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Toric Ideals, Real Toric Varieties, and the Algebraic Moment Map

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Toric ideals, real toric varieties, and the algebraic moment map

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Abstract. This is a tutorial on some aspects of toric varieties related to their potential use in geometric modeling. We discuss projective toric varieties and their ideals, as well as real toric varieties. In particular, we explain the relation between linear precision and a particular linear projection we call the algebraic moment map.

Introduction

We develop further aspects of toric varieties that may be useful in geometric modeling, building on Cox's introduction to toric varieties, *What is a toric variety?* [**Cox03**], which also appears in this volume. Notation and terminology follow that article, with a few small exceptions. This paper is organized into eight sections:

1. Projective Toric Varieties
2. Toric Ideals
3. Linear Projections
4. Rational Varieties
5. Implicit Degree of a Toric Variety
6. The Real Part of a Toric Variety
7. The Double Pillow
8. Linear Precision and the Algebraic Moment Map

1. Projective Toric Varieties

In this tutorial, we study toric varieties as subvarieties of projective space. This differs slightly from Cox's [**Cox03**] presentation, where toric varieties are studied via the abstract toric variety X_Σ of a fan Σ. The resulting loss of generality is compensated by the additional perspective this alternative view provides. Only in the last few sections do we discuss abstract toric varieties.

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A projective toric variety may be given as the closure of the image of a map

$$(\mathbb{C}^*)^n \longrightarrow \mathbb{P}^\ell ,$$

defined by Laurent monomials as in Section 13 of [**Cox03**]. There, the monomials had exponent vectors given by all the integer lattice points in a polytope. Here, we study maps given by *any* set of Laurent monomials.

Our basic data structure will be a list of integer exponent vectors

$$\mathcal{A} := \{\mathbf{m}_0, \mathbf{m}_1, \ldots, \mathbf{m}_\ell\} \subset \mathbb{Z}^n .$$

Such a list gives rise to a map $\varphi_{\mathcal{A}}$ (written φ when $\mathcal{A}$ is understood),

$$\begin{array}{cccll} \varphi_{\mathcal{A}} & : & (\mathbb{C}^*)^n & \longrightarrow & \mathbb{P}^\ell \\ & & \mathbf{t} & \longmapsto & [\mathbf{t}^{\mathbf{m}_0}, \mathbf{t}^{\mathbf{m}_1}, \ldots, \mathbf{t}^{\mathbf{m}_\ell}] . \end{array} \tag{1.1}$$

We explain this notation. Given $\mathbf{t} = (t_1, \ldots, t_n) \in (\mathbb{C}^*)^n$ and an exponent vector $\mathbf{a} = (a_1, \ldots, a_n)$, the monomial $\mathbf{t}^{\mathbf{a}}$ is equal to $t_1^{a_1} t_2^{a_2} \cdots t_n^{a_n}$. In this way, the coordinates of $\mathbb{P}^\ell$ are naturally indexed by the exponent vectors lying in $\mathcal{A}$. The *toric variety* $Y_{\mathcal{A}}$ is the closure in $\mathbb{P}^\ell$ of the image of the map $\varphi_{\mathcal{A}}$. This map $\varphi_{\mathcal{A}}$ gives a parametrization of $Y_{\mathcal{A}}$ by the monomials whose exponents lie in $\mathcal{A}$.

We claim that $Y_{\mathcal{A}}$ is a toric variety as defined in Section 2 of [**Cox03**]. The map

$$(\mathbb{C}^*)^n \ni \mathbf{t} \longmapsto (\mathbf{t}^{\mathbf{m}_0}, \mathbf{t}^{\mathbf{m}_1}, \ldots, \mathbf{t}^{\mathbf{m}_\ell}) \in (\mathbb{C}^*)^{1+\ell}$$

is a homomorphism from the group $(\mathbb{C}^*)^n$ to the group $(\mathbb{C}^*)^{1+\ell}$ of diagonal $(1+\ell)$ by $(1+\ell)$ matrices, which acts on $\mathbb{P}^\ell$. Thus $(\mathbb{C}^*)^n$ acts on $\mathbb{P}^\ell$ via this map. Since scalar matrices (those in $\mathbb{C}^* I_{1+\ell}$) act trivially on $\mathbb{P}^\ell$, this action of $(\mathbb{C}^*)^{1+\ell}$ on $\mathbb{P}^\ell$ factors through the group $(\mathbb{C}^*)^{1+\ell}/\mathbb{C}^* I_{1+\ell} \simeq (\mathbb{C}^*)^\ell$, which is the dense torus in the toric variety $\mathbb{P}^\ell$. Then $Y_{\mathcal{A}}$ is the closure of the image of $(\mathbb{C}^*)^n$ in this torus, that image T acts on $Y_{\mathcal{A}}$, and thus T is the dense torus of $Y_{\mathcal{A}}$.

Suppose that $\mathcal{A} = \Delta \cap \mathbb{Z}^n$, where Δ is a lattice polytope. Then $Y_{\mathcal{A}}$ is the image of the abstract toric variety X_Δ given by the normal fan of Δ under the map of Section 13 in [**Cox03**]. When $\mathcal{A}$ has this form, we write Y_Δ for $Y_{\mathcal{A}}$.

