\mathbf{O})} \mathrm{Gr}_G \rightarrow \mathrm{Gr}_G$ .

It allows one to define the *convolution* of two  $G(\mathbf{O})$  equivariant geometric objects (such as sheaves, or constrictible functions) on  $\mathrm{Gr}_G$ . A famous example of such a structure is the category of  $G(\mathbf{O})$  equivariant perverse sheaves on  $\mathrm{Gr}$  (“Satake category” in the terminology of Beilinson and Drinfeld); this is a semi-simple abelian category, and convolution provides it with a symmetric monoidal structure. By results of [10], [19], [2] this category is identified with the category of (algebraic) representations of the Langlands dual group.

The starting point for the present work was the observation that a similar definition works in another setting, yielding a monoidal structure on the category of  $G(\mathbf{O})$  equivariant *perverse coherent sheaves* on  $\mathrm{Gr}$  (the “coherent Satake category”). The latter is a non-semisimple artinian abelian category, the heart of the middle perversity  $t$ -structure on the derived category of  $G(\mathbf{O})$  equivariant coherent sheaves on  $\mathrm{Gr}_G$ ; existence of this  $t$ -structure is due to the fact that dimensions of all  $G(\mathbf{O})$ -orbits inside a given component of  $\mathrm{Gr}_G$  are of the same parity, cf. [3]. The resulting monoidal category turns out to be non-symmetric, though its Grothendieck ring  $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$  is commutative. One of the results of this paper is a computation of this ring. Along with  $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$  we compute its “graded version”, the ring  $H^{G(\mathbf{O})}(\mathrm{Gr})$  of equivariant homology of  $\mathrm{Gr}$ , where the algebra structure is again provided by convolution.<sup>1</sup> (The ring  $H_{\bullet}^{G(\mathbf{O})}(\mathrm{Gr}_G)$  was essentially computed by Dale Peterson [20], cf. also [15].)

To describe the answer, let  $\check{G}$  be the Langlands dual group to  $G$ , and let  $\check{\mathfrak{g}}$  be its Lie algebra. Consider the *universal centralizers*  $\mathfrak{Z}_{\check{\mathfrak{g}}}^{\check{G}}$  and  $\mathfrak{Z}_{\check{G}}^{\check{\mathfrak{g}}}$ : if we denote by  $C_{\check{G}, \check{\mathfrak{g}}} \subset \check{G} \times \check{\mathfrak{g}}$  (resp.  $C_{\check{\mathfrak{g}}, \check{G}} \subset \check{\mathfrak{g}} \times \check{G}$ ) the locally closed subvariety formed by all the pairs  $(g, x)$  such that  $Ad_g(x) = x$  and  $x$  is regular (resp. all the pairs  $(g_1, g_2)$  such that  $Ad_{g_1}g_2 = g_2$  and

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<sup>1</sup>The two rings are related via the Chern character homomorphism from  $K^{G(\mathbf{O})}(\mathrm{Gr})$  to the completion of  $H^{G(\mathbf{O})}(\mathrm{Gr})$ .

$g_2$  is regular), then  $\mathfrak{Z}_{\mathfrak{g}}^{\check{G}}$  (resp.  $\mathfrak{Z}_{\check{G}}^{\check{G}}$ ) is the categorical quotient  $C_{\check{G}, \mathfrak{g}} // \check{G}$  (resp.  $C_{\check{G}, \check{G}} // \check{G}$ ) with respect to the diagonal adjoint action of  $\check{G}$ .

We identify  $\mathrm{Spec} \left( H_{\bullet}^{G(\mathbf{O})}(\mathrm{Gr}_G) \right)$  with  $\mathfrak{Z}_{\mathfrak{g}}^{\check{G}}$ . Also, we identify  $\mathrm{Spec} \left( K^{G(\mathbf{O})}(\mathrm{Gr}_G) \right)$  with a variant of  $\mathfrak{Z}_{\check{G}}^{\check{G}}$  (the isomorphism  $\mathrm{Spec} \left( K^{G(\mathbf{O})}(\mathrm{Gr}_G) \right) \simeq \mathfrak{Z}_{\check{G}}^{\check{G}}$  holds true iff  $G$  is of type  $E_8$ ).

Notice that  $\mathfrak{Z}_{\mathfrak{g}}^{\check{G}}$  inherits a canonical symplectic structure as a hamiltonian reduction of the cotangent bundle  $\mathbb{T}^*\check{G}$ . Also,  $\mathfrak{Z}_{\check{G}}^{\check{G}}$  inherits a canonical Poisson structure as a  $\mathfrak{q}$ -Hamiltonian reduction of the  $\mathfrak{q}$ -Hamiltonian  $\check{G}$ -space *internal fusion double*  $\mathbf{D}(\check{G})$  (see [1]); this Poisson structure is in fact symplectic iff  $\check{G}$  is simply connected (that is,  $G$  is adjoint).

The corresponding Poisson structures on  $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$ ,  $H^{G(\mathbf{O})}(\mathrm{Gr}_G)$  come from a deformation of these commutative algebras to non-commutative algebras  $H_{\bullet}^{G(\mathbf{O}) \times \mathbb{G}_m}(\mathrm{Gr}_G)$  (resp.  $K^{G(\mathbf{O}) \times \mathbb{G}_m}(\mathrm{Gr}_G)$ ); here  $\mathbb{G}_m$  acts on  $\mathrm{Gr}_G$  by loop rotation. We conjecture that the non-commutative algebra  $H_{\bullet}^{G(\mathbf{O}) \times \mathbb{G}_m}(\mathrm{Gr}_G)$  can also be obtained from the ring of differential operators on  $\check{G}$  by quantum Hamiltonian reduction.

The space  $\mathfrak{Z}_{\mathfrak{g}}^{\check{G}}$  contains an open piece  $\mathfrak{Z}(\check{G})$  which for  $\check{G}$  adjoint (that is, for  $G$  simply connected) is a complexification of the Kostant's phase space of the classical Toda lattice ([14], Theorem 2.6). We remark in passing that Toda lattice also appears in the (apparently related) computations by Givental, Kim and others of quantum cohomology of flag varieties (see e.g. [13]).

Our computation should be compared with (and is to a large extent inspired by) [10] where equivariant cohomology  $H_{G(\mathbf{O})}(\mathrm{Gr}_G)$  were computed<sup>2</sup> in terms of the  $\check{G}$ . (The precise relation between the two computations is spelled out in Remark 2.13).

The second main object considered in the paper is another derived category of coherent sheaves with a convolution monoidal structure, namely the derived category  $D^b \mathrm{Coh}_{\Lambda_G}^{G(\mathbf{O})}(T^* \mathrm{Gr})$  of  $G(\mathbf{O})$ -equivariant coherent sheaves on the cotangent bundle of  $\mathrm{Gr}_G$  supported on the union  $\Lambda_G$  of conormal bundles to the  $G(\mathbf{O})$ -orbits (the definition of involved objects requires extra work since  $\mathrm{Gr}_G$  is infinite dimensional). (In this case we do not find a  $t$ -structure compatible with convolution, so all we get is a monoidal triangulated category). Notice that the singular support of a  $G(\mathbf{O})$ -equivariant  $D$ -module on  $\mathrm{Gr}_G$  is an object of  $\mathrm{Coh}_{\Lambda_G}^{G(\mathbf{O})}(T^* \mathrm{Gr})$ , thus this category can be considered a “classical limit” of the (derived) Satake category. We compute the Grothendieck ring of  $D^b \mathrm{Coh}_{\Lambda_G}^{G(\mathbf{O})}(T^* \mathrm{Gr})$  identifying its spectrum with  $(T \times \check{T})/W$ , where  $T \subset G$ , and  $\check{T} \subset \check{G}$  are Cartan subgroups. This is a singular variety birationally equivalent to  $\mathrm{Spec} \left( K^{G(\mathbf{O})}(\mathrm{Gr}_G) \right)$ . Unlike the latter, the former remains unchanged if we replace  $G$  by  $\check{G}$ . This motivates a conjecture that the corresponding triangulated monoidal categories for  $G$  and  $\check{G}$  are equivalent. The conjecture is compatible with a “classical

<sup>2</sup>Another description for  $H_{G(\mathbf{O})}(\mathrm{Gr}_G)$  is provided by a general result of [16]; in fact, its extension from [17] gives also an answer for  $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$ , and a similar technique can be applied to compute  $H^{G(\mathbf{O})}(\mathrm{Gr}_G)$ . However, this form of the answer does not make the relation to the (dual) group geometry explicit.

limit” of the geometric Langlands conjecture of Beilinson and Drinfeld (see 7.9 below for a more precise statement of the conjecture).

Finally, we remark that the convolution of  $G(\mathbf{O})$ -equivariant perverse coherent sheaves is closely related to the *fusion product* of  $G(\mathbf{O})$ -modules introduced by B. Feigin<sup>3</sup> [6] (see Section 8). In fact, our desire to understand the category  $\mathcal{P}^{G(\mathbf{O})}(\mathrm{Gr}_G)$ , and the work [6] of B. Feigin and S. Loktev, was one of the motivations for the present work.

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## 2. NOTATIONS AND STATEMENTS OF THE RESULTS

**2.1. Kostant slices.**  $G$  is an almost simple algebraic group with the Lie algebra  $\mathfrak{g}$ . We choose a principal  $\mathfrak{sl}_2$  triple  $(e, h, f)$  in  $\mathfrak{g}$ . Let  $\phi : \mathfrak{sl}_2 \rightarrow \mathfrak{g}$  (resp.  $\Phi : SL_2 \rightarrow G$ ) be the corresponding homomorphism. We denote by  $e_G$  (resp.  $f_G$ ) the image  $\Phi \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  (resp.  $\Phi \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ ). We denote by  $\mathfrak{z}(e)$  the centralizer of  $e$  in  $\mathfrak{g}$ , and by  $Z(e)$  (resp.  $Z^0(e)$ ) the centralizer of  $e$  (equivalently, of  $e_G$ ) in  $G$  (resp. its neutral connected component). We denote by  $\Sigma_{\mathfrak{g}} \subset \mathfrak{g}$  (resp.  $\Sigma_G \subset G$ ) the *Kostant slice*  $\mathfrak{z}(e) + f$  (resp.  $Z^0(e) \cdot f_G$ ). It is known that  $\Sigma_{\mathfrak{g}} \subset \mathfrak{g}^{reg}$  (resp.  $\Sigma_G \subset G^{reg}$ ), and the projection to the categorical quotient  $\Sigma_{\mathfrak{g}} \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}/Ad_G = \mathfrak{t}/W$  induces an isomorphism  $\Sigma_{\mathfrak{g}} \simeq \mathfrak{t}/W$ . Similarly, if  $G$  is simply connected, the projection to the categorical quotient  $\Sigma_G \hookrightarrow G \twoheadrightarrow G/Ad_G = T/W$  induces an isomorphism  $\Sigma_G \simeq T/W$ .

**2.2. The universal centralizers.** We consider the locally closed subvariety  $C_{\mathfrak{g}, \mathfrak{g}} \subset \mathfrak{g} \times \mathfrak{g}$  (resp.  $C_{\mathfrak{g}, G} \subset \mathfrak{g} \times G$ ,  $C_{G, \mathfrak{g}} \subset G \times \mathfrak{g}$ ,  $C_{G, G} \subset G \times G$ ) formed by all the pairs  $(x_1, x_2)$  such that  $[x_1, x_2] = 0$  and  $x_2$  is regular (resp. all the pairs  $(x, g)$  such that  $Ad_g(x) = x$  and  $g$  is regular; all the pairs  $(g, x)$  such that  $Ad_g(x) = x$  and  $x$  is regular; all the pairs  $(g_1, g_2)$  such that  $Ad_{g_1}(g_2) = g_2$  and  $g_2$  is regular). The categorical quotients with respect to the diagonal adjoint action of  $G$  are denoted respectively  $C_{\mathfrak{g}, \mathfrak{g}}//G = \mathfrak{Z}_{\mathfrak{g}}^{\mathfrak{g}}$ ,  $C_{\mathfrak{g}, G}//G = \mathfrak{Z}_{\mathfrak{g}}^G$ ,  $C_{G, \mathfrak{g}}//G = \mathfrak{Z}_G^{\mathfrak{g}}$ ,  $C_{G, G}//G = \mathfrak{Z}_G^G$ . The projections to the second (regular) factor are denoted by  $\varpi : \mathfrak{Z}_{\mathfrak{g}}^{\mathfrak{g}} \rightarrow \mathfrak{g}^{reg}/G = \mathfrak{t}/W$ ,  $\varpi : \mathfrak{Z}_{\mathfrak{g}}^G \rightarrow G^{reg}/G = T/W$ ,  $\varpi : \mathfrak{Z}_G^{\mathfrak{g}} \rightarrow \mathfrak{g}^{reg}/G = \mathfrak{t}/W$ ,  $\varpi : \mathfrak{Z}_G^G \rightarrow G^{reg}/G = T/W$ . In all the four cases  $\varpi$  is flat.

We consider the restrictions of our centralizer varieties to the Kostant slices:  $C_{\mathfrak{g}, \mathfrak{g}}^{\Sigma} = C_{\mathfrak{g}, \mathfrak{g}} \cap (\mathfrak{g} \times \Sigma_{\mathfrak{g}})$ ,  $C_{\mathfrak{g}, G}^{\Sigma} = C_{\mathfrak{g}, G} \cap (\mathfrak{g} \times \Sigma_G)$ ,  $C_{G, \mathfrak{g}}^{\Sigma} = C_{G, \mathfrak{g}} \cap (G \times \Sigma_{\mathfrak{g}})$ ,  $C_{G, G}^{\Sigma} = C_{G, G} \cap (G \times \Sigma_G)$ .

<sup>3</sup>The relation between convolution and fusion was known to B. Feigin since 1997.

Then the locally closed embedding  $C_{\mathfrak{g},\mathfrak{g}}^\Sigma \hookrightarrow C_{\mathfrak{g},\mathfrak{g}} \twoheadrightarrow \mathfrak{Z}_{\mathfrak{g}}^{\mathfrak{g}}$  induces an isomorphism  $C_{\mathfrak{g},\mathfrak{g}}^\Sigma \simeq \mathfrak{Z}_{\mathfrak{g}}^{\mathfrak{g}}$ . Similarly, we have isomorphisms  $C_{G,\mathfrak{g}}^\Sigma \simeq \mathfrak{Z}_{\mathfrak{g}}^G$  and (for simply connected  $G$ )  $C_{\mathfrak{g},G}^\Sigma \simeq \mathfrak{Z}_G^{\mathfrak{g}}$ ,  $C_{G,G}^\Sigma \simeq \mathfrak{Z}_G^G$ .

Thus both  $\mathfrak{Z}_{\mathfrak{g}}^{\mathfrak{g}} \rightarrow \mathfrak{t}/W$  and  $\mathfrak{Z}_G^{\mathfrak{g}} \rightarrow T/W$  (for simply connected  $G$ ) are the sheaves of abelian Lie algebras, while both  $\mathfrak{Z}_{\mathfrak{g}}^G \rightarrow \mathfrak{t}/W$  and  $\mathfrak{Z}_G^G \rightarrow T/W$  (for simply connected  $G$ ) are the sheaves of abelian Lie groups.

**2.3. Isogenies.** The center  $Z(G)$  acts naturally on  $\mathfrak{Z}_G^{\mathfrak{g}}$  (resp.  $\mathfrak{Z}_{\mathfrak{g}}^G$ ) by  $z(x, g) = (x, zg)$  (resp.  $z(g, x) = (zg, x)$ ). The center  $Z(G)$  acts on  $\mathfrak{Z}_G^G$  on both sides:  $z_1(g_1, g_2)z_2 = (z_1g_1, z_2g_2)$ . Let  $\tilde{G}$  denote the universal cover of  $G$ . Then the fundamental group  $\pi_1(G)$  is embedded into  $Z(\tilde{G})$ , and we have  $\mathfrak{Z}_G^{\mathfrak{g}} = \pi_1(G) \backslash \mathfrak{Z}_{\tilde{G}}^{\mathfrak{g}}$ ,  $\mathfrak{Z}_{\mathfrak{g}}^G = \pi_1(G) \backslash \mathfrak{Z}_{\tilde{G}}^G$ ,  $\mathfrak{Z}_G^G = \pi_1(G) \backslash \mathfrak{Z}_{\tilde{G}}^G / \pi_1(G)$ .

**2.4. Symplectic structures.** We fix an invariant identification  $\mathfrak{g} \simeq \mathfrak{g}^*$ , hence  $\mathfrak{t} \simeq \mathfrak{t}^*$ . Then  $\mathfrak{g} \times \mathfrak{g}$  gets identified with  $\mathfrak{g} \times \mathfrak{g}^* = T^*\mathfrak{g}$  (the cotangent bundle), and  $G \times \mathfrak{g}$  gets identified with  $G \times \mathfrak{g}^* = T^*G$ . After this  $\mathfrak{Z}_{\mathfrak{g}}^{\mathfrak{g}}$  (resp.  $\mathfrak{Z}_{\mathfrak{g}}^G$ ) can be viewed as a hamiltonian reduction of  $T^*\mathfrak{g}$  (resp.  $T^*G$ ); thus it inherits a canonical symplectic structure.

Identifying  $\mathfrak{g} \times G$  with  $\mathfrak{g}^* \times G = T^*G$  we can view  $\mathfrak{Z}_G^{\mathfrak{g}}$  as a hamiltonian reduction of  $T^*G$  as well; thus it inherits a canonical Poisson structure. Note that  $\mathfrak{Z}_G^{\mathfrak{g}}$  is smooth and symplectic iff  $G$  is simply connected. We have symplectic isomorphisms  $\mathfrak{Z}_{\mathfrak{g}}^{\mathfrak{g}} \simeq T^*(\mathfrak{t}/W)$ , and (in case  $G$  is simply connected)  $\mathfrak{Z}_G^{\mathfrak{g}} \simeq T^*(T/W)$ .

Note that  $\mathfrak{Z}_G^{\mathfrak{g}}$  and  $\mathfrak{Z}_{\mathfrak{g}}^G$  share a common open piece  $Z(G)$  formed by the classes of pairs  $(g, x)$  where both  $g$  and  $x$  are regular. The canonical symplectic structures agree on  $\mathfrak{Z}_G^{\mathfrak{g}} \supset Z(G) \subset \mathfrak{Z}_{\mathfrak{g}}^G$ . Note also that for adjoint  $G$  the space  $Z(G)$  contains (a complexification of) the Kostant's phase space  $\mathfrak{Z}(G)$  of the classical Toda lattice [14], and the embedding  $\mathfrak{Z}(G) \hookrightarrow \mathfrak{Z}_{\mathfrak{g}}^G$  is given by the Theorem 2.6 of *loc. cit.*

A. Alexeev, A. Malkin and E. Meinrenken introduced in [1] Example 6.1 the  $q$ -Hamiltonian  $G$ -space *internal fusion double*  $\mathbf{D}(G)$ . Its  $q$ -Hamiltonian reduction is  $\mathfrak{Z}_G^G$ , so it inherits a canonical Poisson structure. For a simply connected  $G$  the space  $\mathfrak{Z}_G^G$  is smooth and symplectic.

**2.5. Affine blow-ups.** The set of roots of  $G$  (resp.  $\tilde{G}$ ) is denoted by  $R$  (resp.  $\tilde{R}$ ). We will view  $\alpha \in R$  (resp.  $\tilde{\alpha} \in \tilde{R}$ ) as a homomorphism  $\mathfrak{t} \rightarrow \mathbb{C}$  (resp.  $\mathfrak{t} \rightarrow \mathbb{C}$ ) or as a homomorphism  $T \rightarrow \mathbb{C}^*$  (resp.  $\tilde{T} \rightarrow \mathbb{C}^*$ ) depending on a context. Also, for a root  $\alpha \in R$  we denote by  ${}^1\alpha$  (resp.  ${}^2\alpha$ ) the linear function on  $\mathfrak{t} \times \mathfrak{t}$  obtained as a composition of  $\alpha$  with the projection on the first (resp. second) factor.

