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Breather lattice and its stabilization for the modified Korteweg-de Vries equation

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We obtain an exact solution for the breather lattice solution of the modified Korteweg-de Vries (MKdV) equation. Numerical simulation of the breather lattice demonstrates its instability due to the breather-breather interaction. However, such multi-breather structures can be stabilized through the concurrent application of ac driving and viscous damping terms.

I. INTRODUCTION

There are many physical systems where the 1+1 MKdV equation [1,2] appears, e.g. phonons in anharmonic lattices [3], ion acoustic solitons [4] and van Alfvén waves in collisionless plasma [5], Schottky barrier transmission lines [6] as well as in the models of traffic congestion [7]. A subclass of hyperbolic surfaces [8], slag-metallic bath interfaces [9], curve motion [10], meandering ocean jets [11] and other models in fluid mechanics [12] are also related to the MKdV equation. Furthermore, it has been shown that the dynamics of thin elastic rods can also be reduced to the MKdV equation [13]. This equation is also of special interest due to its integrability in the context of nonlinear soliton bearing systems [1,2]. From a physical perspective it is therefore important to examine novel classes of solutions of such partial differential equations and their potential relevance in this diverse class of applications.

A particularly interesting type of solution is the so called breather lattice solution. Breathers are spatially localized and temporally periodic solutions which are of significant relevance to localization type phenomena in optics, condensed matter physics and biophysics. For a representative set of reviews of the continuously increasing volume of work in this area, see e.g. [14]. As can be seen in these investigations, typically analytical expressions for breather type solutions are unavailable and such solutions have to be traced by means of numerical methods. However, in some cases and particularly for integrable models, such solutions may exist in closed form. One such example is the breather lattice solution of the sine-Gordon equation which was presented in explicit analytic form in [15]. Such a solution, albeit unstable, is very important because it can be stabilized in more realistic contexts of driving and damping. In addition, it is useful in extracting the asymptotic breather-breather (exponential) interaction by means of energy methods as demonstrated in [16].

It is naturally worthwhile to enquire whether similar extended pattern solutions are available in closed form in other models. Of particular interest are models of the MKdV (and KdV) variety not only due to an abundance of the corresponding applications but also due to the fundamental differences of such integrable models with nonlinear Klein-Gordon equations, such as sine-Gordon. The latter have Lorentz invariance that permits the breathers to be either standing or travelling. In the former case such invariances are absent and breathers can only be sustained in a travelling wave form. Furthermore, the dissipation type effects are introduced by very different operators in the two cases. Thus, for reasons of physical applicability and mathematical tractability, it is very important to identify similar solutions of the breather lattice type in models of the KdV family such as the MKdV and examine their stability. This is the main objective of the present work.

Our presentation is structured as follows. In Sec. II we present the explicit MKdV breather lattice solution in terms of elliptic functions, retrieve the well known single breather limit and analyze the algebraic and physical conditions for the existence of such a solution. In Sec. III we numerically study the stability of the unperturbed MKdV breather lattice and display its instability. In Sec. IV, we demonstrate that such a solution can be stabilized by viscous effects when combined with appropriate driving to sustain the breathers and finally in Sec. V we summarize our findings and present our conclusions.

II. BREATHER LATTICE SOLUTION

The modified Korteweg-de Vries equation for a field $u(x,t)$

$$u_t + 6u^2 u_x + u_{xxx} = 0 \ , \tag{1}$$

can be transformed into

$$(1+\phi^2)(\phi_t+\phi_{xxx})+6\phi_x(\phi_x^2-\phi\phi_{xx})=0\ , \tag{2}$$

where

$$u=v_x, \quad \phi=\tan(v/2), \quad and \quad v\to 0 \quad as \quad |x|\to\infty\ . \tag{3}$$

The single breather solution of MKdV is known to be [1]

$$u(x,t)=-2\frac{\partial}{\partial x}\tan^{-1}\left(\frac{c\sin(ax+bt+a_0)}{a\cosh(cx+dt+c_0)}\right)\ , \tag{4}$$

with

$$b=a(a^2-3c^2)\ , \quad d=c(3a^2-c^2)\ , \tag{5}$$

and a_0, c_0 are arbitrary constants.