Example 1.2. Consider the three lattice polytopes

$$\begin{aligned} &[n], \text{ the line segment } [0,n] \subset \mathbb{R} , \\ &\triangle_n, \text{ the triangle } \{(x,y) \in \mathbb{R}^2 \mid 0 \le x, y,\ x+y \le n\} , \quad \text{and} \\ &\square_{m,n}, \text{ the rectangle } \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le m,\ 0 \le y \le n\} . \end{aligned}$$

The maps φ for these polytopes are

$$\begin{array}{ccl} t & \longmapsto & [1, t, t^2, \ldots, t^n] \in \mathbb{P}^n , \\ (s,t) & \longmapsto & [1, s, t, s^2, st, t^2, \ldots, s^n, s^{n-1}t, \ldots, t^n] \in \mathbb{P}^{\binom{n+2}{2}} , \quad \text{and} \\ (s,t) & \longmapsto & [1, s, \ldots, s^m, t, st, \ldots, s^m t, \ldots, t^n, st^n, s^m t^n] \in \mathbb{P}^{mn} . \end{array}$$

and the resulting projective toric varieties are known (see [**Har92**]) as

$$\begin{array}{rcl} Y_{[n]} & = & \text{the rational normal curve in } \mathbb{P}^n , \\ Y_{\triangle_n} & = & \text{the Veronese embedding of } \mathbb{P}^2 \text{ in } \mathbb{P}^{\binom{n+2}{2}} , \quad \text{and} \\ Y_{\square_{m,n}} & = & \text{the Segre embedding of } \mathbb{P}^1 \times \mathbb{P}^1 \text{ in } \mathbb{P}^{mn} \text{ of bidegree } m, n . \end{array}$$

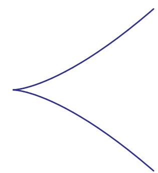
In geometric modeling these projective toric varieties give rise to, respectively, Bézier curves, rational Bézier triangles of degree n, and tensor product surfaces of bidegree (m, n).

EXAMPLE 1.3. Let $n = 1$ and $\mathcal{A} = \{0, 2, 3\}$. Then the map (1.1) is

$$t \longmapsto [1, t^2, t^3]$$

whose image $Y_{\mathcal{A}}$ is the cuspidal cubic

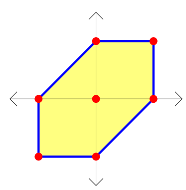
$$\mathbf{V}(x_0x_2^2 - x_1^3) \quad =$$



which is the non-normal toric variety of Example 3.2 in [**Cox03**].

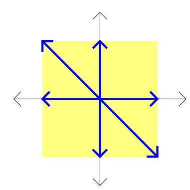
EXAMPLE 1.4. Let Δ be the hexagon which is the convex hull of the six column vectors $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}-1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}-1\\-1\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\-1\end{smallmatrix}\right]$. We depict Δ and its normal fan Σ_Δ.

$\Delta =$



and

$\Sigma_\Delta =$



Then $\Delta \cap \mathbb{Z}^2$ consists of these six vectors (the vertices of Δ) together with the origin $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$. Thus $Y_\Delta = \overline{\{[1, t, st, s, t^{-1}, s^{-1}t^{-1}, s^{-1}] \mid s, t \in \mathbb{C}^\times\}} \subset \mathbb{P}^6$.

REMARK 1.5. Suppose that the origin $\mathbf{0}$ is an element of $\mathcal{A}$ and that $m_0 = \mathbf{0}$. Then the image of the map φ of (1.1) lies in the *principal affine part* of $\mathbb{P}^\ell$

$$U_0 \ := \ \{x \in \mathbb{P}^\ell \mid x_0 \neq 0\} \ \simeq \ \mathbb{C}^\ell\,,$$

whose coordinates are $[1, x_1, x_2, \ldots, x_\ell]$. Thus $U_0 \cap Y_{\mathcal{A}}$ is an affine toric variety. In this case, the dimension of the projective toric variety $Y_{\mathcal{A}}$ is equal to the dimension of the linear span of the exponent vectors $\mathcal{A}$.

2. Toric Ideals

The *toric ideal* $I_{\mathcal{A}}$ is the homogeneous ideal of polynomials whose vanishing defines the projective toric variety $Y_{\mathcal{A}} \subset \mathbb{P}^\ell$. Equivalently, $I_{\mathcal{A}}$ is the ideal of all the homogeneous polynomials vanishing on $\varphi_{\mathcal{A}}((\mathbb{C}^*)^n)$. Our description of $I_{\mathcal{A}}$ follows the presentation in Sturmfels's book, *Gröbner bases and convex polytopes* [**Stu96**].

Let $[x_0, x_1, \ldots, x_\ell]$ be homogeneous coordinates for $\mathbb{P}^\ell$ with x_j corresponding to the monomial $\mathbf{t}^{\mathbf{m}_j}$ in the map $\varphi_{\mathcal{A}}$ (1.1), where $\mathcal{A} = \{\mathbf{m}_0, \mathbf{m}_1, \ldots, \mathbf{m}_\ell\}$. A monomial $\mathbf{x}^{\mathbf{u}}$ in these coordinates has an exponent vector $\mathbf{u} \in \mathbb{N}^{1+\ell}$. Restricting the monomial $\mathbf{x}^{\mathbf{u}}$ to $\varphi_{\mathcal{A}}(t_1, \ldots, t_n) = [\mathbf{t}^{\mathbf{m}_0}, \mathbf{t}^{\mathbf{m}_2}, \ldots, \mathbf{t}^{\mathbf{m}_\ell}]$ yields the monomial

$$\mathbf{t}^{u_0\mathbf{m}_0 + u_1\mathbf{m}_1 + \cdots + u_\ell\mathbf{m}_\ell}\ .$$

This exponent vector is $\mathcal{A}\mathbf{u}$, where we consider $\mathcal{A}$ to be the matrix whose columns are the exponent vectors in $\mathcal{A}$. For the hexagon of Example 1.4, this is

$$\begin{pmatrix} 0 & 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & -1 \end{pmatrix}$$

This discussion shows that a homogeneous binomial $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ with $\mathcal{A}\mathbf{u} = \mathcal{A}\mathbf{v}$ vanishes on $\varphi_{\mathcal{A}}((\mathbb{C}^*)^n)$ and hence lies in the toric ideal $I_{\mathcal{A}}$. In fact, the toric ideal $I_{\mathcal{A}}$ is the linear span of these binomials.

