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ON QUIVER VARIETIES AND AFFINE GRASSMANNIANS OF TYPE A

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Abstract. We construct Nakajima's quiver varieties of type A in terms of affine Grassmannians of type A. This gives a compactification of quiver varieties and a decomposition of affine Grassmannians into a disjoint union of quiver varieties. Consequently, singularities of quiver varieties, nilpotent orbits and affine Grassmannians are the same in type A. The construction also provides a geometric framework for skew $(GL(m), GL(n))$ duality and identifies the natural basis of weight spaces in Nakajima's construction with the natural basis of multiplicity spaces in tensor products which arises from affine Grassmannians.

Dedicated to Igor Frenkel on the occasion of his 50-th birthday

1. Preliminaries

1.1. **Quiver varieties of type A.** We recall Nakajima's construction of simple representations of $SL(n)$, cf. [N1, N2]. Let $I = \{1, \ldots, n-1\}$ be the set of vertices and H be the set of arrows of the Dynkin quiver of type A_{n-1}. For an arrow $h \in H$ we denote by $h' \in I$ and $h'' \in I$ its initial and terminal vertices. For a pair v, d in $\mathbb{Z}^I_{\geq 0}$ take $\mathbb{C}$-vector spaces V_i and D_i of dimensions $\dim V_i = v_i$ and $\dim D_i = d_i$, $i \in I$. Consider the affine space

$$M(v,d) = \bigoplus_{h\in H} \mathrm{Hom}(V_{h'}, V_{h''}) \oplus \bigoplus_{i\in I} \mathrm{Hom}(D_i, V_i) \oplus \bigoplus_{i\in I} \mathrm{Hom}(V_i, D_i)$$

with the natural action of the group $G(V) = \prod_{i\in I} GL(V_i)$. Let $\mathbf{m} : M(v,d) \to \mathfrak{g}(V)$ be the corresponding moment map into the Lie algebra $\mathfrak{g}(V)$. Denote $\Lambda(v,d) = \mathbf{m}^{-1}(0)$.

Nakajima's quiver variety $\mathfrak{M}(v,d)$ is the geometric quotient of $\Lambda^s(v,d)$ by $G(V)$, where $\Lambda^s(v,d)$ is the set of all stable points in $\Lambda(v,d)$ (so $\Lambda^s(v,d)/G(V)$ is the set of $\mathbb{C}$-points of $\mathfrak{M}(v,d)$). The quiver variety $\mathfrak{M}_0(v,d) = \Lambda(v,d)//G(V)$ is the invariant theory quotient (the spectrum of the $G(V)$-invariant functions). There is a natural projective map $p : \mathfrak{M}(v,d) \to \mathfrak{M}_0(v,d)$, cf. [N2], and following Maffei [M], denote its image by $\mathfrak{M}_1(v,d) = p(\mathfrak{M}(v,d)) \subseteq \mathfrak{M}_0(v,d)$. Finally, let $\mathfrak{L}(v,d) \overset{\text{def}}{=} p^{-1}(0) \subseteq \mathfrak{M}(v,d)$ and denote by $\mathcal{H}(\mathfrak{L}(v,d))$ its top-dimensional Borel-Moore homology.

1.2. **Theorem.** [N2, *10.ii*] The space $\oplus_v \mathcal{H}(\mathfrak{L}(v,d))$ has the structure of a simple $SL(n)$-module with the highest weight d (i.e., $\sum_I d_i\omega_i$ for the fundamental weights ω_i). The summand $\mathcal{H}(\mathfrak{L}(v,d))$ is the weight space for the weight $d - Cv$, where C is the Cartan matrix of type A_{n-1}.