We consider the following affine blow-up of  $\mathfrak{t} \times \mathfrak{t}$  at the diagonal walls:  $\mathfrak{B}_{\mathfrak{g}}^{\mathfrak{g}} = \text{Spec}(\mathbb{C}[\mathfrak{t} \times \mathfrak{t}, \frac{{}^1\alpha}{2}, \alpha \in R])$ . We also set  $\mathfrak{B}_{\mathfrak{g}}^G = \text{Spec}(\mathbb{C}[\mathfrak{t} \times T, \frac{{}^1\alpha}{2\alpha-1}, \alpha \in R])$ ;  $\mathfrak{B}_G^G = \text{Spec}(\mathbb{C}[T \times T, \frac{{}^1\alpha-1}{2}, \alpha \in R])$ ,  $\mathfrak{B}_G^{\tilde{G}} = \text{Spec}(\mathbb{C}[\tilde{T} \times T, \frac{{}^1\tilde{\alpha}-1}{2}, \alpha \in R])$ ; and let  $\mathfrak{B}_{\mathfrak{g}}^{\mathfrak{g}} = \mathfrak{B}_{\mathfrak{g}}^{\mathfrak{g}}/W$ ,  $\mathfrak{B}_G^G = \mathfrak{B}_G^G/W$ ,  $\mathfrak{B}_G^{\tilde{G}} = \mathfrak{B}_G^{\tilde{G}}/W$  (thus  $\mathfrak{B}_{\mathfrak{g}}^{\mathfrak{g}} = \text{Spec}(\mathbb{C}[\mathfrak{t} \times \mathfrak{t}, \frac{{}^1\alpha}{2}, \alpha \in R]^W)$ , etc.). We denote by  $\varpi$  the projection of  $\mathfrak{B}$  to the second factor; thus we have  $\varpi : \mathfrak{B}_{\mathfrak{g}}^{\mathfrak{g}} \rightarrow \mathfrak{t}/W$ ,  $\mathfrak{B}_G^G \rightarrow T/W$ ,  $\mathfrak{B}_{\mathfrak{g}}^G \rightarrow \mathfrak{t}/W$ ,  $\mathfrak{B}_G^G \rightarrow T/W$ ,  $\mathfrak{B}_G^{\tilde{G}} \rightarrow T/W$ .

**2.6. Poisson structures.** We have the canonical trivializations of the tangent bundles  $\mathbb{T}(\mathfrak{t} \times \mathfrak{t}) = (\mathfrak{t} \times \mathfrak{t}) \times (\mathfrak{t} \times \mathfrak{t})$ ,  $\mathbb{T}(\mathfrak{t} \times T) = (\mathfrak{t} \times T) \times (\mathfrak{t} \times \mathfrak{t})$ ,  $\mathbb{T}(T \times \mathfrak{t}) = (T \times \mathfrak{t}) \times (\mathfrak{t} \times \mathfrak{t})$ ,  $\mathbb{T}(T \times T) = (T \times T) \times (\mathfrak{t} \times \mathfrak{t})$ ,  $\mathbb{T}(T \times \check{T}) = (T \times \check{T}) \times (\mathfrak{t} \times \check{\mathfrak{t}})$ . Making use of the identification  $\check{\mathfrak{t}} = \mathfrak{t}^* \simeq \mathfrak{t}$  we obtain the  $W$ -invariant symplectic structures on the above varieties. Thus the above affine blow-ups carry the rational Poisson structures (regular off the discriminants  $\mathbf{D} \subset \mathfrak{B}$ ).

**Proposition 2.7.** *The Poisson structure on  $\mathfrak{B}_{\mathfrak{g}}^{\mathfrak{a}} - \mathbf{D}$  (resp.  $\mathfrak{B}_G^{\mathfrak{a}} - \mathbf{D}$ ,  $\mathfrak{B}_{\mathfrak{g}}^G - \mathbf{D}$ ,  $\mathfrak{B}_G^G - \mathbf{D}$ ,  $\mathfrak{B}_G^{\check{G}} - \mathbf{D}$ ) extends to the global Poisson structure; it is a symplectic structure if the corresponding variety is smooth.*

**Proposition 2.8.** *We are in the setup of 2.5.*

- a)  $\varpi$  is flat if  $G$  is simply connected;
- b) There are natural identifications  $\mathfrak{B}_{\mathfrak{g}}^{\mathfrak{a}} \simeq \mathfrak{Z}_{\mathfrak{g}}^{\mathfrak{a}}$ ,  $\mathfrak{B}_G^{\mathfrak{a}} \simeq \mathfrak{Z}_G^{\mathfrak{a}}$ ,  $\mathfrak{B}_{\mathfrak{g}}^G \simeq \mathfrak{Z}_{\mathfrak{g}}^G$ ,  $\mathfrak{B}_G^G \simeq \mathfrak{Z}_G^G$ ,  $\mathfrak{B}_G^{\check{G}} \simeq \mathfrak{Z}_G^{\check{G}}$  commuting with  $\varpi$ .
- c) If  $G$  is simply laced and adjoint, we have an identification  $\mathfrak{B}_G^{\check{G}} \simeq Z(\check{G}) \backslash \mathfrak{Z}_G^{\check{G}}$  commuting with  $\varpi$ ;
- d) If  $G$  is simply laced and simply connected, we have an identification  $\mathfrak{B}_G^{\check{G}} \simeq \mathfrak{Z}_G^{\check{G}}/Z(G)$  commuting with  $\varpi$ ;
- e) The above identifications respect the Poisson structures.

**2.9. Flat group sheaves.** We consider the functor  $\mathfrak{F}_{\mathfrak{g}}^{\mathfrak{a}}$  on the category  $\text{Flat}_{\mathfrak{t}/W}$  of schemes flat over  $\mathfrak{t}/W$  to the category of sets, sending a test scheme  $S$  to the set of  $W$ -invariant morphisms  $(\text{Mor}(S \times_{\mathfrak{t}/W} \mathfrak{t}, \mathfrak{t}))^W$ . Similarly, we consider the functor  $\mathfrak{F}_G^{\mathfrak{a}}$  on the category  $\text{Flat}_{T/W}$  sending a test scheme  $S$  to the set of  $W$ -invariant morphisms  $(\text{Mor}(S \times_{T/W} T, \mathfrak{t}))^W$ . Also, we consider the functor  $\mathfrak{F}_{\mathfrak{g}}^G$  on the category  $\text{Flat}_{\mathfrak{t}/W}$  sending a test scheme  $S$  to the set of  $W$ -invariant morphisms  $(\text{Mor}(S \times_{\mathfrak{t}/W} \mathfrak{t}, T))_0^W \subset (\text{Mor}(S \times_{\mathfrak{t}/W} \mathfrak{t}, T))^W$  subject to the condition (cf. [5] 4.2)

$$(1) \quad \alpha(f(\alpha^{-1}(0))) = 1 \quad \forall \alpha \in R.$$

(note that the  $W$ -invariance condition automatically implies  $\alpha(f(\alpha^{-1}(0))) = \pm 1 \quad \forall \alpha \in R$ .)

Furthermore, we consider the functor  $\mathfrak{F}_G^G$  on the category  $\text{Flat}_{T/W}$  sending a test scheme  $S$  to the set of  $W$ -invariant morphisms  $(\text{Mor}(S \times_{T/W} T, T))_0^W \subset (\text{Mor}(S \times_{T/W} T, T))^W$  subject to the condition

$$(2) \quad \alpha(f(\alpha^{-1}(1))) = 1 \quad \forall \alpha \in R.$$

(note that the  $W$ -invariance condition automatically implies  $\alpha(f(\alpha^{-1}(1))) = \pm 1 \quad \forall \alpha \in R$ .)

Finally, we consider the functor  $\mathfrak{F}_G^{\check{G}}$  on the category  $\text{Flat}_{T/W}$  sending a test scheme  $S$  to the set of  $W$ -invariant morphisms  $(\text{Mor}(S \times_{T/W} T, \check{T}))_0^W \subset (\text{Mor}(S \times_{T/W} T, \check{T}))^W$  subject to the condition

$$(3) \quad \check{\alpha}(f(\alpha^{-1}(1))) = 1 \quad \forall \alpha \in R.$$

(note that the  $W$ -invariance condition automatically implies  $\check{\alpha}(f(\alpha^{-1}(1))) = \pm 1 \forall \alpha \in R$ .)

The following Proposition is a generalization of [5] 11.6.

**Proposition 2.10.** *Assume that  $G$  is simply connected. The functor  $\mathfrak{F}_{\mathfrak{g}}^{\mathfrak{g}}$  (resp.  $\mathfrak{F}_G^{\mathfrak{g}}$ ,  $\mathfrak{F}_{\mathfrak{g}}^G$ ,  $\mathfrak{F}_G^{\check{G}}$ ) is representable by the scheme  $\mathfrak{B}_{\mathfrak{g}}^{\mathfrak{g}}$  (resp.  $\mathfrak{B}_G^{\mathfrak{g}}$ ,  $\mathfrak{B}_{\mathfrak{g}}^G$ ,  $\mathfrak{B}_G^{\check{G}}$ ).*

**2.11. Equivariant Borel-Moore Homology.** For the definition of convolution in equivariant Borel-Moore Homology we refer the reader to [4] 2.7, 8.3 or [18] Chapter 2.

We have  $H_{\bullet}^{G(\mathcal{O})}(pt) = H_{G(\mathcal{O})}^{\bullet}(pt) = \mathbb{C}[\mathfrak{t}/W]$ , and  $H_{\bullet}^{G(\mathcal{O}) \times \mathbb{G}_m}(pt) = H_{G(\mathcal{O}) \times \mathbb{G}_m}^{\bullet}(pt) = \mathbb{C}[\mathfrak{t}/W][\hbar]$  where  $\hbar$  is the generator of  $H_{\mathbb{G}_m}^2(pt)$ . We will consider the  $\mathbb{C}[\mathfrak{t}/W]$ -algebra (resp.  $\mathbb{C}[\mathfrak{t}/W][\hbar]$ -algebra) (with respect to convolution)  $H_{\bullet}^{G(\mathcal{O})}(\mathrm{Gr}_G)$  (resp.  $H_{\bullet}^{G(\mathcal{O}) \times \mathbb{G}_m}(\mathrm{Gr}_G)$ ). Note that setting  $\hbar = 0$  in  $H_{\bullet}^{G(\mathcal{O}) \times \mathbb{G}_m}(\mathrm{Gr}_G)$  we obtain  $H_{\bullet}^{G(\mathcal{O})}(\mathrm{Gr}_G)$ ; indeed for any group  $H$ , a space  $X$  with an  $H \times \mathbb{G}_m$  action, and an  $H \times \mathbb{G}_m$ -equivariant complex  $\mathcal{F}$  on  $X$  we have a long exact sequence  $\dots \rightarrow H_{H \times \mathbb{G}_m}^{i-2}(X, \mathcal{F}) \xrightarrow{\hbar} H_{H \times \mathbb{G}_m}^i(X, \mathcal{F}) \rightarrow H_H^i(X, \mathcal{F}) \rightarrow H_{H \times \mathbb{G}_m}^{i-1}(X, \mathcal{F}) \rightarrow \dots$  coming from the principal  $\mathbb{G}_m$ -bundle  $E(H \times \mathbb{G}_m) \times_H X \rightarrow E(H \times \mathbb{G}_m) \times_{H \times \mathbb{G}_m} X$ ; if the space of  $H \times \mathbb{G}_m$ -equivariant cohomology is  $\hbar$ -torsion free, then we get  $H_H^{\bullet}(X, \mathcal{F}) = H^{\bullet}(X, \mathcal{F})|_{\hbar=0}$ .

**Theorem 2.12.** *a) The algebra  $H_{\bullet}^{G(\mathcal{O})}(\mathrm{Gr}_G)$  is commutative;*

*b) Its spectrum together with the projection onto  $\mathfrak{t}/W = \check{\mathfrak{t}}/W$  is naturally isomorphic to  $\mathfrak{Z}_{\mathfrak{g}}^{\check{G}} \xrightarrow{\cong} \check{\mathfrak{t}}/W$ ;*

*c) The Poisson structure on  $H_{\bullet}^{G(\mathcal{O})}(\mathrm{Gr}_G)$  arising from the  $\hbar$ -deformation  $H_{\bullet}^{G(\mathcal{O}) \times \mathbb{G}_m}(\mathrm{Gr}_G)$ , corresponds under the above identification to the Poisson structure of 2.4 on  $\mathfrak{Z}_{\mathfrak{g}}^{\check{G}}$ .*

**Remark 2.13.** The equivariant cohomology ring  $H_{G(\mathcal{O})}^{\bullet}(\mathrm{Gr}_G, \mathbb{C}) = H_{G(\mathcal{O})}^{\bullet}(\mathrm{Gr}_G)$  was computed by V. Ginzburg [10]. More precisely, the projection to the second (regular) factor  $\mathfrak{Z}_{\mathfrak{g}}^{\check{G}} \rightarrow \check{\mathfrak{g}}^{reg} // \check{G} = \check{\mathfrak{t}}/W$  makes  $\mathfrak{Z}_{\mathfrak{g}}^{\check{G}}$  a sheaf of abelian Lie algebras. V. Ginzburg identifies  $H_{G(\mathcal{O})}^{\bullet}(\mathrm{Gr}_G)$  with the global sections of the relative universal enveloping algebra  $U_{\check{\mathfrak{t}}/W}(\mathfrak{Z}_{\mathfrak{g}}^{\check{G}})$ . One can easily check that this result is compatible with our Theorem 2.12(b) as follows. For a group scheme  $A$  over a base  $S$  one has a natural pairing  $U(\mathfrak{a}) \times \mathcal{O}(A) \rightarrow \mathcal{O}(S)$  where  $U(\mathfrak{a})$  is the enveloping (over  $\mathcal{O}(S)$ ) of the Lie algebra of  $A$ ; the pairing sends  $(\xi, f)$  to  $\xi(f)$  restricted to the identity of  $A$ . On the other hand, for a compact (or ind-compact)  $H$ -space  $X$  we have a pairing  $H_H^{\bullet}(X) \times H_{\bullet}^H(X) \rightarrow H_H^{\bullet}(pt)$  induced by the action of cohomology on homology, and the push-forward map in Borel-Moore homology  $H_{\bullet}^H(X) \rightarrow H_H^{\bullet}(pt)$ . The isomorphisms of [10] and of Theorem 2.12 take the first pairing into the second one.

**2.14. Equivariant  $K$ -theory.** For the definition of convolution in equivariant  $K$ -theory we refer the reader to Chapter 5 of [4].

We have  $K^{G(\mathcal{O})}(pt) = \mathbb{C}[T/W]$ , and  $K^{G(\mathcal{O}) \times \mathbb{G}_m}(pt) = \mathbb{C}[T/W][q^{\pm 1}]$ . We will consider the  $\mathbb{C}[T/W]$ -algebra (resp.  $\mathbb{C}[T/W][q^{\pm 1}]$ -algebra) (with respect to convolution)

$K^{G(\mathbf{O})}(\mathrm{Gr}_G)$  (resp.  $K^{G(\mathbf{O}) \times \mathbb{G}_m}(\mathrm{Gr}_G)$ ). Note that setting  $q = 1$  in  $K^{G(\mathbf{O}) \times \mathbb{G}_m}(\mathrm{Gr}_G)$  we obtain  $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$ .

**Theorem 2.15.** *a) The algebra  $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$  is commutative;*

*b) Its spectrum together with the projection onto  $T/W$  is naturally isomorphic to  $\mathfrak{B}_G^{\check{G}} \xrightarrow{\varpi} T/W$ ;*

*c) The Poisson structure on  $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$  arising from the  $q$ -deformation  $K^{G(\mathbf{O}) \times \mathbb{G}_m}(\mathrm{Gr}_G)$ , corresponds under the above identification to the Poisson structure of 2.7 on  $\mathfrak{B}_G^{\check{G}}$  in case the latter variety is smooth, i.e.  $G$  is simply connected.*

### 3. CALCULATIONS IN RANK 1

In this section  $G = SL_2$ , and  $\check{G} = PGL_2$ . The Weyl group  $W = \mathbb{Z}/2\mathbb{Z}$ , the Cartan torus  $T = \mathbb{G}_m = \mathbb{C}^*$  with a coordinate  $z$ , and the only simple root  $\alpha(z) = z^2$ . The dual torus  $\check{T} = \mathbb{G}_m = \mathbb{C}^*$  with a coordinate  $t$ , and  $\check{\alpha}(t) = t$ . The Cartan Lie algebra  $\mathfrak{t} = \mathbb{C}$  with a coordinate  $x = \alpha(x)$ . We fix a  $\sqrt{-1}$ .

3.1.  $\mathfrak{Z}_G^G$  and  $\mathfrak{B}_G^G$ . We choose the standard  $\mathfrak{sl}_2$ -triple  $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ ,  $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ ,  $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ . Then the Kostant slice  $\Sigma_G = \left\{ \begin{pmatrix} a-1 & a-2 \\ 1 & 1 \end{pmatrix}, a \in \mathbb{C} \right\}$ .

One checks that a matrix  $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$  commutes with  $\begin{pmatrix} a-1 & a-2 \\ 1 & 1 \end{pmatrix}$  iff  $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \sqrt{-1} \begin{pmatrix} (1-a)c+b & (2-a)c \\ -c & b-c \end{pmatrix}$  for  $b, c \in \mathbb{C}$ . Then the condition  $\det = 1$  reads as

$$(4) \quad 1 = abc - b^2 - c^2.$$

Thus,  $\mathfrak{Z}_G^G$  is identified with a hypersurface  $\mathfrak{S}$  in  $\mathbb{A}^3$  given by the equation (4). The left (resp. right) multiplication by  $-1 \in Z(SL_2)$  is an involution  $\iota$  (resp.  $j$ ) on  $\mathfrak{S}$  given by  $\iota(a, b, c) = (a, -b, -c)$  (resp.  $j(a, b, c) = (-a, b, -c)$ ). Hence,  $\mathfrak{Z}_G^{\check{G}} = \iota \backslash \mathfrak{S} / j$ .

Generically, we can diagonalize two commuting matrices simultaneously, that is, there is  $g \in SL_2$  such that  $g\sqrt{-1} \begin{pmatrix} (1-a)c+b & (2-a)c \\ -c & b-c \end{pmatrix} g^{-1} = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}$  and  $g \begin{pmatrix} a-1 & a-2 \\ 1 & 1 \end{pmatrix} g^{-1} = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$  for some  $y, z \in \mathbb{G}_m = \mathbb{C}^* = T$  defined up to simultaneous inversion. Then we have

$$(5) \quad a = z + z^{-1}, \quad b = \frac{-\sqrt{-1}}{2} \left( y + y^{-1} + \frac{(y - y^{-1})(z + z^{-1})}{z - z^{-1}} \right), \quad c = -\sqrt{-1} \frac{y - y^{-1}}{z - z^{-1}}.$$

We conclude that  $\mathbb{C}[\mathfrak{S}] = \mathbb{C}[y^{\pm 1}, z^{\pm 1}, \frac{y-y^{-1}}{z-z^{-1}}]^W$  where the nontrivial element  $w \in W$  acts by  $w(y, z) = (y^{-1}, z^{-1})$ . We can rewrite  $\mathbb{C}[y^{\pm 1}, z^{\pm 1}, \frac{y-y^{-1}}{z-z^{-1}}]^W$  as  $\mathbb{C}[y^{\pm 1}, z^{\pm 1}, \frac{y^2-1}{z^2-1}]^W$  to manifest its coincidence with  $\mathbb{C}[\mathfrak{B}_G^G]$ . All in all, we have  $\mathfrak{B}_G^G \simeq \mathfrak{S} \simeq \mathfrak{Z}_G^G$ . Since we can



identify  $\check{T}$  with  $T/Z(G)$ , the identifications  $\mathfrak{B}_G^{\check{G}} \simeq \mathfrak{S}/j$ ,  $\mathfrak{B}_G^{\check{G}} \simeq \iota \backslash \mathfrak{S}$ ,  $\mathfrak{B}_G^{\check{G}} \simeq \iota \backslash \mathfrak{S}/j \simeq \mathfrak{Z}_G^{\check{G}}$  follow immediately.