THEOREM 2.1. *The toric ideal $I_{\mathcal{A}}$ is the linear span of all homogeneous binomials $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ with $\mathcal{A}\mathbf{u} = \mathcal{A}\mathbf{v}$.*

We obtain a more elegant description of $I_{\mathcal{A}}$ if the row space of the matrix $\mathcal{A}$ contains the vector $(1, \ldots, 1)$, for then the homogeneity of the binomial $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ follows from $\mathcal{A}\mathbf{u} = \mathcal{A}\mathbf{v}$. It is often useful to force this condition as follows.

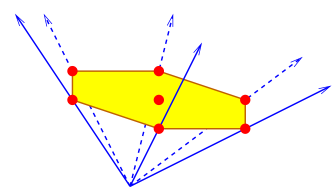
Given a list $\mathcal{A}$ of exponent vectors in $\mathbb{Z}^n$, the *lift* $\mathcal{A}^+$ of $\mathcal{A}$ to $1 \times \mathbb{Z}^n$ is obtained be prepending a component of 1 to each vector in $\mathcal{A}$. That is,

$$\mathcal{A}^+ \ := \ \{(1, \mathbf{m}) \mid \mathbf{m} \in \mathcal{A}\}$$

The matrix $\mathcal{A}^+$ is obtained from the matrix $\mathcal{A}$ by adding a new top row of 1s. For the hexagon, this is

$$\mathcal{A}^+ \ = \ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & -1 \end{pmatrix} .$$

Here is the lifted hexagon (shaded)



Then $Y_{\mathcal{A}^+} = \overline{\{[r, rt, rst, rs, rt^{-1}, rs^{-1}t^{-1}, rs^{-1}] \mid r, s, t \in \mathbb{C}^\times\}} = Y_{\Delta}$.

This lifting does not alter the projective toric variety. Indeed,

$$\begin{aligned} \varphi_{\mathcal{A}^+}(t_0, t_1, \ldots, t_n) \ &= \ [t_0\mathbf{t}^{\mathbf{m}_0}, t_0\mathbf{t}^{\mathbf{m}_1}, \ldots, t_0\mathbf{t}^{\mathbf{m}_\ell}] \\ &= \ [\mathbf{t}^{\mathbf{m}_0}, \mathbf{t}^{\mathbf{m}_1}, \ldots, \mathbf{t}^{\mathbf{m}_\ell}] \\ &= \ \varphi_{\mathcal{A}}(t_1, \ldots, t_n)\,, \end{aligned}$$

and so $Y_{\mathcal{A}} = Y_{\mathcal{A}^+}$. The dimension of $Y_{\mathcal{A}}$ is one less than the dimension of the linear span of $\mathcal{A}^+$. Since $I_{\mathcal{A}} = I_{\mathcal{A}^+}$, we have

THEOREM 2.2. *The toric ideal $I_{\mathcal{A}}$ is the linear span of all binomials $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ with $\mathcal{A}^+\mathbf{u} = \mathcal{A}^+\mathbf{v}$.*

If $\mathbf{u} \in \mathbb{Z}^{1+\ell}$, then we may write $\mathbf{u}$ uniquely as $\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^-$, where $\mathbf{u}^+, \mathbf{u}^- \in \mathbb{N}^{1+\ell}$ but $\mathbf{u}^+$ and $\mathbf{u}^-$ have no non-zero components in common. For example, if $\mathbf{u} = (1, 3, 2, -2, 2, -4)$, then $\mathbf{u}^+ = (1, 3, 2, 0, 2, 0)$ and $\mathbf{u}^- = (0, 0, 0, 2, 0, 4)$. We describe a smaller set of binomials that generate $I_{\mathcal{A}}$. Let $\ker(\mathcal{A}) \subset \mathbb{Z}^{1+\ell}$ be the kernel of the matrix $\mathcal{A}$.

COROLLARY 2.3. $I_{\mathcal{A}} \ = \ \langle x^{\mathbf{u}^+} - x^{\mathbf{u}^-} \mid \mathbf{u} \in \mathit{ker}(\mathcal{A}^+)\rangle$.

Algorithms for computing toric ideals are implemented in the computer algebra systems Macaulay 2 [**Mac2**] and Singular [**SING**]. There are no simple formulas for a finite set of generators of a general toric ideal.

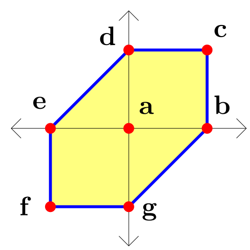
On the other hand, *quadratic binomials* in a toric ideal do have a simple geometric interpretation. Suppose that we have a relation of the form

$$\mathbf{a}+\mathbf{b} \;=\; \mathbf{c}+\mathbf{d}, \qquad \text{for } \mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\in\Delta\,. \tag{2.4}$$

Such a relation comes from coincident midpoints of two line segments between lattice points in Δ. A relation (2.4) gives a vector $\mathbf{u}\in \mathit{ker}(\mathcal{A})$ whose entries are 1 in the coordinates corresponding to $\mathbf{a}$ and $\mathbf{b}$ and -1 in the coordinates corresponding to $\mathbf{c}$ and $\mathbf{d}$. The corresponding generator of the ideal $I_{\mathcal{A}}$ is $ab-cd$, where a is the variable corresponding to the vector $\mathbf{a}$, b the variable corresponding to $\mathbf{b}$, and etc.