1.3. **From** $SL(n)$ **to** $GL(n)$**.** We may consider $\oplus_v \mathcal{H}(\mathfrak{L}(v,d))$ as a representation $W_{\check{\lambda}}$ of $GL(n)$ with highest weight $\check{\lambda}$, where $\check{\lambda} = \check{\lambda}(d) = (\check{\lambda}_1, \check{\lambda}_2, \ldots, \check{\lambda}_n)$ is a partition of $N = \sum_{j=1}^{n-1} j d_j$ defined as follows: $\check{\lambda}_i = \sum_{j=i}^{n} d_j$ (here $d_n = 0$). Then $\mathcal{H}(\mathfrak{L}(v,d))$ is the weight space $W_{\check{\lambda}}(a)$, where $a_i = v_{n-1} + \sum_{j=i}^{n} (d - Cv)_j$ (here $(d-Cv)_n = 0$), cf. [N1, 8.3].

1.4. **Affine Grassmannians of type A.** We recall the construction of representations of $G = GL(m)$ in terms of its affine Grassmannian $\mathcal{G}_G$, cf. [L1, G2, MV]. Let V be a vector space with a basis $\{e_1, \ldots, e_m\}$ and $V((z)) = V \otimes_{\mathbb{C}} \mathbb{C}((z)) \supseteq L_0 = V \otimes_{\mathbb{C}} \mathbb{C}[[z]]$. A *lattice* in $V((z))$ is an $\mathbb{C}[[z]]$-submodule L of $V((z))$ such that $L \otimes_{\mathbb{C}[[z]]} \mathbb{C}((z)) = V((z))$. The affine Grassmannian $\mathcal{G}_G$ is an ind-scheme whose $\mathbb{C}$-points can be described as all lattices in $V((z))$ or as $G((z))/G[[z]]$. Its connected components $\mathcal{G}_{(N)}$ are indexed by integers $N \in \mathbb{Z}$, and if $N \geq 0$ then $\mathcal{G}_{(N)}$ contains $\mathcal{G}_N = \{$lattices L in $V((z))$ such that $L_0 \subseteq L, \dim L/L_0 = N\}$. To a dominant coweight $\lambda \in \mathbb{Z}^m$ of G, one attaches the lattice $L_\lambda = \oplus_1^m \mathbb{C}[[z]] \cdot z^{-\lambda_i} e_i$. The $G[[z]]$-orbits $\mathcal{G}_\lambda$ in $\mathcal{G}_G$ are parameterized by the dominant coweights λ via $\mathcal{G}_\lambda = G[[z]] \cdot L_\lambda$. Finally, we denote by $L^{<0}G$ the congruence subgroup of the group ind-scheme $G[z^{-1}]$ i.e., the kernel of the evaluation $z^{-1} \mapsto 0$.

The intersection homology of the closure $\overline{\mathcal{G}}_\lambda$ is a realization of the representation V_λ, and the convolution of IC-sheaves corresponds to the tensor products of representations, cf. [G2, MV].

1.5. **Resolution of singularities.** The closure $\overline{\mathcal{G}}_\mu$ of the orbit $\mathcal{G}_\mu$ in $\mathcal{G}_N$ has a natural resolution. The $G[[z]]$-orbits in $\mathcal{G}_N$ correspond to μ's which may be considered as partitions of N (into at most m parts). Any permutation $a = (a_1, \ldots, a_n)$ of the partition $\check{\mu}$ dual to μ defines a convolution space $\widetilde{\mathcal{G}}_\mu^a = \mathcal{G}_{\omega_{a_1}} * \cdots * \mathcal{G}_{\omega_{a_n}}$, where ω_k is the k-th fundamental coweight of G, and a resolution of singularities $\pi = \pi_\mu^a : \widetilde{\mathcal{G}}_\mu^a \to \overline{\mathcal{G}}_\mu$, cf. [MV].

2. Nilpotent cones of type A

2.1. n**-flags** [G1, CG]**.** Let us fix a vector space D of dimension N. Let $\mathcal{N} = \mathcal{N}(D)$ be the nilpotent cone in $\mathrm{End}(D)$. The connected components $\mathcal{F}^{n,a}$ of the variety of n-step flags in D are parameterized by all $a \in \mathbb{Z}_{\geq 0}^n$ such that $N = \sum_{i=1}^n a_i$:

$$\mathcal{F}^{n,a} = \{0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \ldots \subseteq F_n = D \mid \dim F_i - \dim F_{i-1} = a_i\}.$$

Its cotangent bundle is $\widetilde{\mathcal{N}}^{n,a} = T^* \mathcal{F}^{n,a} = \{(u, F) \in \mathcal{N} \times \mathcal{F}^a \mid u(F_i) \subseteq F_{i-1}\}$. Denote by $\mathbf{m}_a : \widetilde{\mathcal{N}}^{n,a} \to \mathcal{N}$ the projection onto the first factor.