Let us now try to obtain the breather lattice solution of the MKdV equation. To that end, we start with the ansatz

$$u(x,t)=-2\frac{\partial}{\partial x}\tan^{-1}\phi(x,t)\ , \tag{6}$$

with

$$\phi(x,t)=\alpha\,\mathrm{sn}(ax+bt+a_0,k)\,\mathrm{dn}(cx+dt+c_0,m)\ , \tag{7}$$

where $\mathrm{sn}(x,k)$ and $\mathrm{dn}(x,m)$ are Jacobi elliptic functions with modulus k and m, respectively. This ansatz is inspired by the derivative relationship between a single breather solution of the sine-Gordon equation and that of MKdV [Eq. (4)] as well as by the functional similarity of the sine-Gordon breather lattice solution [15,16]. Substituting the ansatz (7) in Eq. (2) and upon lengthy algebraic manipulations, we find that Eq. (7) is indeed the MKdV breather lattice solution, provided that

$$a^4k=c^4(1-m)\ , \quad \alpha=-\frac{c}{a}\ , \tag{8}$$

$$b=a[a^2(1+k)-3c^2(2-m)]\ , \quad d=c[3a^2(1+k)-(2-m)c^2]\ . \tag{9}$$

As expected, in the limit $m\to 1, k\to 0$, the breather lattice solution (7) reduces to the single breather solution (4) and the relations between c and d as well as between a and b reduce to the ones given by Eq. (5).

On physical grounds (i.e., to have solutions with a definite spatial periodicity), it is natural to demand that the periods of the $\mathrm{sn}(x,k)$ and $\mathrm{dn}(x,m)$ functions must be spatially commensurate, i.e., in addition to conditions (8) and (9), we must also demand that [17]

$$4K(k)/a=2K(m)/c\ , \tag{10}$$

where K(k) is the complete elliptic integral of the first kind. Note, however, that since the MKdV breather is always moving, it need not have temporal commensurability. Combining conditions (8) and (10) yields

$$16kK^4(k)=(1-m)K^4(m)\ , \tag{11}$$

implying thereby that m and k are not independent. For example, note that as expected with $m\to 1$, $k\to 0$ the breather lattice solution reduces to the single breather solution (4). However, as $m\to 0$, $\mathrm{dn}(x,m)=1$ and then it is easily shown that an exact (nonlinear) travelling wave solution is given by

$$\phi(x,t)=\sqrt{k}\,\mathrm{sn}(ax+a^3t[1+k-6\sqrt{k}]+a_0,k)\ . \tag{12}$$

Summarizing, since there are four relations among the six parameters a,b,c,d,k,m, we have obtained a two-parameter family of breather lattice solutions. A plot of the (exact) breather lattice of Eq. (6) for $m=0.5$, $c=1$ is given in the top left panel of Fig. 1.