Often these simple relations suffice. When $n=2$, Koelman [**Koe93**] showed that the ideal of a toric surface Y_Δ is generated by such quadratic binomials if the polygon Δ has more than 3 lattice points on its boundary. Also, if a lattice polytope $\Delta\subset\mathbb{R}^n$ has the form $n\Delta'$, for a (smaller) lattice polytope Δ', then the toric ideal I_Δ is generated by such quadratic binomials for Δ [**BGT97**].

Example 2.5. Consider the ideal I_Δ of the toric variety Y_Δ, where Δ is the hexagon of Example 1.4. Label the exponent vectors in Δ as indicated below.



There are 12 relations of the form (2.4): 6 involving the midpoint of the segment connecting $\mathbf{a}$ and one of the remaining vectors and 6 for the 3 antipodal pairs of points, which all share the midpoint $\mathbf{a}$. Translating these into quadratic relations gives 12 quadratic binomials in the toric ideal I_Δ

$$\begin{gathered} ab-cg,\; ac-bd,\; ad-ce,\; ae-df,\; af-ge,\; ag-bf,\\ a^2-be,\; a^2-gd,\; a^2-cf,\; be-cf,\; be-gd,\; cf-gd\,. \end{gathered}$$

By Koelman's Theorem, these generate I_Δ as Δ has 6 vertices.

3. Linear Projections

Linear projections are key to the relationship between the toric varieties Y_Δ introduced in Section 1 and Krasauskas's toric patches [**Kra02**]. In geometric modeling, linear projections are encoded in the language of weights and control points. In algebraic geometry, linear projections provide the link between projective toric varieties $Y_{\mathcal{A}}$ and general rational varieties.

Given $1+\ell$ vectors $\mathbf{p}_0,\mathbf{p}_1,\ldots,\mathbf{p}_\ell\in\mathbb{C}^{1+k}$, we have the linear map $\mathbb{C}^{1+\ell}\to\mathbb{C}^{1+k}$

$$\mathbf{x} \;=\; (x_0,x_1,\ldots,x_\ell) \;\longmapsto\; x_0\mathbf{p}_0+x_1\mathbf{p}_1+\cdots+x_\ell\mathbf{p}_\ell \;\in\; \mathbb{C}^{1+k}\,, \tag{3.1}$$

represented by the matrix whose columns are the vectors $\mathbf{p}_i$. Let $E:=\{\mathbf{x}\in\mathbb{C}^{1+\ell}\mid 0=\sum_i x_i\mathbf{p}_i\}$ be the kernel of this map.

Let $\mathbb{P}(E)$ be the linear subspace of $\mathbb{P}^\ell$ corresponding to E. Then (3.1) induces a map π from the difference $\mathbb{P}^\ell-\mathbb{P}(E)$ to $\mathbb{P}^k$, called a *linear projection* with center of projection $\mathbb{P}(E)$ (or central projection from $\mathbb{P}(E)$). We write

$$\pi\,:\,\mathbb{P}^\ell \dashrightarrow \mathbb{P}^k\,.$$

(In algebraic geometry, a broken arrow is used to represent such a *rational map*—a function that is not defined on all of $\mathbb{P}^\ell$.) The *control points* $\mathbf{b}_0, \ldots, \mathbf{b}_\ell$ of this projection are the images in $\mathbb{P}^k$ of the vectors $\mathbf{p}_i$.

Given a subvariety $Y \subset \mathbb{P}^\ell$ that does not meet the center $\mathbb{P}(E)$, the linear projection π restricts to give a map $\pi\colon Y \to \mathbb{P}^k$. Points where $\mathbb{P}(E)$ meets Y are called *basepoints* of the projection $Y \dashrightarrow \mathbb{P}^k$.

EXAMPLE 3.2. A projective toric variety $Y_{\mathcal{A}} \subset \mathbb{P}^k$ as defined in Section 1 is the image of the projective toric variety $Y_\Delta \subset \mathbb{P}^\ell$, where Δ is the convex hull of the exponent vectors $\mathcal{A} \subset \mathbb{Z}^n$. Recall that the coordinates of $\mathbb{P}^k$ are naturally indexed by the elements of $\mathcal{A}$ and those of $\mathbb{P}^\ell$ by $\Delta \cap \mathbb{Z}^n$. The projection simply 'forgets' the coordinates of points whose index does not lie in $\mathcal{A}$. What is not immediate from the definitions is that the projection

$$Y_\Delta \longrightarrow Y_{\mathcal{A}} \tag{3.3}$$

has no basepoints.

To see this, note that the center of this projection is defined by the vanishing of all coordinates of $\mathbb{P}^\ell$ indexed by elements of $\mathcal{A}$. Any point $\mathbf{m} \in \Delta$ that is not a vertex is a positive rational combination of the vertices $\mathbf{v}$ of Δ. That is, there are positive integers $d_{\mathbf{m}}$ and $d_{\mathbf{v}}$ for each vertex $\mathbf{v}$ of Δ such that

$$d_{\mathbf{m}} \cdot \mathbf{m} \;=\; \sum_{\mathbf{v}} d_{\mathbf{v}} \cdot \mathbf{v}\,,$$

and so we have the binomial in the toric ideal I_Δ

$$x_{\mathbf{m}}^{d_{\mathbf{m}}} \;-\; \prod_{\mathbf{v}} x_{\mathbf{v}}^{d_{\mathbf{v}}}$$

In particular, if a point $\mathbf{x} \in Y_\Delta$ has a nonvanishing $\mathbf{m}$th coordinate ($x_{\mathbf{m}} \neq 0$), then some vertex coordinates $x_{\mathbf{v}}$ must also be nonvanishing. This shows that the map (3.3) has no basepoints.