**3.2.  $\mathfrak{Z}_{\mathfrak{g}}^G$  and  $\mathfrak{B}_{\mathfrak{g}}^G$ .** The Kostant slice  $\Sigma_{\mathfrak{g}} = \left\{ \begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix}, \delta \in \mathbb{C} \right\}$ . One checks that a matrix  $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$  commutes with  $\begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix}$  iff  $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} \xi & \delta\eta \\ \eta & \xi \end{pmatrix}$  for  $\xi, \eta \in \mathbb{C}$ . Then the condition  $\det = 1$  reads as

$$(6) \quad 1 = \xi^2 - \delta\eta^2.$$

Thus,  $\mathfrak{Z}_{\mathfrak{g}}^G$  is identified with a hypersurface  $\mathfrak{S}'$  in  $\mathbb{A}^3$  given by the equation (6). The action of  $-1 \in Z(SL_2)$  is an involution  $\iota$  on  $\mathfrak{S}'$  given by  $\iota(\delta, \xi, \eta) = (\delta, -\xi, -\eta)$ . Hence,  $\mathfrak{Z}_{\mathfrak{g}}^{\check{G}} = \iota \backslash \mathfrak{S}'$ .

Generically, we can diagonalize two commuting matrices simultaneously, that is, there is  $g \in SL_2$  such that  $g \begin{pmatrix} \xi & \delta\eta \\ \eta & \xi \end{pmatrix} g^{-1} = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}$  and  $g \begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix} g^{-1} = \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}$  for some  $y \in \mathbb{G}_m = \mathbb{C}^* = T$ ,  $x \in \mathbb{C} = \mathfrak{t}$ , defined up to  $(y, x) \mapsto (y^{-1}, -x)$ . Then we have

$$\delta = x^2, \quad \xi = \frac{y + y^{-1}}{2}, \quad \eta = \frac{y - y^{-1}}{2x}.$$

We conclude that  $\mathbb{C}[\mathfrak{S}'] = \mathbb{C}[y^{\pm 1}, x, \frac{y-y^{-1}}{x}]^W$  where the nontrivial element  $w \in W$  acts by  $w(y, x) = (y^{-1}, -x)$ . We can rewrite  $\mathbb{C}[y^{\pm 1}, x, \frac{y-y^{-1}}{x}]^W$  as  $\mathbb{C}[y^{\pm 1}, x, \frac{y^2-1}{x}]^W$  to manifest its coincidence with  $\mathbb{C}[\mathfrak{B}_{\mathfrak{g}}^G]$ . All in all, we have  $\mathfrak{B}_{\mathfrak{g}}^G \simeq \mathfrak{S}' \simeq \mathfrak{Z}_{\mathfrak{g}}^G$ . Since we can identify  $\check{T}$  with  $T/Z(G)$ , the identification  $\mathfrak{B}_G^{\check{G}} \simeq \iota \backslash \mathfrak{S}' \simeq \mathfrak{Z}_G^{\check{G}}$  follows immediately.

**3.3.  $\mathfrak{Z}_G^{\mathfrak{a}}$  and  $\mathfrak{B}_G^{\mathfrak{a}}$ .** Recall the Kostant slice  $\Sigma_G = \left\{ \begin{pmatrix} a-1 & a-2 \\ 1 & 1 \end{pmatrix}, a \in \mathbb{C} \right\}$ . One checks that a traceless matrix  $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & -x_{11} \end{pmatrix}$  commutes with  $\begin{pmatrix} a-1 & a-2 \\ 1 & 1 \end{pmatrix}$  iff  $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & -x_{11} \end{pmatrix} = \zeta \begin{pmatrix} 2-a & 4-2a \\ -2 & a-2 \end{pmatrix}$  for  $\zeta \in \mathbb{C}$ .

Thus,  $\mathfrak{Z}_G^{\mathfrak{a}}$  is identified with  $\mathbb{A}^2$  with coordinates  $a, \zeta$ . The action of  $-1 \in Z(SL_2)$  is an involution  $j$  on  $\mathbb{A}^2$  given by  $j(a, \zeta) = (-a, -\zeta)$ . Hence,  $\mathfrak{Z}_G^{\check{\mathfrak{a}}} = \mathbb{A}^2/j$ .

Generically, we can diagonalize two commuting matrices simultaneously, that is, there is  $g \in SL_2$  such that  $g\zeta \begin{pmatrix} 2-a & 4-2a \\ -2 & a-2 \end{pmatrix} g^{-1} = \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}$  and  $g \begin{pmatrix} a-1 & a-2 \\ 1 & 1 \end{pmatrix} g^{-1} = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$  for some  $x \in \mathbb{C} = \mathfrak{t}$ ,  $z \in \mathbb{G}_m = \mathbb{C}^* = T$  defined up to  $(x, z) \mapsto (-x, z^{-1})$ . Then we have

$$a = z + z^{-1}, \quad \zeta = \frac{x}{z - z^{-1}}.$$

We conclude that  $\mathbb{C}[\mathbb{A}^2] = \mathbb{C}[x, z^{\pm 1}, \frac{x}{z-z^{-1}}]^W$  where the nontrivial element  $w \in W$  acts by  $w(x, z) = (-x, z^{-1})$ . We can rewrite  $\mathbb{C}[x, z^{\pm 1}, \frac{x}{z-z^{-1}}]^W$  as  $\mathbb{C}[x, z^{\pm 1}, \frac{x}{z^2-1}]^W$  to manifest its coincidence with  $\mathbb{C}[\mathfrak{B}_G^{\mathfrak{g}}]$ . All in all, we have  $\mathfrak{B}_G^{\mathfrak{g}} \simeq \mathbb{A}^2 \simeq \mathfrak{Z}_G^{\mathfrak{g}}$ . Since we can identify  $\check{T}$  with  $T/Z(G)$ , the identification  $\mathfrak{B}_G^{\mathfrak{g}} \simeq \mathbb{A}^2/j \simeq \mathfrak{Z}_G^{\mathfrak{g}}$  follows immediately.

**3.4.  $\mathfrak{Z}_G^{\mathfrak{g}}$  and  $\mathfrak{B}_G^{\mathfrak{g}}$ .** Recall the Kostant slice  $\Sigma_{\mathfrak{g}} = \left\{ \begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix}, \delta \in \mathbb{C} \right\}$ . One checks that a traceless matrix  $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & -x_{11} \end{pmatrix}$  commutes with  $\begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix}$  iff  $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & -x_{11} \end{pmatrix} = \begin{pmatrix} 0 & \delta\theta \\ \theta & 0 \end{pmatrix}$  for  $\theta \in \mathbb{C}$ . Thus,  $\mathfrak{Z}_G^{\mathfrak{g}}$  is identified with  $\mathbb{A}^2$  with coordinates  $\delta, \theta$ .

Generically, we can diagonalize two commuting matrices simultaneously, that is, there is  $g \in SL_2$  such that  $g \begin{pmatrix} 0 & \delta\theta \\ \theta & 0 \end{pmatrix} g^{-1} = \begin{pmatrix} u & 0 \\ 0 & -u \end{pmatrix}$  and  $g \begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix} g^{-1} = \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}$  for some  $u, x \in \mathbb{C} = \mathfrak{t}$ , defined up to  $(u, x) \mapsto (-u, -x)$ . Then we have

$$\delta = x^2, \theta = \frac{u}{x}.$$

We conclude that  $\mathbb{C}[\mathbb{A}^2] = \mathbb{C}[u, x, \frac{u}{x}]^W$  where the nontrivial element  $w \in W$  acts by  $w(u, x) = (-u, -x)$ . Hence we get an identification  $\mathfrak{B}_G^{\mathfrak{g}} \simeq \mathbb{A}^2 \simeq \mathfrak{Z}_G^{\mathfrak{g}}$ .

**3.5.  $\mathfrak{B}_G^G$  and  $\mathfrak{F}_G^G$ .** Recall the setup of Proposition 2.10. We will prove that the functor  $\mathfrak{F}_G^G$  is representable by the scheme  $\mathfrak{B}_G^G$ ; the other parts of the Proposition are proved absolutely similarly, as well as the Proposition for  $G$  replaced by  $\check{G}$ . For a scheme  $S$  flat over  $\mathfrak{t}/W$  we will denote by  $S_{\mathfrak{t}}$  the cartesian product  $S \times_{\mathfrak{t}/W} \mathfrak{t}$ . Our usual coordinate  $x$  on  $\mathfrak{t}$  gives rise to the same named function on  $S_{\mathfrak{t}}$ . The nontrivial element  $w \in W$  acts by the involution of  $S_{\mathfrak{t}}$ . Finally, we denote by  $(\mathfrak{B}_G^G)_{S_{\mathfrak{t}}}$  the affine blow-up of  $S_{\mathfrak{t}} \times T$ , that is  $S_{\mathfrak{t}} \times_{\mathfrak{t}} \mathfrak{B}_G^G$ . Clearly,  $w$  acts as an involution of  $(\mathfrak{B}_G^G)_{S_{\mathfrak{t}}}$ .

Note that the condition (1) is void in the case under consideration. Given a  $w$ -equivariant morphism  $f : S_{\mathfrak{t}} \rightarrow T = \mathbb{G}_m$  we see that  $f^2 - 1$  is divisible by  $x$ , hence  $f$  lifts uniquely to a section  $\hat{f}$  of  $(\mathfrak{B}_G^G)_{S_{\mathfrak{t}}}$  over  $S_{\mathfrak{t}}$ . Evidently,  $\hat{f}$  is  $w$ -invariant. If we consider  $\hat{f}$  as a closed subscheme of  $(\mathfrak{B}_G^G)_{S_{\mathfrak{t}}}$ , then  $\hat{f}/W$  is a closed subscheme of  $(\mathfrak{B}_G^G)_{S_{\mathfrak{t}}}/W = S \times_{\mathfrak{t}/W} \mathfrak{B}_G^G$  which is the graph of a morphism  $\tilde{f} : S \rightarrow \mathfrak{B}_G^G$ .

Conversely, given a morphism  $\tilde{f} : S \rightarrow \mathfrak{B}_G^G$  we consider its graph  $\Gamma_{\tilde{f}}$  as a closed subscheme of  $S \times_{\mathfrak{t}/W} \mathfrak{B}_G^G$ , and then the cartesian product  $\Gamma_{\tilde{f}} \times_{S \times_{\mathfrak{t}/W} \mathfrak{B}_G^G} (\mathfrak{B}_G^G)_{S_{\mathfrak{t}}}$  is a section  $\hat{f}$  of  $(\mathfrak{B}_G^G)_{S_{\mathfrak{t}}}$  over  $S_{\mathfrak{t}}$ . Evidently,  $\hat{f}$  gives rise to a  $w$ -equivariant function  $f : S_{\mathfrak{t}} \rightarrow T$ .

**3.6. A basis in equivariant  $K$ -theory.** We recall a few standard facts about the affine Grassmannians  $\text{Gr}_G$  and  $\text{Gr}_{\check{G}}$ . The  $G(\mathbf{O})$ -orbits (equivalently,  $\check{G}(\mathbf{O})$ -orbits) on  $\text{Gr}_{\check{G}}$  are numbered by nonnegative integers and denoted by  $\text{Gr}_{\check{G}, n}$ ,  $n \in \mathbb{N}$ . The orbits  $\text{Gr}_{\check{G}, 2n}$ ,  $n \in \mathbb{N}$ , form a connected component of  $\text{Gr}_{\check{G}}$  equal to  $\text{Gr}_G$ . The open embedding of an orbit into its closure will be denoted by  $j_n : \text{Gr}_{\check{G}, n} \hookrightarrow \overline{\text{Gr}_{\check{G}, n}}$  or simply by  $j$  if no confusion is likely. We have  $\dim \text{Gr}_{\check{G}, n} = n$ ; in particular,  $\text{Gr}_{\check{G}, 0}$  is a point.

We have  $K^{G(\mathbf{O})}(\mathrm{Gr}_{\check{G},0}) = \mathrm{Rep}(G)$  with a basis  $\mathbf{v}(n)$ ,  $n \in \mathbb{N}$ , formed by the classes of irreducible  $G$ -modules  $\mathcal{V}(n)$ . Also,  $K^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G},0}) = \mathrm{Rep}(\check{G}) \subset \mathrm{Rep}(G)$  has a basis  $\mathbf{v}(2n)$ ,  $n \in \mathbb{N}$ .

For  $m > 0$  the  $G(\mathbf{O})$ -equivariant line bundles in  $\mathrm{Gr}_{\check{G},m}$  are numbered by integers and denoted by  $\mathcal{L}(n)_m$ . Among them, the  $\check{G}(\mathbf{O})$ -equivariant line bundles are exactly  $\mathcal{L}(2n)_m$ ,  $n \in \mathbb{Z}$ . We define  $\mathcal{V}(n)_m$  as  $j_*\mathcal{L}(n)_m[\frac{m}{2}]$ , that is, the (nonderived) direct image to the orbit closure placed in the homological degree  $-\frac{m}{2}$ . Note that since the complement  $\overline{\mathrm{Gr}}_{\check{G},m} - \mathrm{Gr}_{\check{G},m}$  has codimension 2, the above direct image is a coherent sheaf. The degree shift will become clear later. The class  $[\mathcal{L}(n)_m]$  in  $K^{G(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$  will be denoted by  $\mathbf{v}(n)_m$ . Thus, it is natural to denote  $\mathbf{v}(n)$  above by  $\mathbf{v}(n)_0$ ; we will keep both names.

The collection  $\{\mathbf{v}(n)_m : n \in \mathbb{N} \text{ if } m = 0; n \in \mathbb{Z} \text{ if } m \in \mathbb{N} - 0\}$  forms a basis in  $K^{G(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$ . Among this collection, all the  $\mathbf{v}(n)_m$  with  $n$  even (resp.  $m$  even) form a basis in  $K^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$  (resp.  $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$ ).

**3.7. Convolution: commutativity.** In this subsection  $G$  is an arbitrary semisimple group. We prove 2.15 (a). We refer the reader to [7] for the basics of Beilinson-Drinfeld Grassmannian. Recall that  $\mathrm{Gr}_G^{BD} \xrightarrow{\pi} \mathbb{A}^1$  is a flat ind-scheme such that  $\pi^{-1}(\mathbb{A}^1 - 0) = (\mathbb{A}^1 - 0) \times \mathrm{Gr}_G \times \mathrm{Gr}_G$ , while  $\pi^{-1}(0) = \mathrm{Gr}_G$ . We also have the deformed convolution diagram  $\mathrm{Gr}_G^{BD,conv} \xrightarrow{\Pi} \mathrm{Gr}_G^{BD}$  such that  $\Pi$  is an isomorphism over  $\mathbb{A}^1 - 0$ , while over  $0 \in \mathbb{A}^1$  our  $\Pi$  is the usual convolution diagram  $G(\mathbf{F}) \times_{G(\mathbf{O})} \mathrm{Gr}_G \xrightarrow{\Pi_0} \mathrm{Gr}_G$ .

Given two  $G(\mathbf{O})$ -equivariant complexes of coherent sheaves  $\mathcal{A}, \mathcal{B}$  on  $\mathrm{Gr}_G$ , we can form their “deformed convolution” complex  $\mathcal{A} \tilde{\star} \mathcal{B}$  on  $\mathrm{Gr}_G^{BD,conv}$  such that over  $\mathbb{A}^1 - 0$  it is isomorphic to  $\mathcal{O}_{\mathbb{A}^1 - 0} \boxtimes \mathcal{A} \boxtimes \mathcal{B}$ , while over  $0 \in \mathbb{A}^1$  it is isomorphic to the usual twisted product  $\mathcal{A} \times \mathcal{B}$  on the convolution diagram  $G(\mathbf{F}) \times_{G(\mathbf{O})} \mathrm{Gr}_G$ . In addition, if  $\mathcal{A}, \mathcal{B}$  are coherent sheaves, then  $\mathcal{A} \tilde{\star} \mathcal{B}$  is flat over  $\mathbb{A}^1$ . It implies that in the  $K$ -group the class  $[\mathcal{A} \times \mathcal{B}]$  is the *specialization* (see [4] 5.3) of the class  $[\mathcal{O}_{\mathbb{A}^1 - 0} \boxtimes \mathcal{A} \boxtimes \mathcal{B}]$  in the family  $\mathrm{Gr}_G^{BD,conv} \xrightarrow{\pi \circ \Pi} \mathbb{A}^1$ , and also the class  $[\mathcal{A} \star \mathcal{B}] = [\Pi_{0*}(\mathcal{A} \times \mathcal{B})]$  is the specialization of the class  $[\mathcal{O}_{\mathbb{A}^1 - 0} \boxtimes \mathcal{A} \boxtimes \mathcal{B}]$  in the family  $\mathrm{Gr}_G^{BD} \xrightarrow{\pi} \mathbb{A}^1$ . Hence the desired commutativity.

**3.8. Convolution: relations.** We return to the setup of 3.6. Note that  $\mathrm{Gr}_{\check{G},1} \simeq \mathbb{P}^1$ , and  $\mathcal{V}(n)_1$  is the line bundle  $\mathcal{O}(n)$  on  $\mathbb{P}^1$ . The twisted product  $\mathcal{V}(n)_1 \times \mathcal{V}(l)_1$  is the line bundle  $\mathcal{O}(n, l)$  on the 2-dimensional subvariety  $\mathcal{H}_2 \subset \check{G}(\mathbf{F}) \times_{\check{G}(\mathbf{O})} \mathrm{Gr}_{\check{G}}$  isomorphic to the Hirzebruch surface  $\mathbb{P}(\mathcal{O}(2) \oplus \mathcal{O})$  over  $\mathbb{P}^1$ . The projection  $\Pi_0 : \mathcal{H}_2 \rightarrow \mathrm{Gr}_{\check{G},2}$  is the contraction of the  $-2$ -section  $\mathbb{P}^1 \hookrightarrow \mathcal{H}_2$ .

Now it is easy to compute  $\mathbf{v}(n)_1 \star \mathbf{v}(n)_1 = \mathbf{v}(2n)_2$ ,  $\mathbf{v}(1)_1 \star \mathbf{v}(-1)_1 = \mathbf{v}(0)_2 + 1$ . Taking into account the evident relation  $\mathbf{v}(1)_0 \star \mathbf{v}(0)_1 = \mathbf{v}(1)_1 + \mathbf{v}(-1)_1$  we arrive at

$$(7) \quad \mathbf{v}(1)_0 \star \mathbf{v}(0)_1 \star \mathbf{v}(1)_1 = \mathbf{v}(1)_1 \star \mathbf{v}(1)_1 + \mathbf{v}(0)_1 \star \mathbf{v}(0)_1 + 1.$$

A moment of reflection shows that  $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$  is generated as algebra by  $\mathbf{v}(1)_0$ ,  $\mathbf{v}(0)_2 = \mathbf{v}(0)_1 \star \mathbf{v}(0)_1$ ,  $\mathbf{v}(2)_2 = \mathbf{v}(1)_1 \star \mathbf{v}(1)_1$ ,  $\mathbf{v}(1)_2 = \mathbf{v}(1)_1 \star \mathbf{v}(0)_1$  (one has to use that  $\mathbf{v}(k)_{2l} \star \mathbf{v}(n)_{2m} = \mathbf{v}(k+n)_{2l+2m}$  plus the terms supported

on the smaller orbits). Similarly,  $K^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$  is generated as algebra by  $\mathbf{v}(2)_0 = \mathbf{v}(1)_0 \star \mathbf{v}(1)_0 - 1$ ,  $\mathbf{v}(0)_1$ ,  $\mathbf{v}(2)_2 = \mathbf{v}(1)_1 \star \mathbf{v}(1)_1$ ,  $\mathbf{v}(2)_1 = \mathbf{v}(1)_1 \star \mathbf{v}(1)_0 - \mathbf{v}(0)_1$ .