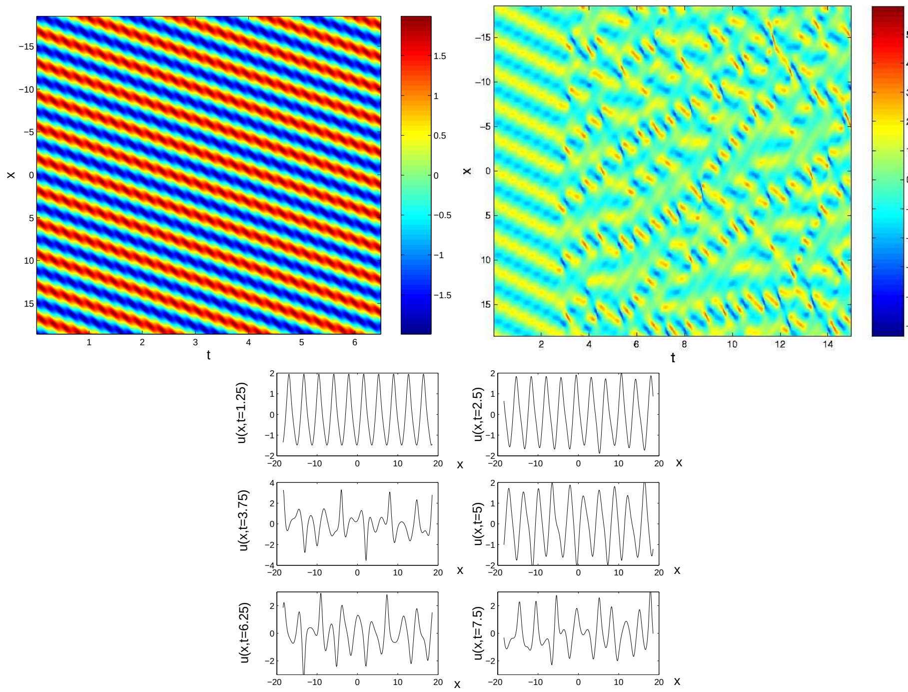

FIG. 1. Pictorial representation of the exact MKdV breather lattice solution in a space-time (x, t) contour plot for $c = 1$, $m = 0.5$ (top left panel). The top right panel shows the time evolution of the same solution under the dynamics of Eq. (13) with $\Delta t = 0.0005$ and $h \approx 0.0927$. These contour plots have dimensionless units for x and t. The bottom panel shows the spatial profiles $u(x, t_0)$ of the solution for various times t_0 before and after the instability develops.

III. NUMERICAL METHODS AND EVOLUTION OF THE BREATHER LATTICE

Based on the breather lattice simulations for the sine-Gordon case [16] it may be natural to expect that the MKdV breather lattice configuration is also unstable. In the numerical simulation of the MKdV problem, we have found that the direct center-difference discretization does a poor job in adequately following the MKdV equation (and in conserving the corresponding integrals of motion). While one can also use the integrable scheme of Ablowitz-Ladik (see e.g., [2,18,19] and references therein), we have followed a different path here in spatially discretizing the partial differential equation and following the integrable discretization scheme of [20] for KdV and adapting it to the case of the MKdV. In particular, the spatially discrete version of our equation reads (with lattice spacing h):

$$\dot{u}_n = -\frac{1}{2h^3}\left(u_{n+2} - 2u_{n+1} + 2u_{n-1} - u_{n-2}\right) - \frac{1}{3h}\left[u_{n+1}^2\left(u_{n+2} + u_{n+1} + u_n\right) - u_{n-1}^2\left(u_n + u_{n-1} + u_{n-2}\right)\right]. \qquad (13)$$

The time integration has been performed by means of a 4th order Runge-Kutta scheme. We used periodic boundary conditions and the initial condition contained an exact breather lattice configuration, matching the periodicity of the finite domain. Hence, the only perturbation to the exact solution came from the numerical discretization of the problem. It should also be noted that in the results mentioned below, the accuracy of the numerical method was monitored by probing the conservation of two quantities, $\sum_n u_n$ and $\sum_n u_n^2$, which emulate the discrete analogs of the mass and the momentum, respectively. Typically the former is conserved (at worst) to 1 part in 10^7, while the latter to 1 part in 10^3.

We found that, as can be seen in the top right panel of Fig. 1, the numerical discretization perturbation grows and eventually destroys the breather lattice configuration in all the cases considered. Various snapshots of the solution $u(x, t)$ are depicted in the bottom panel of Fig. 1.