This discussion also shows that the only basis vectors $[0, \ldots, 0, 1, 0, \ldots, 0]$ contained in a projective toric variety $Y_{\mathcal{A}}$ are those indexed by the extreme points of $\mathcal{A}$—the vertices of the convex hull of $\mathcal{A}$.

EXAMPLE 3.4. Let $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_n$ be vectors in $\mathbb{C}^{1+k}$. Then the image Z of the rational normal curve $Y_{[n]}$ of Example 1.2 under the corresponding linear projection is parametrized by

$$\mathbb{P}^1 \;\ni\; [s,t] \;\longmapsto\; s^n\mathbf{p}_0 + s^{n-1}t\mathbf{p}_1 + \cdots + t^n\mathbf{p}_n \;\in\; \mathbb{P}^k\,. \tag{3.5}$$

The map $Y_{[n]} \to Z$ has a basepoint at $[s,t] \in \mathbb{P}^1$ when the sum in (3.5) vanishes. Since each component of the sum is a homogeneous polynomial in s, t of degree n, this implies that these $1+k$ polynomials share a common factor. When the polynomials have no common factor, Z is a rational curve of degree n.

We consider an example of this when $n = 3$ and $k = 2$. Let $(1,-1,-1)$, $(1,-3,-1)$, $(1,-1,3)$, and $(1,1,-1) \in \mathbb{C}^3$ be the vectors $\mathbf{p}_0, \ldots, \mathbf{p}_3$ which determine a linear projection $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$. Then the image curve Z of the toric variety $Y_{[3]}$ is given parametrically as

$$\begin{aligned} z_0 &= s^3 + s^2t + st^2 + t^3, \\ z_1 &= -s^3 - 3s^2t - st^2 + t^3, \quad \text{and} \\ z_2 &= -s^3 - s^2t + 3st^2 - t^3\,. \end{aligned}$$

If we set $x = z_1/z_0$ and $y = z_2/z_0$ to be coordinates for the principal affine part of $\mathbb{P}^2$, then this has implicit equation

$$y^2(x-1) + 2yx + x^2 + x^3 \;=\; 0\,.$$

We plot the control points $\mathbf{b}_i$ and the curve in Figure 1.

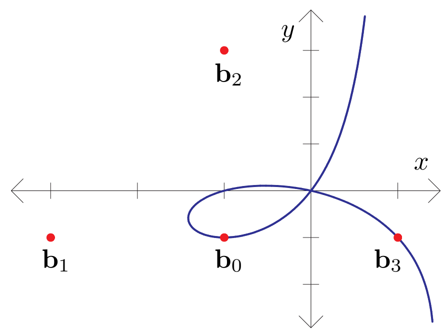


FIGURE 1. A cubic curve

4. Rational Varieties

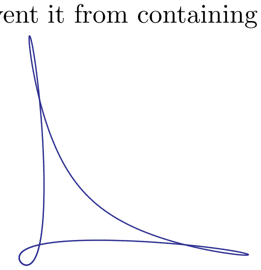
Example 3.2 shows how the toric varieties $Y_{\mathcal{A}}$ and Y_{Δ} are related via special linear projections and Example 3.4 shows how rational curves are related to the rational normal curve. More general linear projections give rational varieties, which are varieties parametrized by some collection of polynomials.

DEFINITION 4.1. A *rational variety* $Z \subset \mathbb{P}^k$ is the image of a projective toric variety $Y_{\mathcal{A}}$ under a linear projection. The composition

$$(\mathbb{C}^*)^n \xhookrightarrow{\varphi_{\mathcal{A}}} Y_{\mathcal{A}} \dashrightarrow Z$$

endows a rational variety Z with a rational parametrization by polynomials whose monomials have exponent vectors in $\mathcal{A}$ and this parametrization is defined for (almost all) points in $(\mathbb{C}^*)^n$.

REMARK 4.2. The class of rational varieties is strictly larger than that of toric varieties. For example, the quartic rational plane curve whose rational parametrization and picture is shown below is not a toric variety—its three singular points prevent it from containing a dense torus.



$$\begin{aligned}
x &= t^4 + 7st^3 + 9s^2t^2 - 7s^3t - 10s^4\\
y &= t^4 - 7st^3 + 9s^2t^2 + 7s^3t - 10s^4\\
z &= 3t^4 - 11s^2t^2 + 80s^4
\end{aligned}$$

This class of rational varieties contains the closures of the images of Bézier curves, triangular Bézier patches, tensor product surfaces, and Krasauskas's toric patches [**Kra02**]. We give another example based upon the hexagon of Example 1.4.