2.2. **A transverse slice to a nilpotent orbit.** Let x be a nilpotent operator on D, with Jordan blocks of sizes $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m)$. We construct a "transverse slice" T_x to the nilpotent orbit $\mathcal{O}_\lambda \subseteq \mathcal{N}$ at x, different from the one considered by Slodowy [S, 7.4]. In some basis $e_{k,i}$, $1 \leq k \leq \lambda_i$, of D, one has $x : e_{k,i} \mapsto e_{k-1,i}$ (we set $e_{0,i} = 0$). Now

$$T_x \overset{\text{def}}{=} \{x + f,\ f \in \operatorname{End}(D) \mid f^{l,j}_{k,i} = 0,\ \text{if } k \neq \lambda_i,\ \text{and } f^{l,j}_{\lambda_i,i} = 0,\ \text{if } l > \lambda_i\},$$

where $f^{l,j}_{k,i} : \mathbb{C}e_{l,j} \to \mathbb{C}e_{k,i}$ are the matrix elements of f in our basis. For a larger orbit $\mathcal{O}_\mu$, any permutation $a = (a_1, \ldots, a_n)$ of the dual partition $\check{\mu}$, gives a resolution $\widetilde{T}^a_x \overset{\text{def}}{=} \mathbf{m}_a^{-1}(T_x \cap \overline{\mathcal{O}}_\mu) \subset \widetilde{\mathcal{N}}^{n,a}$ of the slice $T_{x,\mu} \overset{\text{def}}{=} T_x \cap \overline{\mathcal{O}}_\mu$ to $\mathcal{O}_\lambda$ in $\overline{\mathcal{O}}_\mu$.

3. **Main theorem**

3.1. **From quiver data to affine Grassmannian data.** We start with A_{n-1} quiver data $v, d \in \mathbb{Z}^I_{\geq 0}$ such that $\mathfrak{M}(v,d)$ is nonempty. Take the $SL(n)$-weights d and $d - Cv$, and pass to $GL(n)$-weights $\check{\lambda}$ and a as in subsection 1.3. Now permute a to a partition $\check{\mu} = \check{\mu}(a) = (\check{\mu}_1 \geq \check{\mu}_2 \geq \cdots \geq \check{\mu}_n)$ of $N = \sum_{j=1}^{n-1} j d_j$. Finally, let $\lambda = (\lambda_1, \ldots, \lambda_m)$ and $\mu = (\mu_1, \ldots, \mu_m)$, where $m = \sum_{i=1}^{n-1} d_i$, be the partitions of N (i.e., $GL(m)$-coweights) dual to $\check{\lambda}$ and $\check{\mu}$ respectively.

3.2. **Theorem.** Let N, v, d, a, λ, μ be as above. Let $L_\lambda \in \mathcal{G}_G$ be the lattice corresponding to the coweight λ, and let T_λ be its $L^{<0}G$-orbit. There exist algebraic isomorphisms $\phi, \widetilde{\phi}, \psi, \widetilde{\psi}$ such that the following diagram commutes:

$$\begin{array}{ccccccc}
\mathfrak{M}(v,d) & \xrightarrow[\simeq]{\widetilde{\phi}} & \widetilde{T}^a_x & \xrightarrow[\simeq]{\widetilde{\psi}} & \pi^{-1}(T_\lambda \cap \overline{\mathcal{G}}_\mu) & \xrightarrow{\subseteq} & \widetilde{\mathcal{G}}^a_\mu \\
{\scriptstyle p}\downarrow & & {\scriptstyle \mathbf{m}_a}\downarrow & & {\scriptstyle \pi}\downarrow & & {\scriptstyle \pi}\downarrow \\
\mathfrak{M}_1(v,d) & \xrightarrow[\simeq]{\phi} & T_{x,\mu} & \xrightarrow[\simeq]{\psi} & T_\lambda \cap \overline{\mathcal{G}}_\mu & \xrightarrow{\subseteq} & \overline{\mathcal{G}}_\mu
\end{array} \tag{1}$$