Note that both algebras  $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$  and  $K^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$  lie in the vector space  $K^{G(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$ , and their intersection is the common subalgebra  $K^{\check{G}(\mathbf{O})}(\mathrm{Gr}_G)$ . The tensor product algebra  $K^{G(\mathbf{O})}(\mathrm{Gr}_G) \otimes_{K^{\check{G}(\mathbf{O})}(\mathrm{Gr}_G)} K^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$  can be identified as a vector space with  $K^{G(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$ , and then it is generated by the three basic elements  $\mathbf{v}(1)_0, \mathbf{v}(0)_1, \mathbf{v}(1)_1$  subject to the only relation (7).

The comparison of equations (7) and (4) shows that the assignment  $a \mapsto \mathbf{v}(1)_0$ ,  $b \mapsto \mathbf{v}(0)_1$ ,  $c \mapsto \mathbf{v}(1)_1$  establishes an isomorphism  $\mathbb{C}[\mathcal{S}] \simeq K^{G(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$ . It identifies the spectrum of  $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$  with  $\imath \backslash \mathcal{S} \simeq \mathfrak{B}_{\check{G}}^{\check{G}}$ , and the spectrum of  $K^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$  with  $\mathcal{S}/\mathfrak{J} \simeq \mathfrak{B}_{\check{G}}^G$ .

**3.9. Iwahori-equivariant  $K$ -theory.** Let  $I \subset G(\mathbf{O})$  be the Iwahori subgroup. The space  $K^I(\mathrm{Gr}_G) = K^T(\mathrm{Gr}_G) = K^{T(\mathbf{O})}(\mathrm{Gr}_G) = K(T(\mathbf{O}) \backslash G(\mathbf{F})/G(\mathbf{O}))$  is equipped with the two commuting actions:  $K(T(\mathbf{O}) \backslash T(\mathbf{F})/T(\mathbf{O}))$  acts by convolutions on the left, and  $K^G(\mathrm{Gr}_G) = K^{G(\mathbf{O})}(\mathrm{Gr}_G) = K(G(\mathbf{O}) \backslash G(\mathbf{F})/G(\mathbf{O}))$  acts by convolutions on the right. Also,  $W$  acts on  $K^T(\mathrm{Gr}_G)$  commuting with the right action of  $K^G(\mathrm{Gr}_G)$ . Clearly, the algebra  $K(T(\mathbf{O}) \backslash T(\mathbf{F})/T(\mathbf{O}))$  is isomorphic to  $\mathbb{C}[\check{T} \times T]$ . The action of  $W$  on  $K^T(\mathrm{Gr}_G)$  normalizes the action of  $K(T(\mathbf{O}) \backslash T(\mathbf{F})/T(\mathbf{O}))$  and induces the natural (diagonal) action of  $W$  on  $\mathbb{C}[\check{T} \times T]$ .

Our aim in this subsection is to identify the  $K(T(\mathbf{O}) \backslash T(\mathbf{F})/T(\mathbf{O})) \rtimes W - K^G(\mathrm{Gr}_G)$ -bimodule  $K^T(\mathrm{Gr}_G)$  with the  $\mathbb{C}[\check{T} \times T] \rtimes W - \mathbb{C}[\mathfrak{B}_{\check{G}}^{\check{G}}]$ -bimodule  $\mathbb{C}[\mathfrak{B}_{\check{G}}^{\check{G}}]$  (and similarly for  $G$  replaced by  $\check{G}$ ). As in 3.8, it suffices to identify the  $K(T(\mathbf{O}) \backslash \check{T}(\mathbf{F})/\check{T}(\mathbf{O})) \rtimes W - K^G(\mathrm{Gr}_{\check{G}})$ -bimodule  $K^T(\mathrm{Gr}_{\check{G}})$  with the  $\mathbb{C}[\check{T} \times T] \rtimes W - \mathbb{C}[\mathfrak{B}_{\check{G}}^G]$ -bimodule  $\mathbb{C}[\mathfrak{B}_{\check{G}}^G]$ .

Note that  $K^G(\mathrm{Gr}_{\check{G}}) \subset K^T(\mathrm{Gr}_{\check{G}})$ , and the  $K^G(\mathrm{Gr}_{\check{G}})$ -module  $K^T(\mathrm{Gr}_{\check{G}})$  is free of rank 2 with the generators 1,  $z$  where  $z$  is the generator of  $K^T(pt) = \mathbb{C}[T]$  (so that, e.g.  $\mathbf{v}(1)_0 = z + z^{-1}$ ). Furthermore,  $\mathbb{C}[y^{\pm 1}, z^{\pm 1}] = \mathbb{C}[\check{T} \times T] = K(T(\mathbf{O}) \backslash \check{T}(\mathbf{F})/\check{T}(\mathbf{O})) \subset K^T(\mathrm{Gr}_{\check{G}})$ , and one can check

$$(8) \quad y^{-1} = \sqrt{-1}(\mathbf{u}_0 - \mathbf{u}_2), \quad y = \sqrt{-1}(\mathbf{v}(0)_1 - \mathbf{v}(2)_1 + \mathbf{u}_2 - \mathbf{u}_0)$$

where  $\mathbf{u}_0 \in K^T(\mathrm{Gr}_{\check{G}})$  (resp.  $\mathbf{u}_2$ ) is the class of the irreducible skyscraper sheaf supported at the one-point Iwahori orbit in  $\mathrm{Gr}_{\check{G},1} = \mathbb{P}^1$  with the trivial action of  $T$  (resp. with the action of  $T$  given by  $z^2$ ), and placed in the homological degree  $-\frac{1}{2}$ . Hence

$$(9) \quad y + y^{-1} = \sqrt{-1}(2\mathbf{v}(0)_1 - \mathbf{v}(1)_0 \star \mathbf{v}(1)_1), \quad y - y^{-1} = \sqrt{-1}(z - z^{-1})\mathbf{v}(1)_1.$$

Comparing (9) with (5) we get the desired identification of the  $K(T(\mathbf{O}) \backslash \check{T}(\mathbf{F})/\check{T}(\mathbf{O})) \rtimes W - K^G(\mathrm{Gr}_{\check{G}})$ -bimodule  $K^T(\mathrm{Gr}_{\check{G}})$  with the  $\mathbb{C}[y^{\pm 1}, z^{\pm 1}] \rtimes W - \mathbb{C}[y^{\pm 1}, z^{\pm 1}, \frac{y-y^{-1}}{z-z^{-1}}]$ -bimodule  $\mathbb{C}[y^{\pm 1}, z^{\pm 1}, \frac{y-y^{-1}}{z-z^{-1}}]$ .

**3.10. Borel-Moore Homology.** For an arbitrary semisimple  $G$  one proves the commutativity of  $H_{\bullet}^{G(\mathbf{O})}(\mathrm{Gr}_G)$  (Theorem 2.12 a) exactly as in 3.7 using the Beilinson-Drinfeld Grassmannian and the *specialization* in Borel-Moore Homology (see [4] 2.6.30).

For  $\check{G} = PGL_2$ , let us denote by  $\delta \in H_{\check{G}(\mathbf{O})}^4(pt, \mathbb{Z}) = H_4^{\check{G}(\mathbf{O})}(pt, \mathbb{Z})$  the generator of the equivariant (co)homology. Furthermore, we denote by  $\eta$  (resp.  $\xi$ ) the generator of  $H_{-2}^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G},1}, \mathbb{Z})$  (resp. the generator of  $H_0^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G},1}, \mathbb{Z})$ ). Then it is easy to see that  $\delta, \xi, \eta$  generate  $H_{\bullet}^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$  (while  $\delta, \xi^2, \eta^2, \xi\eta$  generate the subalgebra  $H_{\bullet}^{G(\mathbf{O})}(\mathrm{Gr}_G)$ ), and we claim that

$$(10) \quad 1 = \xi^2 - \delta\eta^2.$$

In effect, this is an equality in  $H_0^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G},2})$ . Since  $\mathrm{Gr}_{\check{G},2}$  is rationally smooth,  $H_0^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G},2}) = H_{\check{G}(\mathbf{O})}^4(\mathrm{Gr}_{\check{G},2})$ . Let us denote by  $\mathbf{B}\mathrm{Gr}_{\check{G},2} \xrightarrow{p} \mathbf{B}\check{G}(\mathbf{O})$  the associated fibre bundle over the classifying space of  $\check{G}(\mathbf{O})$  with the fiber  $\mathrm{Gr}_{\check{G},2}$ . Then  $1 \in H_{\check{G}(\mathbf{O})}^4(\mathrm{Gr}_{\check{G},2}) = H^4(\mathbf{B}\mathrm{Gr}_{\check{G},2})$  is the Poincaré dual class of the codimension 2 cycle  $\mathbf{B}\check{G}(\mathbf{O}) = \mathbf{B}\mathrm{Gr}_{\check{G},0} \hookrightarrow \mathbf{B}\mathrm{Gr}_{\check{G},2}$ , and  $\delta\eta^2 = p^*\delta$ .

Recall the convolution morphism  $\Pi_0 : \mathcal{H}_2 \rightarrow \mathrm{Gr}_{\check{G},2}$  of 3.8. This is a morphism of  $\check{G}(\mathbf{O})$ -varieties, and we denote by  $\Pi_0 : \mathbf{B}\mathcal{H}_2 \rightarrow \mathbf{B}\mathrm{Gr}_{\check{G},2}$  the corresponding morphism of associated fibre bundles. Note that (additively)  $H^{\bullet}(\mathbf{B}\mathcal{H}_2) = H^{\bullet}(\mathbf{B}\mathrm{Gr}_{\check{G},1}) \otimes H^{\bullet}(\mathbf{B}\check{G}(\mathbf{O})) + H^{\bullet}(\mathbf{B}\mathrm{Gr}_{\check{G},1})$ . Recall that  $\xi$  is the generator of  $H_0^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G},1}) = H_{\check{G}(\mathbf{O})}^2(\mathrm{Gr}_{\check{G},1}) = H^2(\mathbf{B}\mathrm{Gr}_{\check{G},1})$ . Finally, we have  $\xi^2 = \Pi_{0*}(\xi \otimes \xi)$ . Now (10) follows easily.

Comparing the sizes of  $H_{\bullet}^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$  and  $\mathbb{C}[\delta, \xi, \eta]/(\xi^2 - \delta\eta^2 - 1)$  we conclude that they are isomorphic. The comparison with the equation (6) establishes an isomorphism  $H_{\bullet}^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G}}) \simeq \mathbb{C}[S']$ , and identifies the spectrum of  $H_{\bullet}^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$  with  $S' \simeq \mathfrak{Z}_{\mathfrak{g}}^G$ , and the spectrum of  $H_{\bullet}^{G(\mathbf{O})}(\mathrm{Gr}_G)$  with  $\iota \setminus S' \simeq \mathfrak{Z}_{\mathfrak{g}}^G$ .

#### 4. CENTRALIZERS AND BLOW-UPS

The aim of this section is a proof of Proposition 2.8. We will consider  $\mathfrak{B}_G^G$  and  $\mathfrak{Z}_G^G$ , the other cases being similar. Till the further notice  $G$  is assumed simply connected.

**Lemma 4.1.**  $\varpi : \mathfrak{B}_G^G \rightarrow T/W$  is flat.

*Proof* It suffices to prove that the first projection of  $\mathfrak{B}_G^G$  to  $T$  is smooth (recall that  $\mathfrak{B}_G^G$  is defined as  $\mathrm{Spec}(\mathbb{C}[T \times T, \frac{1}{2}\frac{\alpha-1}{\alpha-1}, \alpha \in R])$ ). In effect, then  $\mathbb{C}[T \times T, \frac{1}{2}\frac{\alpha-1}{\alpha-1}, \alpha \in R]$  is a flat  $\mathbb{C}[T]$ -module; hence it is a flat  $\mathbb{C}[T]^W$ -module (since  $\mathbb{C}[T]$  is free over  $\mathbb{C}[T]^W$ , see [21]). Finally,  $\mathbb{C}[T \times T, \frac{1}{2}\frac{\alpha-1}{\alpha-1}, \alpha \in R]^W$  is a direct summand of a flat  $\mathbb{C}[T]^W$ -module  $\mathbb{C}[T \times T, \frac{1}{2}\frac{\alpha-1}{\alpha-1}, \alpha \in R]$ ; hence it is flat.

The affine blow-up  $\mathfrak{B}_G^G$  is the result of the following successive blow up of  $T \times T$ . We choose an ordering  $\alpha_1, \dots, \alpha_\nu$  of the set of positive roots  $R^+$ . We define  $\mathfrak{B}_1$  as the

blow up of  $T \times T$  at the diagonal wall  ${}^1\alpha_1 = {}^2\alpha_1 = 1$  with the proper preimage of the divisor  ${}^1\alpha_1 = 1$  removed. We define  $\mathfrak{B}_2$  as the blow up of  $\mathfrak{B}_1$  at the proper transform of the diagonal wall  ${}^1\alpha_2 = {}^2\alpha_2 = 1$  with the proper preimage of the divisor  ${}^1\alpha_2 = 1$  removed. Going on like this we construct  $\mathfrak{B}_\nu$ ; evidently, it coincides with  $\mathfrak{B}_G^\bullet$ .

Note that at each step the center of the blow-up is smooth over the corresponding wall  ${}^2\alpha_i = 1$  in  $T$  by the following Claim. Thus the desired flatness assertion follows inductively from the

*Claim.* Let  $p : X \rightarrow Y$  be a smooth morphism of smooth varieties; let  $X' \subset X$  be a subvariety such that  $Y' = f(X') \subset Y$  is a smooth hypersurface, and  $p : X' \rightarrow Y'$  is also smooth. Then the blow-up  $\text{Bl}_{X'} X$  with the proper preimage of the divisor  $p^{-1}(p(X'))$  removed is smooth over  $Y$ .

The smoothness is checked in the formal neighbourhoods of points by direct calculation in coordinates. This completes the proof of the lemma.

**4.2. The simultaneous resolution.** Recall that  $\{(g, B) : g \in B\} = \overset{\bullet}{G} \xrightarrow{p} G$  is the Grothendieck simultaneous resolution; here  $B$  is a Borel subgroup, and  $p(g, B) = g$ . We also have the projection  $\varrho : \overset{\bullet}{G} \rightarrow T$  to the abstract Cartan, which we identify with  $T$ ; namely,  $\varrho(g, B) = g \pmod{\text{rad}(B)}$ . The preimage  $p^{-1}(\Sigma_G) \subset \overset{\bullet}{G}$  is identified with  $T$  by  $\varrho$ . We denote by  $\overset{\bullet}{\mathfrak{Z}}_G^G \subset G \times \overset{\bullet}{G}$  the subset of triples  $(g_1, g_2, B)$  such that  $Ad_{g_1} = g_2$  and  $(g_2, B) \in p^{-1}(\Sigma_G)$ . Note that necessarily  $g_1 \in B$  (as well as  $g_2 \in B$ ); hence we have the projections  $\varrho_1, \varrho_2 : \overset{\bullet}{\mathfrak{Z}}_G^G \rightarrow T$ ; namely,  $\varrho_i(g_1, g_2, B) = g_i \pmod{\text{rad}(B)}$ .

The natural projection  $\overset{\bullet}{\mathfrak{Z}}_G^G \rightarrow \overset{\bullet}{\mathfrak{Z}}_G^G$  (forgetting  $B$ ) is a Galois  $W$ -covering. Finally,  $\varrho_2 : \overset{\bullet}{\mathfrak{Z}}_G^G \rightarrow T$  is flat.

**4.3. The proof of Proposition 2.8.** In order to identify  $\overset{\bullet}{\mathfrak{Z}}_G^G$  and  $\mathfrak{B}_G^G$  it suffices to identify their Galois  $W$ -coverings  $\overset{\bullet}{\mathfrak{Z}}_G^G \rightarrow T$  and  $\mathfrak{B}_G^G \rightarrow T$  in an equivariant way. Let  $\mathbf{D} \subset T$  denote the discriminant, so that  $T - \mathbf{D} = T^{reg}$ . Let  $\Delta \in \mathbb{C}[T]^W$  denote the product  $\prod_{\alpha \in R} (\alpha - 1)$ , so that  $\mathbf{D}$  is the divisor cut out by  $\Delta$ .

Evidently, both  $\overset{\bullet}{\mathfrak{Z}}_G^G|_{T^{reg}}$  and  $\mathfrak{B}_G^G|_{T^{reg}}$  are isomorphic to  $T \times T^{reg}$ . Hence both  $\mathbb{C}[\overset{\bullet}{\mathfrak{Z}}_G^G]$  and  $\mathbb{C}[\mathfrak{B}_G^G]$  are the flat  $\mathbb{C}[T]$ -modules embedded into  $\mathbb{C}[T \times T](\Delta^{-1})$ . We must prove that the identification of  $\overset{\bullet}{\mathfrak{Z}}_G^G|_{T^{reg}}$  and  $\mathfrak{B}_G^G|_{T^{reg}}$  extends to the identification over the whole  $T$ . To this end it suffices to check that the identification extends over the codimension 1 points of  $T$  (indeed, for a flat quasi-coherent sheaf  $\mathcal{F}$  on a normal irreducible scheme we have  $\mathcal{F} \xrightarrow{\sim} j_* j^* \mathcal{F}$  if  $j$  is an open imbedding with complement of codimension 2). Let  $g \in T$  be a regular point of  $\mathbf{D}$ ; that is,  $g$  is a semisimple element of  $G$  such that the centralizer  $Z(g)$  has semisimple rank 1.

We must construct an isomorphism between localizations  $(\overset{\bullet}{\mathfrak{Z}}_G^G)_g$  and  $(\mathfrak{B}_G^G)_g$  which is compatible with the above isomorphism at the generic point. To this end note that the embedding of reductive groups  $Z(g) \hookrightarrow G$  (note that  $Z(g)$  is connected since  $G$  is

simply connected) induces the morphisms  $\mathfrak{Z}_{Z(g)}^{Z(g)} \rightarrow \mathfrak{Z}_G^G$  and  $\mathfrak{B}_{Z(g)}^{Z(g)} \rightarrow \mathfrak{B}_G^G$  which become isomorphisms after localizations:  $\left(\mathfrak{Z}_{Z(g)}^{Z(g)}\right)_g \simeq \left(\mathfrak{Z}_G^G\right)_g$  and  $\left(\mathfrak{B}_{Z(g)}^{Z(g)}\right)_g \simeq \left(\mathfrak{B}_G^G\right)_g$ . Now the desired identification  $\left(\mathfrak{Z}_{Z(g)}^{Z(g)}\right)_g \simeq \left(\mathfrak{B}_{Z(g)}^{Z(g)}\right)_g$  follows from the calculations in 3.1.

This completes the identification  $\mathfrak{Z}_G^G \simeq \mathfrak{B}_G^G$  for a simply connected  $G$ . Evidently, this identification respects the left and right actions of the center  $Z(G)$ , so the isomorphism for an arbitrary  $G$  follows from the one for its universal cover. The other isomorphisms in 2.8 (b) are proved in a similar way.