To further understand the instability, we performed runs for different values of the two relevant parameters of the solution, namely c and m. Three typical cases are shown in Fig. 2. The particular numerical experiments are chosen to illustrate the characteristic dependences of the instability. We expect from the experience of other nonlinear wave equations with interacting breather structure (see e.g., [16] for sine-Gordon and [21] for nonlinear Schrödinger type models) that the instability is caused by the interaction between the breathers, which is exponential in their separation. From Eq. (10), the separation between the breathers is given by $S = 2K(m)/c$. The top panels of Fig. 2 correspond to two cases with different c and m but with the same $S \equiv S_0$. Clearly the instability develops at very similar times and verifies the dependence of the growth rate on the inter-breather separation S_0. The bottom panel shows a case with $S = 2S_0$. Notice that these typical results have been verified by additional numerical experiments. Furthermore, the ratio of the breather separations in the cases of the top right panel of Fig. 1, top panels of Fig. 2 and the bottom panel of Fig. 2 is 1:2:4 while the corresponding (approximate) instability onset times have a ratio of 2.5:16:150, clearly hinting an exponential dependence of the instability onset on S.

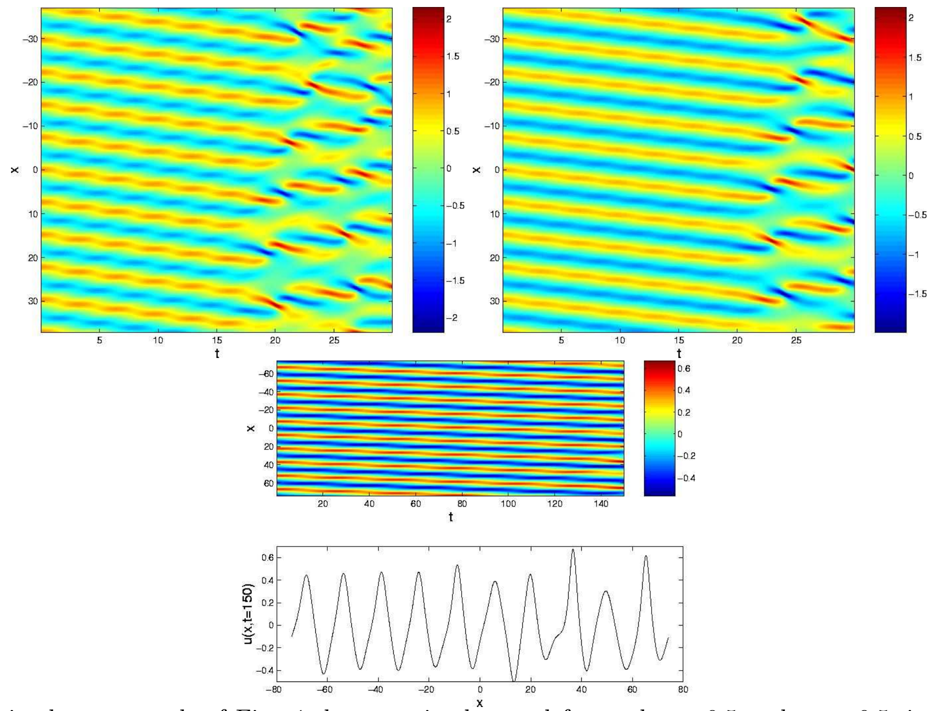

FIG. 2. Same as in the top panels of Fig. 1, but now in the top left panel $c = 0.5$ and $m = 0.5$, in the top right panel $c = 0.4546$ and $m = 0.25$, and in the bottom panel $c = 0.25$ and $m = 0.5$. In the latter case the snapshot of $t = 150$ is also shown to indicate the onset of the instability. The contour plots have dimensionless units for x and t.