and $(\psi \circ \phi)(0) = L_\lambda$. In particular, $\widetilde{\psi} \circ \widetilde{\phi}$ restricts to an isomorphism $\mathfrak{L}(v,d) \simeq \pi^{-1}(L_\lambda)$.

3.3. For $d = (d_1, 0, \ldots, 0)$ and $\lambda = (1, \ldots, 1)$ the theorem above was proven in (or follows immediately from) [L1, N1]. The isomorphisms ϕ (resp. $\widetilde{\phi}$) is analogous to the isomorphism constructed in [N1] (resp. isomorphism conjectured in [N1, 8.6] and constructed in [M] using a result from [L2]). However, our isomorphism ϕ is given by an explicit formula described as follows. Let us think of a point in $\mathfrak{M}_1(v,d)$ as (closed orbit of) a quadruple $(\{B_i\}_{i\in I}, \{\overline{B}_i\}_{i\in I}, \{p_i\}_{i\in I}, \{q_i\}_{i\in I}) \in \Lambda(v,d)$, where $B_i \in \operatorname{Hom}(V_i, V_{i+1})$, $\overline{B}_i \in \operatorname{Hom}(V_{i+1}, V_i)$, $p_i \in \operatorname{Hom}(D_i, V_i)$, and $q_i \in \operatorname{Hom}(V_i, D_i)$. We decompose the vector space D, $\dim D = N$, as a direct sum: $D = \oplus_{1 \leq h \leq j \leq n-1} D^h_j$, cf. [M], where $D^h_j = \mathbb{C}\{e_{h,i} \mid \lambda_i = j\}$, $\dim D^h_j = \dim D_j = d_j$ (notation of 1.1, 2.2, 3.1). Then for any $f \in \operatorname{End}(D)$ we consider its blocks $f_{j,h}^{j',h'} : D_{j'}^{h'} \to D_j^h$. By definition, $\phi(B, \overline{B}, p, q) = x + f \in \mathcal{N}$ (notation of 2.2), where

$$f_{j,h}^{j',h'} = \begin{cases} q_j B_{j-1} \dots B_{h'+1} B_{h'} \overline{B}_{h'} \overline{B}_{h'+1} \dots \overline{B}_{j'-1} p_{j'}, & \text{if } h = j, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

In particular, $\phi(0) = x$.

3.4. **Compactification of quiver varieties.** A compactification of $\mathfrak{M}_1(v,d)$ and $\mathfrak{M}(v,d)$ is given by closures of their respective images under the embeddings $\mathfrak{M}_1(v,d) \hookrightarrow \overline{\mathcal{G}}_\mu$ and $\mathfrak{M}(v,d) \hookrightarrow \widetilde{\mathcal{G}}_\mu^a$.

3.5. **Decomposition.** The theorem implies a decomposition of $\overline{\mathcal{G}}_\mu$ into a disjoint union of quiver varieties

$$\overline{\mathcal{G}}_\mu = \bigsqcup_{\mathcal{G}_\lambda \subseteq \overline{\mathcal{G}}_\mu} \; \bigsqcup_{y \in G \cdot L_\lambda} \mathfrak{M}_0(v,d)_y, \tag{3}$$

where $\mathfrak{M}_0(v,d)_y$ is a copy of quiver variety $\mathfrak{M}_0(v,d)$ for every point $y \in G \cdot L_\lambda$, and v, d are obtained from λ, μ by reversing formulas in subsection 1.3.