To prove 2.8 (c), (d) it suffices to notice that the minimal level (viewed as a  $W$ -equivariant homomorphism  $T \rightarrow \check{T}$ ) for a simply laced simply connected  $G$  identifies  $\check{T}$  with  $T/Z(G)$ ; also,  $\check{G} = G/Z(G)$ .

## 5. $W$ -INVARIANT SECTIONS AND BLOW-UPS

The aim of this section is a proof of Proposition 2.10. We concentrate on the last statement, the other being completely similar.

Let  $T^{reg} \subset T$ ,  $T_\alpha^{reg} \subset T$  be the open subschemes defined by  $T^{reg} = \{t \mid \alpha(t) \neq 1 \text{ for all roots } \alpha\}$ ;  $T_\alpha^{reg} = \{t \mid \beta(t) \neq 1 \text{ for all roots } \beta \neq \alpha\}$ ; and  $\overset{\circ}{T} = \bigcup_{\alpha} T_\alpha^{reg}$  (thus  $T - \overset{\circ}{T}$  has codimension 2 in  $T$  (where the empty subscheme in a curve is considered to be of codimension 2)). Notice that since  $G$  is simply connected the action of  $W$  on  $T^{reg}$  is free.

We start with a

**Lemma 5.1.** *The map  $\mathfrak{B}_G^{\check{G}} \times_T \overset{\circ}{T} \rightarrow \mathfrak{B}_{\check{G}}^{\check{G}}/W \times_{T/W} \overset{\circ}{T}$  is an isomorphism.*

*Proof* Let  $X \rightarrow Y$  be a flat morphism of semi-separated (which means that the diagonal embedding is affine) schemes of finite type over a characteristic zero field, and let a finite group  $W$  act on  $X, Y$  so that the map is  $W$ -equivariant. Assume that  $Y$  is flat over  $Y/W$ . We then claim that the map  $X \rightarrow X/W \times_{Y/W} Y$  is an isomorphism provided that for every Zariski point  $y \in Y$  the action of  $\text{Stab}_W(y)$  on the scheme-theoretic fiber  $X_y$  is trivial (here  $X/W, Y/W$  stand for categorical quotients). To check this claim we can assume  $X$  is affine: by semi-separatedness every  $W$ -invariant subset in  $X$  has a  $W$ -invariant affine neighborhood. Let us first assume also that  $Y/W$  is a point; then (by replacing  $Y$  by its connected component, and  $W$  by the stabilizer of that component) we can assume that  $Y$  is nilpotent. Then  $\mathcal{O}_X$  is free over  $\mathcal{O}_Y$ , and the generators of  $\mathcal{O}_X$  as an  $\mathcal{O}_Y$  module can be chosen to be  $W$ -invariant (by semi-simplicity of the  $W$  action on  $\mathcal{O}_X$ , and triviality of the  $W$ -action on  $\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathbf{k}$ ); since  $\mathcal{O}_Y^W = \mathbf{k}$  (where  $\mathbf{k}$  is the base field) we see that  $\mathcal{O}_X^W \otimes_{\mathcal{O}_Y} \mathbf{k} \xrightarrow{\sim} \mathcal{O}_X$  as claimed. Now for a general  $Y$  we see that the morphism in question is a morphism of flat schemes of finite type over  $Y/W$ , which induces an isomorphism on every fiber; and such a morphism is necessarily an isomorphism.

Now it remains to check that the above conditions hold for  $X = \mathfrak{B}_G^{\check{G}} \times_T \overset{\circ}{T}$ ,  $Y = T$ . For  $y \in T^{reg}$  the stabilizer of  $y$  is trivial, so there is nothing to check. Consider now

$y \in T_\alpha^{reg}$ ,  $y \notin T^{reg}$ . Then the stabilizer of  $y$  is  $\{1, s_\alpha\}$ . The ring of functions on  $\mathfrak{B}_G^{\check{\alpha}}$  is generated by  ${}^1\check{\lambda}$ ,  ${}^2\mu$ ,  $t_\alpha$  where  $\check{\lambda}$ ,  $\mu$  run over weights of  $\check{T}$ ,  $T$  respectively,  $\alpha \in R^+$ , and  $t_\alpha(2\alpha - 1) = {}^1\check{\alpha} - 1$ . We have  $s_\alpha^*({}^1\check{\lambda}) = {}^1\check{\lambda} \cdot ({}^1\check{\alpha})^{\langle -\alpha, \check{\lambda} \rangle}$ ,  $s_\alpha^*({}^2\mu) = {}^2\mu \cdot ({}^2\alpha)^{\langle -\mu, \check{\alpha} \rangle}$ , and  $s_\alpha^*(t_\alpha) = t_\alpha \cdot \frac{{}^2\alpha}{{}^1\check{\alpha}}$ . On the fiber we have  ${}^2\alpha = 1$ , hence  ${}^1\check{\alpha} = 1$ , so the action of  $s_\alpha$  on the fiber is trivial.  $\square$

Proposition 2.10 clearly follows from the (ii)  $\iff$  (iv) part of the next

**Proposition 5.2.** *Let  $S \rightarrow T/W$  be a flat morphism, and set  $\phi : S \times_{T/W} T^{reg}/W \rightarrow (\check{T} \times T)/W$  be a  $T^{reg}/W$ -morphism. Then the following are equivalent:*

- (i)  $\phi$  extends to a morphism  $S \times_{T/W} \mathring{T}/W \rightarrow \mathfrak{B}_G^{\check{\alpha}} \times_{T/W} \mathring{T}$ .
- (ii)  $\phi$  extends to a morphism  $S \rightarrow \mathfrak{B}_G^{\check{\alpha}}$ .
- (iii) For every  $\alpha \in R$  the morphism  $\phi \times id_{T^{reg}} : S \times_{T/W} T^{reg} \rightarrow \check{T} \times T^{reg}$  extends to a morphism  $S \times_{T/W} T_\alpha^{reg} \rightarrow \check{T} \times T_\alpha^{reg}$  such that (3) holds.
- (iv)  $\phi \times id_{T^{reg}} : S \times_{T/W} T^{reg} \rightarrow \check{T} \times T^{reg}$  extends to a morphism  $S \times_{T/W} T \rightarrow \check{T} \times T$ , such that (3) holds for every  $\alpha \in R$ .

*Proof* It is enough to assume that  $S$  is affine. Indeed, a morphism from  $S$  extends iff its restriction to every affine open in  $S$  does, because compatibility on intersections follows from uniqueness of such an extension; this uniqueness follows from flatness: if  $S$  is flat affine, then tensoring the injection  $\mathcal{O} \rightarrow j_*\mathcal{O}$  with  $\mathcal{O}_S$  we get an imbedding  $\mathcal{O}_S \hookrightarrow j_*j^*\mathcal{O}_S$ , where  $j$  stands for the imbedding  $T^{reg}/W \rightarrow T/W$ , or  $T^{reg} \rightarrow T$ . So we will assume  $S$  affine from now on.

(iv)  $\Rightarrow$  (iii) and (ii)  $\Rightarrow$  (i) are obvious.

To check that (iii)  $\Rightarrow$  (iv) we tensor (over  $\mathcal{O}_{T/W}$ ) the exact sequence of  $\mathcal{O}_T$ -modules

$$(11) \quad 0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}_{T^{reg}} \rightarrow \bigoplus_{\alpha} (\mathcal{O}_{T_\alpha^{reg}}/\mathcal{O}_T)$$

with  $\mathcal{O}_S$ . The resulting exact sequence shows that a regular function on  $S \times_{T/W} T^{reg}$  extends to a regular function on  $S \times_{T/W} T$  iff it extends to  $S \times_T T_\alpha^{reg}$  for all  $\alpha$ . Applying this observation to  $(\phi \times id)^*(f|_{\check{T} \times T^{reg}})$  for each regular function  $f$  on  $\check{T} \times T$  we see that (iii) implies extendability of  $\phi \times id$  to  $S \times_{T/W} T$ . It is also clear that (3) holds if it holds on  $\mathring{T}$ .

Verification of (i)  $\Rightarrow$  (ii) is similar (with (11) replaced by the  $W$ -invariant part of (11)).

It remains to check (i)  $\iff$  (iii). If (i) holds, i.e.  $\phi$  extends to a map  $S \times_{T/W} \mathring{T}/W \rightarrow \mathfrak{B}_G^{\check{\alpha}} \times_{T/W} \mathring{T}$  then we can take the fiber product of this map with  $id_{\mathring{T}}$  over  $T/W$ . By Lemma 5.1 it yields a map  $S \times_{T/W} \mathring{T} \rightarrow \mathfrak{B}_G^{\check{\alpha}} \times_T \mathring{T}$ , which can be composed with the projection  $\mathfrak{B}_G^{\check{\alpha}} \rightarrow \check{T} \times T$  to produce a map  $S \times_{T/W} \mathring{T} \rightarrow \check{T} \times \mathring{T}$ . It is clear that this map satisfies (3), because the image of the map  $\mathfrak{B}_G^{\check{\alpha}} \rightarrow \check{T} \times T$  intersected with  $\check{T} \times \text{Ker}({}^2\alpha)$  is contained in  $\text{Ker}({}^1\check{\alpha}) \times T$ .



Conversely, if (iii) holds then restricting the given map  $S \times_{T/W} \overset{\circ}{T} \rightarrow \check{T} \times \overset{\circ}{T}$  to  $S \times_{T/W} (\text{Ker}(\alpha) \cap \overset{\circ}{T})$  we get a map into  $\text{Ker}(\check{\alpha}) \times T$  (this is immediate from (3)). This means that the map lifts to a map into  $\overset{\bullet}{\mathfrak{B}}_{\check{G}}$ . Replacing both the source and the target by their quotients by  $W$  we get the map required in (i).  $\square$

## 6. $K$ -THEORY AND BLOW-UPS

The aim of this section is a proof of Proposition 2.15. Recall that 2.15 (a) was already proved in 3.7.  $G$  is assumed simply connected till the further notice.

**6.1. Reminder on the affine Grassmannians.** Let  $X = X_G$  be the lattice of characters of  $T$ , and let  $Y = Y_G$  be the lattice of cocharacters of  $G$ . Note that  $X_G = Y_{\check{G}}$ ,  $Y_G = X_{\check{G}}$ . Let  $X^+ \subset X$  (resp.  $Y^+ \subset Y$ ) be the cone of dominant weights (resp. dominant coweights). It is well known that the  $G(\mathbf{O})$ -orbits in  $\text{Gr}_G$  are numbered by the dominant coweights:  $\text{Gr}_G = \bigsqcup_{\check{\lambda} \in Y^+} \text{Gr}_{G, \check{\lambda}}$ . The adjacency relation of orbits corresponds to the standard partial order on coweights:  $\overline{\text{Gr}}_{G, \check{\lambda}} = \bigsqcup_{\check{\mu} \leq \check{\lambda}} \text{Gr}_{G, \check{\mu}}$ . The open embedding  $\text{Gr}_{G, \check{\lambda}} \hookrightarrow \overline{\text{Gr}}_{G, \check{\lambda}}$  will be denoted by  $j_{\check{\lambda}}$  or simply by  $j$  if no confusion is likely. The dimension  $\dim(\text{Gr}_{G, \check{\lambda}}) = \langle 2\rho, \check{\lambda} \rangle$  where  $2\rho = \sum_{\alpha \in R^+} \alpha$ , and  $\langle \cdot, \cdot \rangle : X \times Y \rightarrow \mathbb{Z}$  is the canonical perfect pairing.

Recall that the  $T$ -fixed points in  $\text{Gr}_G$  are naturally numbered by  $Y$ ; a point  $\check{\mu}$  lies in an orbit  $\text{Gr}_{G, \check{\lambda}}$  iff  $\check{\mu}$  lies in the  $W$ -orbit of  $\check{\lambda}$ . Each  $G(\mathbf{O})$ -orbit  $\text{Gr}_{G, \check{\lambda}}$  is partitioned into Iwahori orbits isomorphic to affine spaces and numbered by  $\check{\mu} \in W\check{\lambda}$ . Hence the basics of [4] Chapter 5 are applicable in our situation.

In particular,  $K^T(\text{Gr}_{G, \check{\lambda}})$  is a free  $K^T(pt)$ -module, and  $K^{G(\mathbf{O})}(\text{Gr}_{G, \check{\lambda}}) = K^G(\text{Gr}_{G, \check{\lambda}})$  is a free  $K^G(pt)$ -module (recall that  $K^T(pt) = \mathbb{C}[T]$ , and  $K^G(pt) = \mathbb{C}[T/W]$ ). Moreover, the natural map  $K^T(pt) \otimes_{K^G(pt)} K^G(\text{Gr}_{G, \check{\lambda}}) \rightarrow K^T(\text{Gr}_{G, \check{\lambda}})$  is an isomorphism, and  $K^G(\text{Gr}_{G, \check{\lambda}}) = K^T(\text{Gr}_{G, \check{\lambda}})^W$ , cf. [4] 6.1.22.

Since  $K^{T(\mathbf{O})}(\text{Gr}_G) = K^T(\text{Gr}_G)$  (resp.  $K^{G(\mathbf{O})}(\text{Gr}_G) = K^G(\text{Gr}_G)$ ) is filtered by the support in  $G(\mathbf{O})$ -orbit closures, with the associated graded  $\bigoplus_{\check{\lambda} \in Y^+} K^T(\text{Gr}_{G, \check{\lambda}})$  (resp.  $\bigoplus_{\check{\lambda} \in Y^+} K^G(\text{Gr}_{G, \check{\lambda}})$ ), we arrive at the following

**Lemma 6.2.**  *$K^{T(\mathbf{O})}(\text{Gr}_G) = K^T(\text{Gr}_G)$  is a flat  $K^T(pt)$ -module, and  $K^{G(\mathbf{O})}(\text{Gr}_G) = K^G(\text{Gr}_G)$  is a flat  $K^G(pt)$ -module. Moreover, the natural map  $K^T(pt) \otimes_{K^G(pt)} K^G(\text{Gr}_G) \rightarrow K^T(\text{Gr}_G)$  is an isomorphism, and  $K^G(\text{Gr}_G) = (K^T(\text{Gr}_G))^W$ .*

**6.3. Localization.** The space  $K^T(\text{Gr}_G) = K^{T(\mathbf{O})}(\text{Gr}_G) = K(T(\mathbf{O}) \backslash G(\mathbf{F}) / G(\mathbf{O}))$  is equipped with the two commuting actions:  $K(T(\mathbf{O}) \backslash T(\mathbf{F}) / T(\mathbf{O}))$  acts by convolutions on the left, and  $K^G(\text{Gr}_G) = K^{G(\mathbf{O})}(\text{Gr}_G) = K(G(\mathbf{O}) \backslash G(\mathbf{F}) / G(\mathbf{O}))$  acts by convolutions on the right. Also,  $W$  acts on  $K^T(\text{Gr}_G)$  commuting with the right action of  $K^G(\text{Gr}_G)$ . Clearly, the algebra  $K(T(\mathbf{O}) \backslash T(\mathbf{F}) / T(\mathbf{O}))$  is isomorphic to  $\mathbb{C}[\check{T} \times T]$ . The action of  $W$  on  $K^T(\text{Gr}_G)$  normalizes the action of  $K(T(\mathbf{O}) \backslash T(\mathbf{F}) / T(\mathbf{O}))$  and induces the natural (diagonal) action of  $W$  on  $\mathbb{C}[\check{T} \times T]$ .

Let  $g$  be a general (regular) element of  $T$ . Then the fixed point set  $(\mathrm{Gr}_G)^g = (\mathrm{Gr}_G)^T = Y$  coincides with the image of the embedding  $\mathrm{Gr}_T \hookrightarrow \mathrm{Gr}_G$ . According to Thomason Localization Theorem (see e.g. [4] 5.10), after localization,  $(K^T(\mathrm{Gr}_G))_g$  becomes a free rank one  $(K(T(\mathbf{O}) \setminus T(\mathbf{F})/T(\mathbf{O})))_g$ -module. This means that after restriction to  $T^{reg} \subset T = \mathrm{Spec}(K^T(pt))$  we have an isomorphism  $K^T(\mathrm{Gr}_G)|_{T^{reg}} \simeq \mathbb{C}[\check{T} \times T]|_{T^{reg}}$  compatible with the natural  $W$ -actions. The localized algebra  $K^{G(\mathbf{O})}(\mathrm{Gr}_G)|_{T^{reg}/W}$  is embedded into  $(\mathrm{End}_{K(T(\mathbf{O}) \setminus T(\mathbf{F})/T(\mathbf{O}))|_{T^{reg}}}(K^T(\mathrm{Gr}_G)|_{T^{reg}}))^W$ . According to Lemma 6.2,  $K^G(\mathrm{Gr}_G) = (K^T(\mathrm{Gr}_G))^W$ ; hence this embedding is an isomorphism, and we have  $K^{G(\mathbf{O})}(\mathrm{Gr}_G)|_{T^{reg}/W} \simeq \mathbb{C}[\check{T} \times T]^W|_{T^{reg}/W}$ .

Hence both  $\mathbb{C}[\mathfrak{B}_G^{\check{G}}]$  and  $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$  are the flat  $\mathbb{C}[T]^W$ -modules embedded into  $\mathbb{C}[\check{T} \times T](\Delta^{-1})$  (see 4.3). We must prove that the identification of  $\mathbb{C}[\mathfrak{B}_G^{\check{G}}]|_{T^{reg}/W}$  and  $K^{G(\mathbf{O})}(\mathrm{Gr}_G)|_{T^{reg}/W}$  extends to the identification over the whole  $T/W$ . To this end it suffices to check that the identification extends over the codimension 1 points of  $T/W$ . Let  $g \in T/W$  be a regular point of  $\mathbf{D}$ ; that is,  $g$  is represented by a semisimple element of  $G$  such that the centralizer  $Z(g)$  has semisimple rank 1.

We must prove that the localizations  $\mathbb{C}[\mathfrak{B}_G^{\check{G}}]_g$  and  $(K^{G(\mathbf{O})}(\mathrm{Gr}_G))_g$  are isomorphic. To this end it suffices to identify  $\mathbb{C}[\check{T} \times T, \frac{1}{2\alpha-1}, \alpha \in R]_g$  (which we denote by  $\mathbb{C}[\mathfrak{B}_G^{\check{G}}]_g$  for short) and  $(K^T(\mathrm{Gr}_G))_g$ . Note that the embedding of reductive groups  $Z(g) \hookrightarrow G$  (the neutral connected component) induces the isomorphism  $\mathrm{Gr}_{Z(g)} = (\mathrm{Gr}_G)^g \hookrightarrow \mathrm{Gr}_G$ . According to Thomason Localization Theorem, we have an isomorphism of localizations  $(K^T(\mathrm{Gr}_{Z(g)}))_g \simeq (K^T(\mathrm{Gr}_G))_g$ . Finally, the isomorphism  $K^T(\mathrm{Gr}_{Z(g)}) \simeq \mathbb{C}[\mathfrak{B}_{Z(g)}^{\check{Z}(g)}]$  follows from the calculations in 3.8, 3.9, and together with the evident isomorphism of localizations  $\mathbb{C}[\mathfrak{B}_{Z(g)}^{\check{Z}(g)}]_g \simeq \mathbb{C}[\mathfrak{B}_G^{\check{G}}]_g$  establishes the desired isomorphism  $(K^T(\mathrm{Gr}_G))_g \simeq \mathbb{C}[\mathfrak{B}_G^{\check{G}}]_g$ .

This completes the proof of 2.15 (b).