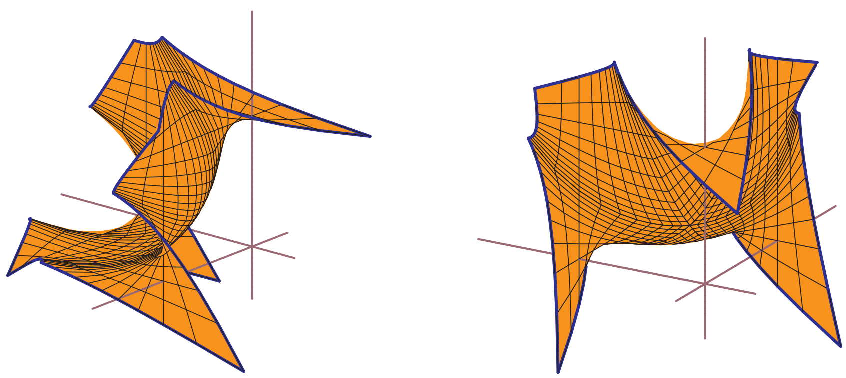
Example 4.3. Consider a projection $\mathbb{P}^6 \dashrightarrow \mathbb{P}^3$ where the points $\mathbf{p}_i$ corresponding to the vertices $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}-1\\0\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}-1\\-1\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}0\\-1\end{smallmatrix}\right]$ of the hexagon are the following points in $\mathbb{C}^4$ taken in order:

$$(1,1,0,0),\ (1,1,1,0),\ (1,0,1,0),\ (1,0,1,1),\ (1,0,0,1),\ (1,1,0,1)$$

and suppose that the center of the hexagon corresponds to the point $(1,-1,-1,-1)$. In coordinates $[w,x,y,z]$ for $\mathbb{P}^3$, this has the rational parametrization.

$$\begin{aligned}
w &= 1+s+st+t+s^{-1}+s^{-1}t^{-1}+t^{-1}\\
x &= -1+s+st+t^{-1}\\
y &= -1+st+t+s^{-1}\\
z &= -1+s^{-1}+s^{-1}t^{-1}+t^{-1}
\end{aligned}$$

Here are two views of (part of) the resulting rational surface and the axes.



As a subset of $\mathbb{P}^3$, this is defined by a the vanishing of a single homogeneous polynomial of degree 6 with 72 terms

$$\begin{aligned}
&112w^6-240(x+y+z)w^5+296(xy+xz+yz)w^4+216(x^2+y^2+z^2)w^4\\
&-92(x^3+y^3+z^3)w^3-124(x^2y+xy^2+x^2z+xz^2+y^2z+yz^2)w^3-568xyzw^3\\
&+4(x^4+y^4+z^4)w^2+70(x^3y+xy^3+x^3z+xz^3+y^3z+yz^3)w^2\\
&-125(x^2y^2+x^2z^2+y^2z^2)w^2+272(x^2yz+xy^2z+xyz^2)w^2\\
&-2(x^4y+xy^4+x^4z+xz^4+y^4z+yz^4)w-141(x^3yz+xy^3z+xyz^3)w\\
&+35(x^3y^2+x^2y^3+x^3z^2+x^2z^3+y^3z^2+y^2z^3)w-7(x^2y^2z+x^2yz^2+xy^2z^2)w\\
&+5(x^4yz+xy^4z+xyz^4)+19(x^3y^2z+x^3yz^2+x^2y^3z+x^2yz^3+xy^3z^2+xy^2z^3)\\
&-50x^2y^2z^2-13(x^3y^3+x^3z^3+y^3z^3)-2(x^4y^2+x^2y^4+x^4z^2+x^2z^4+y^4z^2+y^2z^4)
\end{aligned}$$

The symmetry of this polynomial in the variables x,y,z is due to the symmetry of the hexagon and of the control points.

In these two examples, the toric ideals of the varieties $Y_{\mathcal{A}}$ were generated by quadratic binomials, while the resulting rational varieties were hypersurfaces defined by polynomials of degrees 4 and 6 respectively. These examples show that the ideal of a rational variety may be rather complicated. Nevertheless, this ideal can be computed quite reasonably either from the original toric ideal and the projection or from the resulting parametrization. (See Sections 3.2 and 3.3 of [**CLO97**] for details.)

5. Implicit Degree of a Toric Variety

The (implicit) degree of a hypersurface (e.g. planar curve or a surface in $\mathbb{P}^3$) is the degree of its implicit equation. Similarly, the degree of a rational curve $\mathbf{f}\colon \mathbb{P}^1 \to \mathbb{P}^k$ is the degree of the components of its parametrization $\mathbf{f}$. Projective varieties with greater dimension and codimension also have a degree that is well-behaved under linear projection, and this degree is readily determined for toric varieties.

Definition 5.1. Let $X \subset \mathbb{P}^\ell$ be an algebraic variety of dimension n. The *degree* of X, $\deg(X)$, is the number of points common to X and to a general linear subspace L of dimension $\ell - n$. Such a linear subspace is defined by n linear equations and so the degree of X is also the number of (complex) solutions to n general linear equations on X.

Remark 5.2. This notion of degree agrees with the usual notions for hypersurfaces and for rational curves. Suppose that $X \subset \mathbb{P}^\ell$ is a hypersurface with implicit equation $f = 0$ where f has degree d. Restricting f to a line L and identifying L with $\mathbb{P}^1$ gives a polynomial of degree d on $\mathbb{P}^1$ whose zeroes correspond to the points of $X \cap L$. If L is general then there will be d distinct roots of this polynomial, showing the equality $\deg(X) = \deg(f)$, as $\deg(X)$ is $\#L \cap X$ which equals $d = \deg(f)$.

Similarly, suppose that $X \subset \mathbb{P}^\ell$ is a rational curve of degree d. Then it has a parametrization $\mathbf{f}\colon \mathbb{P}^1 \to X \subset \mathbb{P}^\ell$ where the components of $\mathbf{f}$ are homogeneous polynomials of degree d. A hyperplane L in $\mathbb{P}^\ell$ is defined by a single linear equation $\Lambda(\mathbf{x}) = 0$. Then the points of $X \cap L$ correspond to the zeroes of the polynomial $\Lambda(\mathbf{f})$, which has degree d (when L does not contain the image of $\mathbf{f}$).