IV. STABILIZATION

While the results of the previous section indicate that very long-lived breather lattice configurations can be achieved by appropriate parameter selection, it is natural to enquire whether by mechanisms of ac driving and damping it is possible to fully stabilize such configurations. We have thus examined the following driven-damped MKdV equation

$$u_t + u_{xxx} + 6u^2 u_x = \beta u_{xx} + F_0 \sin\left(\frac{\pi}{K(m)}(cx + \omega t)\right), \tag{14}$$

where the "viscosity" coefficient was fixed to $\beta = 5$ while F_0 was varied. Notice that the periodicity of the driver was chosen to match one of the unperturbed breather lattice configurations. For small values of F_0, the viscosity damps the breather amplitude. However, for sufficiently large driving amplitudes (such as the one used in Fig. 3) the driver can lead to the stabilization of an asymmetric lattice configuration. We also note that the ac drive was motivated by earlier studies, e.g. [22] (and the references therein). In addition, we point out that lattice configurations propagating in the opposite direction can be stabilized if the velocity of the driver is reversed (results not shown here).

We note that the value of the viscosity coefficient $\beta = 5$ and the amplitude of the driver $F_0 = 5$ used in Fig. 3 to stabilize the breather lattice are relatively large. In this sense the stabilized (numerical) breather lattice in Fig. 3 is different from the exact solution of the MKdV equation given in Eqs. (6) and (7). Thus the physical origin of the stabilized solution may be different from the original solution of the MKdV equation.

We should also remark that variation of the value of β will not significantly modify the results. I.e., even for much smaller values of β the multi-breather configuration is destroyed at a finite time and the resulting profile does not closely resemble the initial condition. However, the relaxation time to the final state depends considerably on the exact value of the viscosity coefficient and is longer for smaller β.

V. CONCLUSION

Inspired by the exact breather lattice solution of the sine-Gordon equation [15,16] we used an ansatz to find a corresponding solution of the modified Korteweg-de Vries equation. We determined the conditions under which the ansatz becomes an exact solution of MKdV and showed how it degenerates to the single MKdV breather solution in the appropriate limit. We then used this exact expression to derive additional lattices of nonlinear travelling waves. The MKdV breather lattice is a genuinely propagating solution in constrast to the sine-Gordon solution which can be static. Our numerical experiments (by means of a novel numerical scheme) indicated that the MKdV breather lattice solution is unstable; however, it can be stabilized by inclusion of damping and ac driving. The results presented here may be relevant to numerous physical phenomena such as jamming in traffic flow [7], fluid dynamics [12] and collisionless plasmas [5].

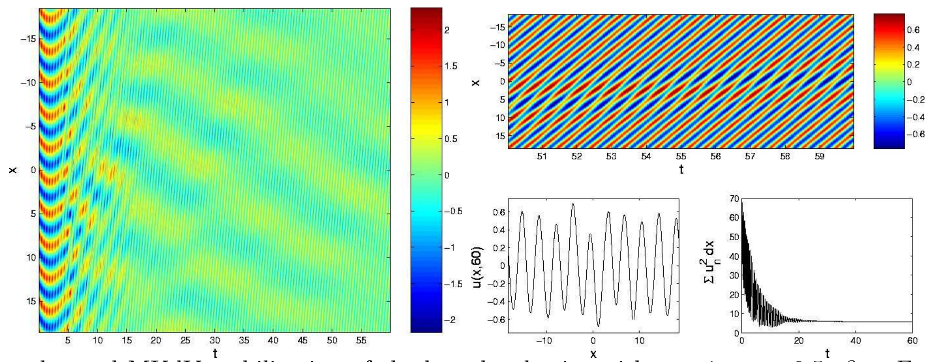

FIG. 3. The driven-damped MKdV stabilization of the breather lattice with $c = 1$, $m = 0.5$, $\beta = F_0 = 5$. The left panel shows the space-time contour plot. The right panel top subplot shows a detail of the left plot to indicate the stabilization of the configuration. These contour plots have dimensionless units for x and t. The bottom left subplot shows the spatial profile of the solution at $t = 60$, while the bottom right subplot depicts the relaxational time evolution of a global property such as the discrete analog of the continuum momentum.

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