**6.4. Comparison of Poisson structures.** In order to compare the Poisson structures on  $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$  and  $\mathbb{C}[\mathfrak{B}_G^{\check{G}}]$  it suffices to identify them on the open subset  $K^{G(\mathbf{O})}(\mathrm{Gr}_G)|_{T^{reg}/W} = \mathbb{C}[\mathfrak{B}_G^{\check{G}}]|_{T^{reg}/W} = \mathbb{C}[\check{T} \times T^{reg}]^W$ . The space

$$K^{T \times \mathbb{G}_m}(\mathrm{Gr}_G) = K^{T(\mathbf{O}) \times \mathbb{G}_m}(\mathrm{Gr}_G) = K(T(\mathbf{O}) \times \mathbb{G}_m \setminus G(\mathbf{F}) \times \mathbb{G}_m/G(\mathbf{O}) \times \mathbb{G}_m)$$

is equipped with the two commuting actions:  $K(T(\mathbf{O}) \times \mathbb{G}_m \setminus T(\mathbf{F}) \times \mathbb{G}_m/T(\mathbf{O}) \times \mathbb{G}_m)$  acts by convolutions on the left, and

$$K^{G \times \mathbb{G}_m}(\mathrm{Gr}_G) = K^{G(\mathbf{O}) \times \mathbb{G}_m}(\mathrm{Gr}_G) = K(G(\mathbf{O}) \times \mathbb{G}_m \setminus G(\mathbf{F}) \times \mathbb{G}_m/G(\mathbf{O}) \times \mathbb{G}_m)$$

acts by convolutions on the right. Also,  $W$  acts on  $K^{T(\mathbf{O}) \times \mathbb{G}_m}(\mathrm{Gr}_G)$  commuting with the right action of  $K^{G(\mathbf{O}) \times \mathbb{G}_m}(\mathrm{Gr}_G)$ . Clearly, the algebra

$K(T(\mathbf{O}) \times \mathbb{G}_m \backslash T(\mathbf{F}) \times \mathbb{G}_m / T(\mathbf{O}) \times \mathbb{G}_m)$  is isomorphic to the group algebra  $\mathbb{C}[\Gamma]$  of the following Heisenberg group  $\Gamma$ .

It is a  $\mathbb{Z}$ -central extension of  $Y \times X$  with the multiplication (written multiplicatively)

$$(q^{n_1}, e^{\check{\lambda}_1}, e^{\mu_1}) \cdot (q^{n_2}, e^{\check{\lambda}_2}, e^{\mu_2}) = (q^{n_1+n_2+\langle \mu_1, \check{\lambda}_2 \rangle}, e^{\check{\lambda}_1+\check{\lambda}_2}, e^{\mu_1+\mu_2})$$

where  $\langle \cdot, \cdot \rangle : X \times Y \rightarrow \mathbb{Z}$  is the canonical perfect pairing.

Finally, the action of the Weyl group  $W$  on  $K^{T(\mathbf{O}) \times \mathbb{G}_m}(\mathrm{Gr}_G)$  normalizes the action of  $K(T(\mathbf{O}) \times \mathbb{G}_m \backslash T(\mathbf{F}) \times \mathbb{G}_m / T(\mathbf{O}) \times \mathbb{G}_m)$  and induces the natural (diagonal) action of  $W$  on  $\mathbb{C}[\Gamma]$ . From this we deduce, exactly as in 6.3, that  $K^{G(\mathbf{O}) \times \mathbb{G}_m}(\mathrm{Gr}_G)|_{T^{reg}/W} \simeq \mathbb{C}[\Gamma]|_{T^{reg}/W}$ . It follows that the Poisson structure on  $K^{G(\mathbf{O})}(\mathrm{Gr}_G)|_{T^{reg}/W}$  coincides with the standard Poisson structure on  $\mathbb{C}[\check{T} \times T^{reg}]^W$ .

This completes the proof of 2.15 (c).

**6.5. The case of non simply connected  $G$ .** For general  $G$  let  $\tilde{G}$  denote its universal cover, and let  $\tilde{T}$  stand for the Cartan of  $\tilde{G}$ . Note that the dual torus is  $\tilde{T}/\pi_1(G)$ . As in 6.3, we have  $K^G(\mathrm{Gr}_G) = (\mathrm{End}_{K(T(\mathbf{O}) \backslash T(\mathbf{F})/T(\mathbf{O}))}(K^T(\mathrm{Gr}_G)))^W$ , so it suffices to identify the  $K(T(\mathbf{O}) \backslash T(\mathbf{F})/T(\mathbf{O})) \times W = \mathbb{C}[\check{T} \times T] \times W$ -module  $K^T(\mathrm{Gr}_G)$  with  $\mathbb{C} \left[ \check{T} \times T, \frac{1-\alpha}{2\alpha-1}, \alpha \in R \right] = \mathrm{Spec} \mathbb{C}[\check{\mathfrak{B}}_{\tilde{G}}]$ . We do this by reduction to the known case of  $\tilde{G}$ .

Evidently, the  $K(T(\mathbf{O}) \backslash T(\mathbf{F})/T(\mathbf{O})) \times W = \mathbb{C}[\check{T} \times T] \times W$ -module  $K^T(\mathrm{Gr}_G)$  equals  $\mathbb{C}[\check{T} \times T] \times W \otimes_{\mathbb{C}[(\tilde{T}/\pi_1(G)) \times T] \times W} K^T(\mathrm{Gr}_{\tilde{G}})$ . On the other hand, it follows from 6.3 that the  $K(T(\mathbf{O}) \backslash \tilde{T}(\mathbf{F})/\tilde{T}(\mathbf{O})) \times W = \mathbb{C}[(\tilde{T}/\pi_1(G)) \times T] \times W$ -module  $K^T(\mathrm{Gr}_{\tilde{G}})$  equals the invariants of  $\pi_1(G)$  in  $K^{\tilde{T}}(\mathrm{Gr}_{\tilde{G}})$ , that is  $\mathbb{C} \left[ (\tilde{T}/\pi_1(G)) \times \tilde{T}, \frac{1-\alpha}{2\alpha-1}, \alpha \in R \right]^{\pi_1(G)} = \mathbb{C} \left[ (\tilde{T}/\pi_1(G)) \times T, \frac{1-\alpha}{2\alpha-1}, \alpha \in R \right]$ .

This completes the proof of 2.15 for general  $G$ .

**6.6. Borel-Moore Homology and blow-ups.** Theorem 2.12 is proved absolutely parallelly to the proof of Theorem 2.15.

## 7. COMPUTATION OF $K_{G(\mathbf{O})}(\Lambda)$ .

**7.1. The affine Grassmannian Steinberg variety.** We denote by  $\mathfrak{u} \subset \mathfrak{g}(\mathbf{O})$  (resp.  $U \subset G(\mathbf{O})$ ) the nilpotent (resp. unipotent) radical. It has a filtration  $\mathfrak{u} = \mathfrak{u}^{(0)} \supset \mathfrak{u}^{(1)} \supset \dots$  by congruence subalgebras. The trivial (Tate) vector bundle  $\underline{\mathfrak{g}}(\mathbf{F})$  with the fiber  $\mathfrak{g}(\mathbf{F})$  over  $\mathrm{Gr}_G$  has a structure of an ind-scheme. It contains a profinite dimensional vector subbundle  $\underline{\mathfrak{u}}$  whose fiber over a point  $g \in \mathrm{Gr}_G$  represented by a compact subalgebra in  $\mathfrak{g}(\mathbf{F})$  is the pronilpotent radical of this subalgebra. The trivial vector bundle  $\underline{\mathfrak{g}}(\mathbf{F}) = \mathfrak{g}(\mathbf{F}) \times \mathrm{Gr}_G$  also contains a trivial vector subbundle  $\mathfrak{u} \times \mathrm{Gr}_G$ .

We will call  $\underline{\mathfrak{u}}$  the *cotangent bundle* of  $\mathrm{Gr}_G$ , and we will call the intersection  $\Lambda := \underline{\mathfrak{u}} \cap (\mathfrak{u} \times \mathrm{Gr}_G)$  the *affine Grassmannian Steinberg variety*. It has a structure of an ind-scheme of ind-infinite type. Namely, if  $p$  stands for the natural projection  $\Lambda \rightarrow \mathrm{Gr}_G$ , then  $\Lambda_{\leq \check{\lambda}} := p^{-1}(\overline{\mathrm{Gr}}_{G, \check{\lambda}})$  is a scheme of infinite type, and  $\Lambda = \bigcup \Lambda_{\leq \check{\lambda}}$ .

Note that for a fixed  $\check{\lambda}$  and  $l \gg 0$  the intersection of fibers of  $\underline{\mathbf{u}}$  over all points of  $\overline{\text{Gr}}_{G, \check{\lambda}}$  (as vector subspaces of  $\mathfrak{g}(\mathbf{F})$ ) contains  $\mathbf{u}^{(l)}$ . Thus  $\mathbf{u}^{(l)}$  acts freely (by fiberwise translations) on  $\Lambda_{\leq \check{\lambda}}$ , and the quotient is a scheme of finite type, to be denoted by  $\Lambda_{\leq \check{\lambda}}^l$ . For  $k > l$  we have evident affine fibrations  $p_l^k : \Lambda_{\leq \check{\lambda}}^k \rightarrow \Lambda_{\leq \check{\lambda}}^l$ , and  $\Lambda_{\leq \check{\lambda}}$  coincides with the inverse limit of this system.

Similarly, the total space of the vector bundle  $\underline{\mathbf{u}}$  (to be denoted by the same symbol) is a union of infinite type schemes  $\underline{\mathbf{u}}_{\leq \check{\lambda}}$ , and for fixed  $\check{\lambda}$  and  $l \gg 0$ , the scheme  $\underline{\mathbf{u}}_{\leq \check{\lambda}}^l$  is the inverse limit of affine fibrations  $p_l^k : \underline{\mathbf{u}}_{\leq \check{\lambda}}^k \rightarrow \underline{\mathbf{u}}_{\leq \check{\lambda}}^l$  ( $k > l$ ). Note that the proalgebraic group  $G(\mathbf{O})$  acts on all the above schemes, and the fibrations  $p_l^k$  are  $G(\mathbf{O})$ -equivariant.

A  $G(\mathbf{O})$ -equivariant coherent sheaf  $\mathcal{F}$  on  $\underline{\mathbf{u}}$  is by definition supported on some  $\underline{\mathbf{u}}_{\leq \check{\lambda}}$ . There, it is defined as a collection of  $G(\mathbf{O})$ -equivariant sheaves  $\mathcal{F}^l$  on  $\underline{\mathbf{u}}_{\leq \check{\lambda}}^l$  for  $l \gg 0$  together with isomorphisms  $(p_l^k)^* \mathcal{F}^l \simeq \mathcal{F}^k$ . We will consider the  $G(\mathbf{O})$ -equivariant coherent sheaves on  $\underline{\mathbf{u}}$  supported on  $\Lambda$ , and  $D^b \text{Coh}_{\Lambda}^{G(\mathbf{O})}(\underline{\mathbf{u}})$  stands for the derived category of such sheaves, and  $K^{G(\mathbf{O})}(\Lambda)$  stands for the  $K$ -group of such sheaves.

**7.2. Convolution in  $D^b \text{Coh}_{\Lambda}^{G(\mathbf{O})}(\underline{\mathbf{u}})$ .** We have a principal  $G(\mathbf{O})$ -bundle  $G(\mathbf{F}) \rightarrow \text{Gr}_G$ . Given a  $G(\mathbf{O})$ -(ind)-scheme  $A$  we can form an associated bundle  $\tilde{A} = G(\mathbf{F}) \times_{G(\mathbf{O})} A \rightarrow \text{Gr}_G$ . Given a coherent  $G(\mathbf{O})$ -equivariant sheaf  $\mathcal{F}$  on  $A$  we can form an associated sheaf  $\tilde{\mathcal{F}}$  on  $\tilde{A}$  as  $G(\mathbf{O})$ -invariants in the direct image of  $\mathcal{O}_{G(\mathbf{F})} \boxtimes \mathcal{F}$  from  $G(\mathbf{F}) \times A$  to  $G(\mathbf{F}) \times_{G(\mathbf{O})} A$ . If  $A = \text{Gr}_G$ , apart from the natural projection  $p_1 : \tilde{A} \rightarrow \text{Gr}_G$ , we have a multiplication map  $G(\mathbf{F}) \times_{G(\mathbf{O})} \text{Gr}_G \rightarrow \text{Gr}_G$ , to be denoted  $p_2$ . Then  $(p_1, p_2)$  identifies  $\widetilde{\text{Gr}}_G$  with  $\text{Gr}_G \times \text{Gr}_G$ . Furthermore,  $\tilde{\underline{\mathbf{u}}}$  is a vector bundle over  $\widetilde{\text{Gr}}_G = \text{Gr}_G \times \text{Gr}_G$  which is naturally identified with  $p_2^* \underline{\mathbf{u}}$ . Thus we have an ind-proper morphism  $p_2 : \tilde{\underline{\mathbf{u}}} \rightarrow \underline{\mathbf{u}}$ .

Note that both  $\tilde{\underline{\mathbf{u}}} = p_2^* \underline{\mathbf{u}}$  and  $p_1^* \underline{\mathbf{u}}$  are subbundles in the trivial (Tate) vector bundle  $\underline{\mathfrak{g}}(\mathbf{F})$  over  $\text{Gr}_G \times \text{Gr}_G$  with the fiber  $\mathfrak{g}(\mathbf{F})$ . Their intersection is naturally identified with  $\tilde{\Lambda}$ . In particular, we have an embedding  $\tilde{\Lambda} \subset p_1^* \underline{\mathbf{u}} \oplus p_2^* \underline{\mathbf{u}}$ , and an ind-proper morphism  $p_2 : \tilde{\Lambda} \rightarrow \underline{\mathbf{u}}$ .

Hence given  $G(\mathbf{O})$ -equivariant coherent sheaves  $\mathcal{F}, \mathcal{G}$  on  $\Lambda$  we can consider the  $G(\mathbf{O})$ -equivariant complex  $\mathcal{F} \star \mathcal{G} := (p_2)_* (p_1^* \mathcal{F} \otimes^L \tilde{\mathcal{G}})$  (tensor product over the structure sheaf of the profinite dimensional vector bundle  $p_1^* \underline{\mathbf{u}} \oplus p_2^* \underline{\mathbf{u}}$ ). Clearly,  $\mathcal{F} \star \mathcal{G}$  is supported on  $\Lambda$ . Hence we get a convolution operation on  $D^b \text{Coh}_{\Lambda}^{G(\mathbf{O})}(\underline{\mathbf{u}})$  and on  $K^{G(\mathbf{O})}(\Lambda)$  once we check that  $p_1^* \mathcal{F} \otimes^L \tilde{\mathcal{G}}$  is bounded.

To this end, note that  $\tilde{\mathcal{G}}$  is flat over the first copy of  $\text{Gr}_G$ , and for some  $\check{\lambda}$  the sheaf  $\mathcal{F}$  is supported on  $\Lambda_{\leq \check{\lambda}}$ , so the tensor product  $p_1^* \mathcal{F} \otimes^L \tilde{\mathcal{G}}$  can actually be computed over the structure sheaf of  $(p_1^* \underline{\mathbf{u}} \oplus p_2^* \underline{\mathbf{u}})|_{\overline{\text{Gr}}_{G, \check{\lambda}} \times \text{Gr}_G} = \underline{\mathbf{u}}_{\leq \check{\lambda}} \times \underline{\mathbf{u}} \subset \underline{\mathbf{u}} \times \underline{\mathbf{u}} = p_1^* \underline{\mathbf{u}} \oplus p_2^* \underline{\mathbf{u}}$ . That is,  $p_1^* \mathcal{F} \otimes^L \tilde{\mathcal{G}}$  is the direct image of  $p_1^* \mathcal{F}|_{\underline{\mathbf{u}}_{\leq \check{\lambda}} \times \underline{\mathbf{u}}} \otimes_{\mathcal{O}_{\underline{\mathbf{u}}_{\leq \check{\lambda}} \times \underline{\mathbf{u}}}}^L \tilde{\mathcal{G}}|_{\underline{\mathbf{u}}_{\leq \check{\lambda}} \times \underline{\mathbf{u}}}$  under the closed embedding  $\underline{\mathbf{u}}_{\leq \check{\lambda}} \times \underline{\mathbf{u}} \hookrightarrow \underline{\mathbf{u}} \times \underline{\mathbf{u}}$ . On the other hand,  $p_1^* \mathcal{F}$  is flat over the second copy of  $\text{Gr}_G$ , while the support of  $\tilde{\mathcal{G}}$  intersected with  $\underline{\mathbf{u}}_{\leq \check{\lambda}} \times \underline{\mathbf{u}}$  is contained in  $\underline{\mathbf{u}}_{\leq \check{\lambda}} \times \underline{\mathbf{u}}_{\leq \check{\mu}}$  for some  $\check{\mu}$ . Hence the

tensor product  $p_1^* \mathcal{F} \otimes^L \tilde{\mathcal{G}}$  can actually be computed over the structure sheaf of  $\underline{\mathbf{u}}_{\leq \tilde{\lambda}} \times \underline{\mathbf{u}}_{\leq \tilde{\mu}}$ . There exists  $l \gg 0$  such that the diagonal fiberwise action of  $\mathbf{u}^{(l)}$  on  $\underline{\mathbf{u}}_{\leq \tilde{\lambda}} \times \underline{\mathbf{u}}_{\leq \tilde{\mu}}$  is free, and both  $p_1^* \mathcal{F}$  and  $\tilde{\mathcal{G}}$  restricted to  $\underline{\mathbf{u}}_{\leq \tilde{\lambda}} \times \underline{\mathbf{u}}_{\leq \tilde{\mu}}$  are  $\mathbf{u}^{(l)}$ -equivariant, that is, they are lifted from the sheaves on  $(\underline{\mathbf{u}}_{\leq \tilde{\lambda}} \times \underline{\mathbf{u}}_{\leq \tilde{\mu}})/\mathbf{u}^{(l)} =: V$ ; we abuse notation by keeping the same names for these sheaves. So the tensor product  $p_1^* \mathcal{F} \otimes^L \tilde{\mathcal{G}}$  can actually be computed as the tensor product of coherent sheaves over the structure sheaf of the profinite dimensional vector bundle  $V$  over the finite dimensional scheme  $\overline{\mathrm{Gr}}_{G, \tilde{\lambda}} \times \overline{\mathrm{Gr}}_{G, \tilde{\mu}}$ .

Now there exists a vector subbundle  $V' \subset V$  such that the quotient  $\overline{V} := V/V'$  is a finite dimensional vector bundle,  $p_1^* \mathcal{F}$  is lifted from  $\overline{V}$ , and the support of  $\tilde{\mathcal{G}}$  in  $V$  projects isomorphically onto its image in  $\overline{V}$ . Moreover, recall that  $p_1^* \mathcal{F}$  is flat over  $\overline{\mathrm{Gr}}_{G, \tilde{\mu}}$ , while  $\tilde{\mathcal{G}}$  is flat over  $\overline{\mathrm{Gr}}_{G, \tilde{\lambda}}$ . Clearly, in this situation  $p_1^* \mathcal{F} \otimes^L \tilde{\mathcal{G}} \in D^b(V)$ . This explains why for  $G(\mathbf{O})$ -equivariant coherent sheaves  $\mathcal{F}, \mathcal{G}$  on  $\Lambda$  the tensor product  $p_1^* \mathcal{F} \otimes^L \tilde{\mathcal{G}}$  is a bounded complex of coherent sheaves on  $p_1^* \underline{\mathbf{u}} \oplus p_2^* \underline{\mathbf{u}}$  supported on  $\tilde{\Lambda}$ . Hence the same is true for the bounded complexes of  $G(\mathbf{O})$ -equivariant coherent sheaves  $\mathcal{F}, \mathcal{G}$  on  $\underline{\mathbf{u}}$  supported on  $\Lambda$ . Thus,  $D^b \mathrm{Coh}_{\Lambda}^{G(\mathbf{O})}(\underline{\mathbf{u}})$  is closed with respect to convolution.