3.6. **Beilinson-Drinfeld Grassmannians.** Recall the moment map $\mathbf{m} : M(v,d) \to \mathfrak{g}(V)$ from subsection 1.1. Any $c = (c_1 \operatorname{Id}_{V_1}, \dots, c_{n-1} \operatorname{Id}_{V_{n-1}})$ in the center of the Lie algebra $\mathfrak{g}(V)$ defines $\Lambda_c(v,d) = \mathbf{m}^{-1}(c)$, and then, as in 1.1, the "deformed" quiver varieties $\mathfrak{M}^c(v,d) = \Lambda_c^s(v,d)/G(V)$ and $\mathfrak{M}_0^c(v,d) = \Lambda_c(v,d)//G(V)$. We expect that in type A our theorem and decomposition (3) extend to a relation between deformed quiver varieties and the Beilinson-Drinfeld Grassmannians, cf. [BD].

For instance, when $d = (d_1, 0, \dots, 0)$ there is an embedding $\mathfrak{M}_0^c(v,d) \hookrightarrow \mathcal{G}_{\mathbb{A}^{(n)}}^{BD}(GL(m))$ of our quiver variety into the fiber of the Beilinson-Drinfeld Grassmannian over the point $(0, c_1, c_1 + c_2, \dots, c_1 + \dots + c_{n-1}) \in \mathbb{A}^{(n)}$.

The proofs and more details will appear in a forthcoming paper.

Another example of a decomposition of an infinite Grassmannian into a disjoint union of quiver varieties can be found in [BGK] (who generalized a result from [W]). A part of adelic Grassmannian is a union of quiver varieties $\mathfrak{M}^c(v,d)$ associated to *affine* quivers of type A.

4. Geometric construction of skew $(GL(n), GL(m))$ duality

4.1. **Skew Howe duality.** Let $V = \mathbb{C}^m$ and $W = \mathbb{C}^n$ be two vector spaces. Then we have the $GL(V) \times GL(W)$-decomposition [H, 4.1.1]:

$$\wedge^N (V \otimes W) = \bigoplus_\lambda V_\lambda \otimes W_{\check{\lambda}}, \tag{4}$$

where λ varies over all partitions of N which fit into the $n \times m$ box, and V_λ and $W_{\check{\lambda}}$ are the corresponding highest weight representation of $GL(m)$ of $GL(n)$. This is essentially equivalent to natural isomorphisms of vector spaces

$$\mathrm{Hom}_{GL(m)}(\wedge^{a_1} V \otimes \cdots \otimes \wedge^{a_n} V, V_\lambda) \simeq W_{\check{\lambda}}(a), \tag{5}$$

where $W_{\check{\lambda}}(a)$ is the weight space corresponding to the weight $a = (a_1, \ldots, a_n)$.

4.2. We construct a based version of the isomorphism (5), i.e., a geometric skew Howe duality. More precisely, with N, v, d, a, λ as in 3.1, we identify the right hand side with $\mathcal{H}(\pi^{-1}(L_\lambda))$ (notation from Theorem 3.2) and the left hand side with $\mathcal{H}(\mathfrak{L}(v, d))$ by Theorem1.2. The identification of irreducible components $\mathrm{Irr}\, \pi^{-1}(L_\lambda) = \mathrm{Irr}\, \mathfrak{L}(v, d)$, which follows from Theorem 3.2, matches the natural basis of the space of intertwiners $\mathrm{Hom}_{GL(m)}(\wedge^{a_1} V \otimes \cdots \otimes \wedge^{a_n} V, V_\lambda)$ arising from the affine Grassmannian construction (i.e., $\mathrm{Irr}\, \pi^{-1}(L_\lambda)$), and the natural basis of the weight space $W_{\check{\lambda}}(a)$ in the Nakajima construction (i.e., $\mathrm{Irr}\, \mathfrak{L}(v, d)$). Altogether:

$$\mathrm{Hom}_{GL(m)}(\wedge^{a_1} V \otimes \cdots \otimes \wedge^{a_n} V, V_\lambda) \simeq \mathcal{H}(\pi^{-1}(L_\lambda)) \simeq \mathcal{H}(\mathfrak{L}(v, d)) \simeq W_{\check{\lambda}}(a).$$

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