If $\pi\colon \mathbb{P}^\ell \dashrightarrow \mathbb{P}^k$ is a surjective linear projection with center $\mathbb{P}(E)$, then the inverse image of a linear subspace $K \subset \mathbb{P}^k$ of dimension $k - n$ is a linear subspace L of $\mathbb{P}^\ell$ of dimension $\ell - n$ that contains the center $\mathbb{P}(E)$. This implies that the degree of a projective variety is reasonably well-behaved under linear projection. We give the precise statement.

Theorem 5.3. *Let $\pi\colon \mathbb{P}^\ell \dashrightarrow \mathbb{P}^k$ be a linear projection with center $\mathbb{P}(E)$ and $Y \subset \mathbb{P}^\ell$. Suppose that $\pi(Y)$ has the same dimension as does Y. Then*

$$\deg(\pi(Y)) \;\leq\; \deg(Y)\,,$$

with equality when the map $\pi\colon Y \to \pi(Y)$ has no basepoints and is one-to-one.

These conditions are satisfied for a general linear projection if $\dim Y < k$.

Thus rational varieties Z that have an injective (one-to-one) parametrization given by a map $\pi\colon Y_{\mathcal{A}} \to Z$ with no basepoints will have the same degree as the projective toric variety $Y_{\mathcal{A}}$. This degree is nicely expressed in terms of the convex hull Δ of the exponent vectors $\mathcal{A}$.

Theorem 5.4. *The implicit degree of a toric variety $Y_{\mathcal{A}}$ is*

$$n!\,\mathit{Vol}(\Delta)\,,$$

where $\mathit{Vol}(\Delta)$ is the usual Euclidean volume of the n-dimensional polytope Δ.

Thus the degree of a rational variety Z parametrized by polynomials whose monomials have exponents from a set $\mathcal{A}$ whose convex hull is Δ is at most $n!\mathrm{Vol}(\Delta)$, with equality when the parametrization $\pi\colon Y_{\mathcal{A}} \to Z$ has no basepoints and is one-to-one (injective). This is what we saw in the examples of Section 4. The polytope of the quartic curve is a line segment of length, and hence volume, 4, while the hexagon whose corresponding monomials parameterize the rational surface of Example 4.3 has area 3, and $2! \cdot 3 = 6$, which is the degree of its implicit equation.

This determination of the degree of the toric variety $Y_{\mathcal{A}}$ is an important result due to Kouchnirenko [**BKK76**] concerning the solutions of *sparse equations*. One of the equivalent definitions of the degree of $Y_{\mathcal{A}}$ is the number of solutions to n $(= \dim(Y_{\mathcal{A}}))$ equations on $Y_{\mathcal{A}} \subset \mathbb{P}^\ell$. Under the parametrization $\varphi_{\mathcal{A}}$ (1.1) of $Y_{\mathcal{A}}$, these linear equations become *Laurent polynomials* on $(\mathbb{C}^*)^n$ whose monomials have exponent vectors in $\mathcal{A}$. Thus the degree of $Y_{\mathcal{A}}$ is equal to the number of solutions in $(\mathbb{C}^*)^n$ to n general Laurent polynomials whose monomials have exponent vectors in $\mathcal{A}$.

This result of Kouchnirenko was generalized by Bernstein [**Ber75**] who determined the number of solutions in $(\mathbb{C}^*)^n$ to n general Laurent polynomials with possibly different sets of exponent vectors. In that, the rôle of the volume is played by the mixed volume. For more, see the contribution of Rojas [**Roj03**] to these proceedings.

6. The Real Part of a Toric Variety

Bézier curves and surface patches in geometric modeling are parametrizations of some of the real part of a rational variety. We discuss the real part of a toric variety and of rational varieties, with respect to their usual real structure. Some toric varieties admit exotic real structures, a topic covered in the article by Delaunay [**Del03**] that also appears in this volume.

Definition 6.1. The (standard) real part of a toric variety is defined by replacing the complex numbers $\mathbb{C}$ by the real numbers $\mathbb{R}$ everywhere in the given definitions.

For example, consider the projective toric variety $Y_{\mathcal{A}}$, defined as a subset of projective space $\mathbb{P}^\ell$ by the toric ideal $I_{\mathcal{A}}$ (equivalently, as the closure of the image of $\varphi_{\mathcal{A}}$ (1.1)). Then the real part $Y_{\mathcal{A}}(\mathbb{R})$ of $Y_{\mathcal{A}}$ is the intersection of $Y_{\mathcal{A}}$ with $\mathbb{RP}^\ell$, that is, the subset of $\mathbb{RP}^\ell$ defined by the toric ideal. Recall that $I_{\mathcal{A}}$ is generated by binomials $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$, which are real polynomials.

Suppose that we have a linear projection $\pi\colon \mathbb{P}^\ell \dashrightarrow \mathbb{P}^k$ defined by real points $\mathbf{p}_i \in \mathbb{R}^{1+k}$. Then the rational variety Z (the image of $Y_{\mathcal{A}}$ under π) has ideal $I(Z)$ generated by real polynomials. The real part $Z(\mathbb{R})$ of Z is the subset of $\mathbb{RP}^k$ defined by the ideal $I(Z)$. This again coincides with the intersection of Z with $\mathbb{RP}^k$. All pictures in this tutorial arise in this fashion. When the map π has no basepoints and $\pi\colon Y_{\mathcal{A}} \to Z$ is one-to-one at almost all points of Z, then $\pi(Y_{\mathcal{A}}(\mathbb{R})) = Z(\mathbb{R})$.