**Theorem 7.3.**  $K^{G(\mathbf{O})}(\Lambda)$  is a commutative algebra isomorphic to  $\mathbb{C}[\check{T} \times T]^W$ .

**Remark 7.4.** Since  $\Lambda_G$  is an affine Grassmannian analogue of the classical Steinberg variety, this result agrees well with the geometric realization of the Cherednik double affine Hecke algebra in [8], [23]. In effect,  $K^{G(\mathbf{O})}(\Lambda_G)$  is the spherical subalgebra of the Cherednik algebra with both parameters trivial:  $q = t = 1$ .

**7.5. Bialynicki-Birula stratifications.** The proof of Theorem 7.3 uses the following lemma on  $K$ -theory of cellular spaces. Let  $M$  be a normal quasiprojective variety equipped with a torus  $H$ -action with finitely many fixed points. We assume that  $M$  is equipped with an  $H$ -invariant stratification  $M = \bigsqcup_{\mu \in M^H} M_{\mu}$  such that each stratum  $M_{\mu}$  contains exactly one  $H$ -fixed point  $\mu$ , and  $M_{\mu}$  is isomorphic to an affine space. For  $\mu \in M^H$  we denote by  $j_{\mu} : M_{\mu} \hookrightarrow M$  the locally closed embedding of the corresponding stratum. We denote by  $i_{\mu} : \mu \hookrightarrow M_{\mu}$  the closed embedding of an  $H$ -fixed point in the corresponding stratum, or in the whole of  $M$  when no confusion is likely. We denote by  $\mu \leq \nu$  the closure relation of strata. We denote by  $M_{\leq \mu} \subset M$  the union  $\bigcup_{\nu \leq \mu} M_{\nu}$ .

Given an  $H$ -equivariant closed embedding of  $M$  into a smooth  $H$ -variety  $M'$  (for the existence see [22]) we denote by  $T^*M$  the restriction of the cotangent bundle  $T^*M'$  to  $M \subset M'$ . We denote by  $\iota : M \hookrightarrow T^*M$  the embedding of the zero section. We also denote by  $i_{\mu}$  the closed embedding of the conormal bundle  $T_{\mu}^*M' \hookrightarrow T^*M$  when no confusion is likely. Finally, we denote by  $\mathcal{L}'$  the union of conormal bundles  $\bigcup_{\mu} T_{M_{\mu}}^*M'$ , and  $j$  stands for the closed embedding  $\mathcal{L}' \hookrightarrow T^*M$ . We denote by  $\mathcal{L}'_{\leq \mu} \subset \mathcal{L}'$  the union  $\bigcup_{\nu \leq \mu} T_{M_{\nu}}^*M'$ ; it is a closed subvariety of  $\mathcal{L}'$ . It has a closed subvariety  $\mathcal{L}'_{< \mu} := \bigcup_{\nu < \mu} T_{M_{\nu}}^*M'$ .

For  $\mu \in M^H$  we have an embedding  $i_{\mu*} : K^H(\mu) \hookrightarrow K^H(M)$ . We have an embedding  $j_* : K^H(\mathcal{L}') \hookrightarrow K^H(T^*M) \xrightarrow{i_*} K^H(M)$ . Indeed, the exact sequences (see [4] Chapter 5)

$$0 \rightarrow K^H(\mathcal{L}'_{<\mu}) \rightarrow K^H(\mathcal{L}'_{\leq\mu}) \rightarrow K^H(T^*_{M_\mu} M') \rightarrow 0,$$

$$0 \rightarrow K^H(T^* M'|_{M_{<\mu}}) \rightarrow K^H(T^* M'|_{M_{\leq\mu}}) \rightarrow K^H(T^* M'|_{M_\mu})$$

give rise to the support filtrations on  $K^H(\mathcal{L}')$  and  $K^H(T^*M)$  with associated graded  $\bigoplus_{\mu \in M^H} K^H(T^*_{M_\mu} M')$  and  $\bigoplus_{\mu \in M^H} K^H(T^* M'|_{M_\mu})$ . Now  $j_*$  is strictly compatible with the support filtrations and clearly injective on the associated graded.

Note that the image  $j_*(K^H(\mathcal{L}')) \subset K^H(M)$  is independent of the choice of the closed embedding  $M \hookrightarrow M'$ . In effect, given another embedding  $M \hookrightarrow \widetilde{M}$ , we can consider the diagonal embedding  $M \hookrightarrow M'' := M' \times \widetilde{M}$ . Clearly, we have a projection  $p : T^* M''|_M \rightarrow T^* M'|_M$  which realizes  $T^* M''|_M$  as a vector bundle over  $T^* M'|_M$ . Moreover, if we denote by  $\mathcal{L}''$  the union of conormal bundles  $\bigcup_{\mu} T^*_{M_\mu} M'' \subset T^* M''|_M$  then  $\mathcal{L}'' = p^{-1}\mathcal{L}'$ . This shows that the images of  $K^H(\mathcal{L}')$  and  $K^H(\mathcal{L}'')$  in  $K^H(M)$  coincide, and thus  $j_*(K^H(\mathcal{L}')) \subset K^H(M)$  is well-defined.

**Lemma 7.6.** *In  $K^H(M)$  we have an equality  $j_*(K^H(\mathcal{L}')) = \bigoplus_{\mu} i_{\mu*}(K^H(\mu))$ .*

*Proof* Let  $K^H(D_M)$  stand for the  $K$ -group of weakly  $H$ -equivariant  $D$ -modules on  $M'$  supported on  $M \subset M'$ . Given such a  $D$ -module and passing to associated graded with respect to a good filtration, we obtain an  $H$ -equivariant coherent sheaf on  $T^*M$ , and this way one obtains a homomorphism  $SS : K^H(D_M) \rightarrow K^H(T^*M) \xrightarrow{i_*} K^H(M)$  (see e.g. [11]). Let  $\delta_\mu$  stand for a  $\delta$ -function  $D$ -module at the point  $\mu \in M^H$  with its obvious  $H$ -equivariance. Then, evidently,  $SS(\delta_\mu)$  generates  $i_{\mu*}(K^H(\mu))$  as a module over  $K^H(pt)$ . Moreover,  $\{SS(j_{\mu!}\mathcal{O}_{M_\mu}), \mu \in M^H\}$  forms a basis of  $j_*(K^H(\mathcal{L}'))$ .

In effect, the closed embedding  $\mathcal{L}'_{<\mu} \hookrightarrow \mathcal{L}'_{\leq\mu}$  gives rise to the exact sequence

$$0 \rightarrow K^H(\mathcal{L}'_{<\mu}) \rightarrow K^H(\mathcal{L}'_{\leq\mu}) \rightarrow K^H(T^*_{M_\mu} M') \rightarrow 0$$

(see [4] Chapter 5), and the image of  $SS(j_{\mu!}\mathcal{O}_{M_\mu})$  in  $K^H(T^*_{M_\mu} M')$  clearly generates it.

So it is enough to check the equality in  $K^H(T^*M)$ :

$$(12) \quad SS(\delta_\mu) = SS(j_{\mu!}\mathcal{O}_{M_\mu}) \cdot (-1)^{\dim M_\mu} \det(T_\mu M_\mu)$$

where  $\det(T_\mu M_\mu)$  is the character of  $H$  (thus an invertible element of  $K^H(pt) = \mathbb{C}[H]$ ) acting in the determinant of the tangent bundle of  $M_\mu$  at  $\mu$ .

To this end note that restriction to the  $H$ -fixed points gives rise to an embedding  $\bigoplus_{\nu} i_{\nu}^* i^* : K^H(T^*M) \hookrightarrow \bigoplus_{\nu} K^H(\nu)$ . This is checked by induction in  $\nu$  using the exact sequences

$$0 \rightarrow K^H(T^* M'|_{M_{<\nu}}) \rightarrow K^H(T^* M'|_{M_{\leq\nu}}) \rightarrow K^H(T^* M'|_{M_\nu}) \rightarrow 0.$$

It is clear that for  $\nu = \mu$  the restrictions  $i_{\mu}^* i^*$  of the LHS and RHS of (12) coincide. We are going to check that for  $\nu \neq \mu$  the restrictions  $i_{\nu}^* i^*$  of the LHS and RHS of (12) both vanish. Evidently,  $i_{\nu}^* i^* SS(\delta_\mu) = 0$ .

Recall that  $i_\nu$  also stands for the closed embedding  $T_\nu^*M' \hookrightarrow T^*M$ , so we just have to check that  $i_\nu^*SS(j_{\mu!}\mathcal{O}_{M_\mu}) = 0 \in K^H(T_\nu^*M')$ . Note that the functor of global sections of  $H$ -equivariant coherent sheaves on the vector space  $T_\nu^*M'$  gives rise to an embedding  $\Gamma : K^H(T_\nu^*M') \hookrightarrow \mathbb{Z}^{X^*(H)}$  where  $X^*(H)$  stands for the lattice of characters of  $H$ . Now for a  $D$ -module  $\mathcal{F}$  we have  $\Gamma(i_\nu^*SS\mathcal{F}) = \mathbf{i}_\nu^*\mathcal{F}$  where  $\mathbf{i}_\nu^*\mathcal{F}$  stands for the fiber at  $\nu \in M$  of the  $H$ -equivariant quasicohherent  $\mathcal{O}_{M'}$ -module  $\mathcal{F}$ . Finally, for  $\mathcal{F} = j_{\mu!}\mathcal{O}_{M_\mu}$  and  $\nu \neq \mu$  we have  $\mathbf{i}_\nu^*j_{\mu!}\mathcal{O}_{M_\mu} = 0$ . This completes the proof of the lemma.

**7.7. Bialynicki-Birula stratification of  $\mathrm{Gr}_G$ .** We consider the stratification of  $\mathrm{Gr}_G$  by the Iwahori orbits  $\mathrm{Gr}_G = \bigsqcup_{\check{\mu} \in Y} \mathrm{Gr}_G^{\check{\mu}}$ . This is a refinement of the stratification by the  $G(\mathbf{O})$ -orbits:  $\mathrm{Gr}_{G,\check{\lambda}} = \bigsqcup_{\check{\mu} \in W\check{\lambda}} \mathrm{Gr}_G^{\check{\mu}}$ . Let us denote by  $\mathfrak{n} \supset \mathfrak{u}$  the nilpotent radical of the Iwahori subalgebra in  $\mathfrak{g}(\mathbf{F})$ . The union of conormal bundles to the Iwahori orbits is the following subvariety  $\Lambda_I$  of the cotangent bundle  $\underline{\mathfrak{u}}$ : by definition,  $\Lambda_I := \underline{\mathfrak{u}} \cap (\mathfrak{n} \times \mathrm{Gr}_G)$ . We have a closed embedding  $\Lambda \subset \Lambda_I$ .

Lemma 7.6 allows us to compute  $K^T(\Lambda_I) = \bigoplus_{\check{\mu} \in Y} K^T(\check{\mu}) \subset K^T(\mathrm{Gr}_G)$ , i.e.  $K^T(\Lambda_I) \simeq \mathbb{C}[\check{T} \times T]$  (note that the natural  $W$ -action on  $K^T(\mathrm{Gr}_G)$  induces the diagonal  $W$ -action on  $\mathbb{C}[\check{T} \times T] \simeq K^T(\Lambda_I) \subset K^T(\mathrm{Gr}_G)$ ). Although Lemma 7.6 was formulated for finite dimensional varieties  $M$ , its proof goes through for  $\mathrm{Gr}_G$  without changes: we only need to have the singular support map  $SS : K^T(D_{\mathrm{Gr}_G}) \rightarrow K^T(\underline{\mathfrak{u}}) \simeq K^T(\mathrm{Gr}_G)$ . For this see [12], [2] (Chapter 15), [8].

The embedding  $\Lambda \hookrightarrow \Lambda_I$  gives rise to the embedding  $K^T(\Lambda) \hookrightarrow K^T(\Lambda_I) \hookrightarrow K^T(\underline{\mathfrak{u}}) = K^T(\mathrm{Gr}_G)$ . Note that  $W$  acts naturally on both  $K^T(\Lambda)$  and  $K^T(\mathrm{Gr}_G)$ , and the embedding  $K^T(\Lambda) \hookrightarrow K^T(\mathrm{Gr}_G)$  is  $W$ -equivariant. Also,  $(K^T(\Lambda))^W = K^G(\Lambda) = K^G(\mathbf{O})(\Lambda)$ . Hence, the image of the embedding  $K^G(\mathbf{O})(\Lambda) \hookrightarrow K^T(\Lambda_I) \simeq \mathbb{C}[\check{T} \times T] \subset K^T(\mathrm{Gr}_G)$  lies in the invariants of the diagonal  $W$ -action on  $\mathbb{C}[\check{T} \times T]$ . Thus to prove Theorem 7.3 we must check that the image of this embedding contains  $\mathbb{C}[\check{T} \times T]^W$ .

We have projections  $\pi : \Lambda \rightarrow \mathrm{Gr}_G$ , and  $\pi_I : \Lambda_I \rightarrow \mathrm{Gr}_G$ . For  $\check{\lambda} \in Y^+$  we denote by  $\Lambda_{\check{\lambda}}$  (resp.  $\Lambda_{\leq \check{\lambda}}$ ,  $\Lambda_{< \check{\lambda}}$ ) the preimage  $\pi^{-1}(\mathrm{Gr}_{G,\check{\lambda}})$  (resp.  $\pi^{-1}(\overline{\mathrm{Gr}}_{G,\check{\lambda}})$ ,  $\pi^{-1}(\overline{\mathrm{Gr}}_{G,\check{\lambda}} - \mathrm{Gr}_{G,\check{\lambda}})$ ). For  $\check{\lambda} \in Y^+$  we denote by  $\Lambda_{I,\check{\lambda}}$  (resp.  $\Lambda_{I,\leq \check{\lambda}}$ ,  $\Lambda_{I,< \check{\lambda}}$ ) the preimage  $\pi_I^{-1}(\mathrm{Gr}_{G,\check{\lambda}})$  (resp.  $\pi_I^{-1}(\overline{\mathrm{Gr}}_{G,\check{\lambda}})$ ,  $\pi_I^{-1}(\overline{\mathrm{Gr}}_{G,\check{\lambda}} - \mathrm{Gr}_{G,\check{\lambda}})$ ). Clearly,  $\Lambda_{< \check{\lambda}}$  (resp.  $\Lambda_{I,< \check{\lambda}}$ ) is closed in  $\Lambda_{\leq \check{\lambda}}$  (resp.  $\Lambda_{I,\leq \check{\lambda}}$ ), with the open complement  $\Lambda_{\check{\lambda}}$  (resp.  $\Lambda_{I,\check{\lambda}}$ ). In  $K$ -groups we have exact sequences (see [4] Chapter 5)

$$0 \rightarrow K^T(\Lambda_{< \check{\lambda}}) \rightarrow K^T(\Lambda_{\leq \check{\lambda}}) \rightarrow K^T(\Lambda_{\check{\lambda}}) \rightarrow 0,$$

$$0 \rightarrow K^T(\Lambda_{I,< \check{\lambda}}) \rightarrow K^T(\Lambda_{I,\leq \check{\lambda}}) \rightarrow K^T(\Lambda_{I,\check{\lambda}}) \rightarrow 0.$$

Thus we obtain a support filtration on  $K^T(\Lambda_I)$  (resp.  $K^T(\Lambda)$ ) with associated graded  $\bigoplus_{\check{\lambda} \in Y^+} K^T(\Lambda_{I,\check{\lambda}})$  (resp.  $\bigoplus_{\check{\lambda} \in Y^+} K^T(\Lambda_{\check{\lambda}})$ ).

We have the embeddings  $K^T(\Lambda_{\check{\lambda}}) \hookrightarrow K^T(\Lambda_{I,\check{\lambda}}) \hookrightarrow K^T(\underline{\mathfrak{u}}|_{\mathrm{Gr}_{\check{\lambda}}}) \simeq K^T(\mathrm{Gr}_{\check{\lambda}})$ . The Weyl group  $W$  acts naturally both on  $K^T(\Lambda_{\check{\lambda}})$  and  $K^T(\mathrm{Gr}_{\check{\lambda}})$ , and to prove Theorem 7.3 it suffices to check that the image of  $(K^T(\Lambda_{\check{\lambda}}))^W$  in  $K^T(\Lambda_{I,\check{\lambda}})$  contains (equivalently, coincides with) the intersection  $K^T(\Lambda_{I,\check{\lambda}}) \cap (K^T(\mathrm{Gr}_{\check{\lambda}}))^W$ .

To this end recall that  $\mathrm{Gr}_{G,\tilde{\lambda}}$  can be  $G$ -equivariantly identified with the total space  $\tilde{\mathcal{B}}$  of a vector bundle over a certain partial flag variety  $\mathcal{B}$  of the group  $G$  (the quotient  $G/P_{\tilde{\lambda}}$  by a parabolic subgroup depending on  $\tilde{\lambda}$ ). The Borel subgroup  $B \subset G$  acts on  $\mathcal{B}$  with finitely many orbits numbered by the cosets of parabolic Weyl subgroup  $W^{\tilde{\lambda}} = W/W_{\tilde{\lambda}}$ ; we have  $\mathcal{B} = \bigsqcup_{w \in W^{\tilde{\lambda}}} \mathcal{B}_w$ . Let us denote by  $\mathcal{L} \subset T^*\mathcal{B}$  the union of conormal bundles  $\mathcal{L} = \bigsqcup_{w \in W^{\tilde{\lambda}}} T_{\mathcal{B}_w}^* \mathcal{B}$ . Let us also denote by  $\tilde{\mathcal{B}}_w$  the preimage of  $\mathcal{B}_w$  in  $\tilde{\mathcal{B}}$  (it coincides with a certain Iwahori orbit  $\mathrm{Gr}_G^{\tilde{\mu}} \subset \mathrm{Gr}_{G,\tilde{\lambda}} = \tilde{\mathcal{B}}$ ). We define  $\tilde{\mathcal{L}} := \bigsqcup_{w \in W^{\tilde{\lambda}}} T_{\tilde{\mathcal{B}}_w}^* \tilde{\mathcal{B}} \subset T^*\tilde{\mathcal{B}}$ . Then there exists a  $G$ -equivariant profinite dimensional vector bundle  $\mathcal{V} \xrightarrow{p} T^*\tilde{\mathcal{B}}$  such that  $\mathcal{V} \simeq \underline{u}|_{\mathrm{Gr}_{\tilde{\lambda}}}$ , and under this isomorphism we have  $\mathcal{V}|_{\tilde{\mathcal{L}}} \simeq \Lambda_{I,\tilde{\lambda}}$ ,  $\mathcal{V}|_{\tilde{\mathcal{B}} \hookrightarrow T^*\tilde{\mathcal{B}}} \simeq \Lambda_{\tilde{\lambda}}$ . Thus to prove Theorem 7.3 it is enough to check that the image of  $(K^T(\tilde{\mathcal{B}}))^W$  in  $K^T(T^*\tilde{\mathcal{B}})$  contains the intersection  $K^T(\tilde{\mathcal{L}}) \cap (K^T(T^*\tilde{\mathcal{B}}))^W$ . Equivalently, we have to check that the image of  $(K^T(\mathcal{B}))^W$  in  $K^T(T^*\mathcal{B})$  contains the intersection  $K^T(\mathcal{L}) \cap (K^T(T^*\mathcal{B}))^W$ . This is the subject of the following lemma.

**Lemma 7.8.** *Let  $\iota : \mathcal{B} \hookrightarrow T^*\mathcal{B}$  denote the embedding of the zero section, and let  $j : \mathcal{L} \hookrightarrow T^*\mathcal{B}$  denote the natural closed embedding. Then  $\iota_*(K^T(\mathcal{B}))^W$  coincides with  $\mathrm{Im}(j_* : K^T(\mathcal{L}) \hookrightarrow K^T(T^*\mathcal{B})) \cap (K^T(T^*\mathcal{B}))^W$ .*

*Proof* For  $w \in W^{\tilde{\lambda}}$  we denote by  $w \in \mathcal{B}_w \subset \mathcal{B}$  the corresponding  $T$ -fixed point. We denote by  $i_w$  the closed embedding  $T_w^*\mathcal{B} \hookrightarrow T^*\mathcal{B}$  (and also the closed embedding  $w \hookrightarrow \mathcal{B}$ , when the confusion is unlikely), and we denote by  $i_w$  the closed embedding  $w \hookrightarrow T^*\mathcal{B}$ . According to Lemma 7.6, the image of  $j_* : K^T(\mathcal{L}) \hookrightarrow K^T(T^*\mathcal{B})$  coincides with the image of  $\bigoplus_{w \in W^{\tilde{\lambda}}} i_w^* : \bigoplus_{w \in W^{\tilde{\lambda}}} K^T(T_w^*\mathcal{B}) \rightarrow K^T(T^*\mathcal{B})$ . We have an embedding  $\bigoplus_{w \in W^{\tilde{\lambda}}} i_w^* : K^T(T^*\mathcal{B}) \hookrightarrow \bigoplus_{w \in W^{\tilde{\lambda}}} K^T(w)$ , and similarly an embedding  $\bigoplus_{w \in W^{\tilde{\lambda}}} i_w^* : K^T(\mathcal{B}) \hookrightarrow \bigoplus_{w \in W^{\tilde{\lambda}}} K^T(w)$ .

Clearly, the  $W$ -invariants project injectively into any direct summand:  $K^G(\mathcal{B}) = (K^T(\mathcal{B}))^W \xrightarrow{i_w^*} K^T(w)$  (resp.  $K^G(T^*\mathcal{B}) = (K^T(T^*\mathcal{B}))^W \xrightarrow{i_w^*} K^T(w)$ ) for any  $w \in W^{\tilde{\lambda}}$ . Thus it suffices to check that for any  $w \in W^{\tilde{\lambda}}$  we have a coincidence  $\mathrm{Im}(i_w^* i_{w*} : K^T(T_w^*\mathcal{B})^W \rightarrow K^T(w)) = \mathrm{Im}(i_w^* j_* \mathrm{Res}_T^G : K^G(\mathcal{B}) \rightarrow K^T(w))$ . Note that if  $w = e$  (the identity coset of  $W_{\tilde{\lambda}}$  in  $W$ ), then the image  $i_e^*(K^T(\mathcal{B}))^W \subset K^T(e)$  (resp.  $i_e^*(K^T(T^*\mathcal{B}))^W \subset K^T(e)$ ) coincides with  $(K^T(e))^{W_{\tilde{\lambda}}} = \mathbb{C}[T]^{W_{\tilde{\lambda}}}$ . Moreover, under identification  $K^T(T_e^*\mathcal{B}) = K^T(e) = \mathbb{C}[T]$ , we have  $K^T(T_e^*\mathcal{B}) \cap (K^T(T^*\mathcal{B}))^W = \mathbb{C}[T]^{W_{\tilde{\lambda}}}$ .

Identifying both  $K^T(T_e^*\mathcal{B})$  and  $K^T(e)$  with  $\mathbb{C}[T]$ , the map  $i_e^* i_{e*}$  is a multiplication by the product  $\Delta_1 = \prod_{k=1}^{\dim \mathcal{B}} (1 - \chi_k)$  where  $\chi_k$  run through the characters of  $T$  in the tangent space  $T_e(T_e^*\mathcal{B}) = T_e^*\mathcal{B}$ . Furthermore, identifying  $K^G(\mathcal{B})$  with  $\mathbb{C}[T]^{W_{\tilde{\lambda}}}$ , and  $K^T(e)$  with  $\mathbb{C}[T]$ , the map  $i_e^* j_* \mathrm{Res}_T^G$  is a multiplication by the product  $\Delta_2 = \prod_{k=1}^{\dim \mathcal{B}} (1 - \chi'_k)$  where  $\chi'_k$  run through the characters of  $T$  in the tangent space  $T_e \mathcal{B}$ . We can arrange the characters  $\chi'_k$  so that we have  $\chi'_k = \chi_k^{-1}$ . Then we see that  $\Delta_1 = \Delta_2 \cdot \prod_{k=1}^{\dim \mathcal{B}} (-\chi_k)$ , so they differ by an invertible function, hence the corresponding images coincide:  $\Delta_1 \cdot \mathbb{C}[T]^{W_{\tilde{\lambda}}} = \Delta_2 \cdot \mathbb{C}[T]^{W_{\tilde{\lambda}}}$ .

This completes the proof of the lemma along with Theorem 7.3.



7.9. In this subsection we describe (without striving for high precision) a conjectural picture motivating Theorem 7.3.

We hope that the isomorphism  $K^{G(\mathbf{O})}(\Lambda_G) = \mathbb{C}[\check{T} \times T]^W = \mathbb{C}[T \times \check{T}]^W = K^{\check{G}(\mathbf{O})}(\Lambda_{\check{G}})$  lifts to an equivalence of monoidal categories  $F : D^b Coh_{\Lambda_G}^{G(\mathbf{O})}(\underline{\mathbf{u}}_G) \simeq D^b Coh_{\Lambda_{\check{G}}}^{\check{G}(\mathbf{O})}(\underline{\mathbf{u}}_{\check{G}})$ . The conjectural equivalence  $F$  is related to the Langlands correspondence in the following way.

Recall that the conjectural (for  $G = GL(n)$  mostly proven in [9]) geometric Langlands correspondence is an equivalence of triangulated categories between the derived category of  $D$ -modules on the stack  $\text{Bun}_G$  of  $G$ -bundles on a given smooth projective curve  $C$ , and the derived category of coherent sheaves on the stack of  $\check{G}$  local systems on the same curve. One might expect its “classical limit” to be an equivalence between the derived categories of coherent sheaves  $L : D(T^* \text{Bun}_G) \simeq D(T^* \text{Bun}_{\check{G}})$  where  $T^* \text{Bun}_G$  is the cotangent bundle to the moduli stack of  $G$ -bundles on  $C$ . Given a point  $c \in C$ , and identifying  $\mathbf{O}$  with the algebra of functions on the formal neighbourhood of  $c$ , one gets an action of  $D^b Coh_{\Lambda_G}^{G(\mathbf{O})}(\underline{\mathbf{u}}_G)$  on  $D(T^* \text{Bun}_G)$ . The “classical limit” of the Hecke eigen-property of geometric Langlands correspondence (see [2]) should be stated in terms of this action; it should say that the global equivalence  $L$  is compatible with our local equivalence  $F$ .

## 8. PERVERSE SHEAVES AND FUSION

We refer the reader to [3] for the definition of perverse equivariant coherent sheaves and related objects.

8.1. Recall the setup of 6.1. Note that all the  $G(\mathbf{O})$ -orbits in a connected component of  $\text{Gr}_G$  have dimensions of the same parity. Thus it makes sense to consider the middle perversity function  $p(\text{Gr}_{G,\check{\lambda}}) = -\frac{1}{2} \dim(\text{Gr}_{G,\check{\lambda}}) = -\langle \rho, \check{\lambda} \rangle$ . It is obviously strictly monotone and comonotone, but at some connected components of  $\text{Gr}_G$  it takes values in half-integers. This means that we consider equivariant complexes formally placed in half-integer homological degrees. The theory of [3] defines the artinian abelian category  $\mathcal{P}^{G(\mathbf{O})}(\text{Gr}_G)$  of perverse  $G(\mathbf{O})$ -equivariant coherent sheaves (with respect to the above middle perversity). Let  $D^{b,G(\mathbf{O})}(\text{Gr}_G)$  denote the bounded derived category of  $G(\mathbf{O})$ -equivariant coherent sheaves on  $\text{Gr}_G$  (with the same convention that the complexes at “odd” connected components are placed in half-integer homological degrees).

Given two complexes  $\mathcal{F}, \mathcal{G} \in D^{b,G(\mathbf{O})}(\text{Gr}_G)$  we have their convolution  $\mathcal{F} \star \mathcal{G} \in D^{b,G(\mathbf{O})}(\text{Gr}_G)$ . Recall that  $\mathcal{F} \star \mathcal{G} = \Pi_{0*}(\mathcal{F} \times \mathcal{G})$  where  $\Pi_0 : G(\mathbf{F}) \times_{G(\mathbf{O})} \text{Gr}_G \rightarrow \text{Gr}_G$  is the convolution diagram, and  $\mathcal{F} \times \mathcal{G}$  is the twisted product of  $\mathcal{F}$  and  $\mathcal{G}$  on  $G(\mathbf{F}) \times_{G(\mathbf{O})} \text{Gr}_G$ .

**Proposition 8.2.** *The convolution preserves perverse sheaves: for  $\mathcal{F}, \mathcal{G} \in \mathcal{P}^{G(\mathbf{O})}(\text{Gr}_G)$  we have  $\mathcal{F} \star \mathcal{G} \in \mathcal{P}^{G(\mathbf{O})}(\text{Gr}_G)$ .*

*Proof* Denote the projection  $G(\mathbf{F}) \rightarrow G(\mathbf{F})/G(\mathbf{O}) = \text{Gr}_G$  by  $p$ , and consider a stratification  $G(\mathbf{F}) \times_{G(\mathbf{O})} \text{Gr}_G = \bigsqcup_{\check{\lambda}, \check{\mu} \in Y^+} p^{-1}(\text{Gr}_{G,\check{\lambda}}) \times_{G(\mathbf{O})} \text{Gr}_{G,\check{\mu}}$ . Clearly,  $\mathcal{F} \times \mathcal{G}$  is smooth (locally free) along this stratification, and perverse (with respect to the middle perversity). According to [19] 2.7, the map  $\Pi_0$  is stratified semismall with respect to

the above stratification. Now the perversity of  $\Pi_{0*}(\mathcal{F} \times \mathcal{G})$  follows in the same manner as in the constructible case, cf. *loc. cit.*

**8.3. The absence of commutativity constraint.** According to Proposition 8.2,  $\mathcal{P}^{G(\mathbf{O})}(\mathrm{Gr}_G)$  acquires the structure of abelian artinian monoidal category. Moreover, according to 2.15 (a), its  $K$ -ring is commutative. Nevertheless,  $\mathcal{P}^{G(\mathbf{O})}(\mathrm{Gr}_G)$  admits no commutativity constraint, as can be seen in the following example.

We recall the setup of 3.6, and consider  $\mathrm{Gr}_{PGL_2}$ . One can check that there are the nonsplit exact sequences in  $\mathcal{P}^{PGL_2(\mathbf{O})}(\mathrm{Gr}_{PGL_2})$ :

$$\begin{aligned} 0 \rightarrow \mathcal{V}(0)_0 \rightarrow \mathcal{V}(0)_1 \star \mathcal{V}(-2)_1 \rightarrow \mathcal{V}(-2)_2 \rightarrow 0 \\ 0 \rightarrow \mathcal{V}(-2)_2 \rightarrow \mathcal{V}(-2)_1 \star \mathcal{V}(0)_1 \rightarrow \mathcal{V}(0)_0 \rightarrow 0 \end{aligned}$$

Thus  $\mathcal{V}(0)_1 \star \mathcal{V}(-2)_1$  and  $\mathcal{V}(-2)_1 \star \mathcal{V}(0)_1$  are nonisomorphic.

**8.4.  $G(\mathbf{O}) \times \mathbb{G}_m$ -equivariant sheaves and fusion.** The orbits of  $G(\mathbf{O}) \times \mathbb{G}_m$  on  $\mathrm{Gr}_G$  coincide with the  $G(\mathbf{O})$ -orbits, so one can consider the abelian artinian monoidal category  $\mathcal{P}^{G(\mathbf{O}) \times \mathbb{G}_m}(\mathrm{Gr}_G)$  of  $G(\mathbf{O}) \times \mathbb{G}_m$ -equivariant coherent perverse sheaves on  $\mathrm{Gr}_G$ . For  $\mathcal{F} \in \mathcal{P}^{G(\mathbf{O}) \times \mathbb{G}_m}(\mathrm{Gr}_G)$  we have  $R\Gamma(\mathrm{Gr}_G, \mathcal{F}) \in D^b(G(\mathbf{O}) \times \mathbb{G}_m - \mathrm{mod})$ .

B. Feigin and S. Loktev define (under certain restrictions) in [6] the *fusion product*  $V_1 \star \dots \star V_k \in G(\mathbf{O}) \times \mathbb{G}_m - \mathrm{mod}$  of  $G(\mathbf{O}) \times \mathbb{G}_m$ -modules  $V_1, \dots, V_k$ . We recall some of their results in case  $G = PGL_2$ .

Let  $V(n)$  be the  $n + 1$ -dimensional  $G(\mathbf{O}) \times \mathbb{G}_m$ -module factoring through  $G(\mathbf{O}) \times \mathbb{G}_m \rightarrow G \times \mathbb{G}_m \rightarrow G$ . Recall the irreducible  $PGL_2(\mathbf{O})$ -equivariant perverse sheaf  $\mathcal{V}(n)_m$  introduced in 3.6. It can be lifted to the same named  $PGL_2(\mathbf{O}) \times \mathbb{G}_m$ -equivariant perverse sheaf, where the action of  $\mathbb{G}_m$  in the fiber over a  $\mathbb{G}_m$ -fixed point in the orbit  $\mathrm{Gr}_{PGL_2, m}$  is set *trivial*. In particular,  $R\Gamma(\mathrm{Gr}_{PGL_2}, \mathcal{V}(n)_1) = V(n) \left[ \frac{1}{2} \right]$  for  $n \geq 0$ .

Now we can reformulate Theorem 2.5 of [6] as follows.

**Proposition 8.5.** *Let  $n_1 \geq n_2 \geq \dots \geq n_k$ . Then*

- (a)  $R\Gamma(\mathrm{Gr}_{PGL_2}, \mathcal{V}(n_1)_1 \star \dots \star \mathcal{V}(n_k)_1)$  is concentrated in degree  $-\frac{k}{2}$ ;
- (b)  $R\Gamma(\mathrm{Gr}_{PGL_2}, \mathcal{V}(n_1)_1 \star \dots \star \mathcal{V}(n_k)_1) \left[ -\frac{k}{2} \right] \simeq V(n_k) \star \dots \star V(n_1)$ .

**8.6. Multiplication table.** According to Proposition 8.5, the calculation of fusion product in  $K(G(\mathbf{O}) \times \mathbb{G}_m - \mathrm{mod})$  is closely related to the ring structure of  $K^{G(\mathbf{O}) \times \mathbb{G}_m}(\mathrm{Gr}_G)$ . Let us formulate the recurrence relations in  $K^{G(\mathbf{O}) \times \mathbb{G}_m}(\mathrm{Gr}_G)$ , compare [6], end of section 2.1. So  $\mathbf{v}(n)_m$  is the class of  $\mathcal{V}(n)_m$  in  $K^{G(\mathbf{O}) \times \mathbb{G}_m}(\mathrm{Gr}_G)$ . We assume that  $n \geq 0$ .

$$(13) \quad q^{-l} \mathbf{v}(l+n)_1 \star \mathbf{v}(l)_1 = q^{-2l} \mathbf{v}(2l+n)_2 + q^2 \mathbf{v}(n-2)_0 + q^4 \mathbf{v}(n-4)_0 + \dots$$

(the last summand being  $q^n \mathbf{v}(0)_0$  if  $n$  is even, and  $q^{n-1} \mathbf{v}(1)_0$  if  $n$  is odd.)

$$(14) \quad q^{-l-2} \mathbf{v}(l-n)_1 \star \mathbf{v}(l)_1 = q^{-2l-2} \mathbf{v}(2l-n)_2 + q^{-2} \mathbf{v}(n-2)_0 + q^{-4} \mathbf{v}(n-4)_0 + \dots$$

(the last summand being  $q^{-n} \mathbf{v}(0)_0$  if  $n$  is even, and  $q^{-n+1} \mathbf{v}(1)_0$  if  $n$  is odd.)

$$(15) \quad \mathbf{v}(l+1)_1^{\star a} \star \mathbf{v}(l)_1^{\star b} = q^{\frac{1}{2}(a(1-a)+l(a+b)(1-a-b))} \mathbf{v}(a+l(a+b))_{a+b}$$

## REFERENCES

- [1] A. Alexeev, A. Malkin, E. Meinrenken, *Lie group valued moment maps*, J. Differential Geom. **48**, No. 3 (1998), 445–495.
- [2] A. Beilinson, V. Drinfeld, *Quantization of the Hitchin’s integrable system and Hecke eigensheaves*, preprint (2000), available at [www.math.uchicago.edu/~benzvi](http://www.math.uchicago.edu/~benzvi).
- [3] R. Bezrukavnikov, *Perverse coherent sheaves (after Deligne)*, preprint math.AG/0005152.
- [4] N. Chriss, V. Ginzburg, *Representation Theory and Complex Geometry*, Birkhäuser, Boston (1997).
- [5] R. Y. Donagi, D. Gaitsgory, *The gerbe of Higgs bundles*, Transform. Groups **7**, No. 2 (2002), 109–153.
- [6] B. Feigin, S. Loktev, *On generalized Kostka polynomials and quantum Verlinde rule*, The AMS Translations **194** (1999), 61–80.
- [7] D. Gaitsgory, *Construction of central elements in the affine Hecke algebra via nearby cycles*, Invent. Math. **144** (2001), 253–280.
- [8] H. Garland, I. Grojnowski, *“Affine” Hecke algebras associated to Kac-Moody groups*, preprint q-alg/9508019.
- [9] D. Gaitsgory, *On a vanishing conjecture appearing in the geometric Langlands correspondence*, preprint, math.AG/0204081.
- [10] V. Ginzburg, *Perverse sheaves on a loop group and Langlands duality*, preprint alg-geom/9511007.
- [11] V. Ginzburg, *Characteristic varieties and vanishing cycles*, Invent. Math. **84**, No. 2 (1986), 327–402.
- [12] M. Kashiwara, T. Tanisaki, *Kazhdan-Lusztig conjecture for affine Lie algebras with negative level*, Duke Math. Journal **77** (1995), 21–62.
- [13] B. Kim, *Quantum cohomology of flag manifolds  $G/B$  and quantum Toda lattices*, Ann. of Math. (2) **149** (1999), no. 1, 129–148.
- [14] B. Kostant, *The solution to a generalized Toda lattice and representation theory*, Adv. in Math. **34** (1979), 195–338.
- [15] B. Kostant, *Flag manifold quantum cohomology, the Toda lattice, and the representation with highest weight  $\rho$* , Selecta Math. (N.S.) **2** (1996), 43–91.
- [16] B. Kostant, S. Kumar, *The nil Hecke ring and cohomology of  $G/P$  for a Kac-Moody group  $G$* , Adv. in Math. **62** (1986), no. 3, 187–237.
- [17] B. Kostant, S. Kumar,  *$T$ -equivariant  $K$ -theory of generalized flag varieties*, J. Differential Geom. **32** (1990), no. 2, 549–603.
- [18] G. Lusztig, *Cuspidal Local systems and Graded Hecke Algebras, II*, CMS Conference Proceedings **16** (1995), 217–275.
- [19] I. Mirković, K. Vilonen, *Perverse sheaves on affine Grassmannians and Langlands duality*, Math. Res. Letters **7** (2000), 13–24.
- [20] D. Peterson, *Quantum cohomology of  $G/P$* , Lecture Course, M.I.T., Spring Term (1997).
- [21] R. Steinberg, *On a theorem of Pittie*, Topology **14** (1975), 173–177.
- [22] H. Sumihiro, *Equivariant completion*, J. Math. Kyoto Univ. **14** (1974), 1–28.
- [23] E. Vasserot, *Induced and simple modules of double affine Hecke algebras*, preprint math.RT/0207127.

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