The reason for this is that when $x \in \mathbb{RP}^k$, the points in $\pi^{-1}(x) \cap Y_{\mathcal{A}}$ are the solution to a system of equations with real coefficients. Since the map π is one-to-one on $Y_{\mathcal{A}}$, this system has a single solution that is necessarily real. If π is not one-to-one, then we may have $\pi(Y_{\mathcal{A}}(\mathbb{R})) \subsetneq Z(\mathbb{R})$. For example, when $\mathcal{A}$ is a line segment of length 2, $Y_{\mathcal{A}}$ is the parabola $\{[1, x, x^2] \mid x \in \mathbb{C}\} \subset \mathbb{P}^2$. The projection to $\mathbb{P}^1$ omitting the second coordinate (which is basepoint-free) is the two-to-one map

$$\mathbb{C} \ni x \mapsto [1, x^2] \in \mathbb{P}^1$$

whose restriction to $\mathbb{R}$ has image the nonnegative part of the real toric variety $\mathbb{RP}^1$.

This description does little to aid our intuition about the real part of a toric variety or a rational variety. We obtain a more concrete picture of the real points of a toric variety $Y_{\mathcal{A}}$ and an alternative construction of $Y_{\mathcal{A}}(\mathbb{R})$ if we first describe the real points of an abstract toric variety X_Σ. For this, we recall the definition of the abstract toric variety X_Σ associated to a fan Σ, as described by Cox [**Cox03**].

Let $\sigma \subset \mathbb{R}^n$ be a strongly convex rational polyhedral cone with dual cone $\sigma^\vee \subset \mathbb{R}^n$. Lattice points $\mathbf{m} \in \sigma^\vee \cap \mathbb{Z}^n$ are exponent vectors of Laurent monomials $\mathbf{t}^{\mathbf{m}}$ defined on $(\mathbb{C}^*)^n$. The affine toric variety U_σ corresponding to σ is constructed by first choosing a finite generating set $\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_\ell$ of the additive semigroup $\sigma^\vee \cap \mathbb{Z}^n$. These define the map

$$\begin{array}{rccc}\varphi\,: & (\mathbb{C}^*)^n & \longrightarrow & \mathbb{C}^\ell \\ & \mathbf{t} & \longmapsto & (\mathbf{t}^{\mathbf{m}_1}, \mathbf{t}^{\mathbf{m}_2}, \ldots, \mathbf{t}^{\mathbf{m}_\ell})\ ,\end{array}$$

and we set U_σ to be the closure of the image of this map. The real part $U_\sigma(\mathbb{R})$ of this affine toric variety is simply the intersection of U_σ with $\mathbb{R}^\ell$.

The intersection $\sigma \cap \tau$ of two cones σ, τ in a fan Σ is a face of each cone and $U_{\sigma\cap\tau}$ is naturally a subset of both U_σ and U_τ. The toric variety X_Σ is obtained by gluing together U_σ and U_τ along their common subset $U_{\sigma\cap\tau}$, for all cones σ, τ in Σ. The real part $X_\Sigma(\mathbb{R})$ of X_Σ is similarly obtained by piecing together the real parts $U_\sigma(\mathbb{R})$ and $U_\tau(\mathbb{R})$ along their common subset $U_{\sigma\cap\tau}(\mathbb{R})$, for all σ, τ in Σ.

Since the origin $\mathbf{0} \in \mathbb{R}^n$ lies in Σ and $\mathbf{0}^\vee = \mathbb{R}^n$, the affine toric variety $U_{\mathbf{0}^\vee}$ is the torus $(\mathbb{C}^*)^n$, which is a common subset of each U_σ. This torus is dense in the toric variety X_Σ and it acts on X_Σ. Similarly, the torus $(\mathbb{R}^*)^n$ is dense in $X_\Sigma(\mathbb{R})$ and it acts on $X_\Sigma(\mathbb{R})$. This torus $(\mathbb{R}^*)^n$ has 2^n components called *orthants*, each identified by the sign vector $\varepsilon \in \{\pm 1\}^n$ recording the signs of coordinates of points in that component. The identity component is the orthant containing the identity, and it has sign vector $(1, 1, \ldots, 1)$. Write $\mathbb{R}^n_>$ for this identity component.

Definition 6.2. The *non-negative part* $X_\geq$ of a toric variety X is the closure (in $X(\mathbb{R})$) of the identity component $\mathbb{R}^n_>$ of $(\mathbb{R}^*)^n$. The *boundary* of $X_\geq$ is defined to be the difference $X_\geq - \mathbb{R}^n_>$.

We could also consider the closures of other components of the torus $(\mathbb{R}^*)^n$, obtaining 2^n other pieces analogous to this non-negative part $X_\geq$. Since the component of $(\mathbb{R}^*)^n$ having sign vector ε is simply $\varepsilon \cdot \mathbb{R}^n_>$, these other pieces are translates of $X_\geq$ by the appropriate sign vector, and hence all are isomorphic. Since $X(\mathbb{R})$ is the closure of $(\mathbb{R}^*)^n$ and each piece $\varepsilon \cdot X_\geq$ is the closure of the orthant $\varepsilon \cdot \mathbb{R}^n_>$, we obtain a concrete picture of $X(\mathbb{R})$: it is pieced together from 2^n copies of this non-negative part $X_\geq$ glued together along their common boundaries.

The non-negative part of the toric variety $Y_{\mathcal{A}}$ is simply the intersection of $Y_{\mathcal{A}}$ with the non-negative part of the ambient projective space $\mathbb{P}^\ell$, those points with non-negative homogeneous coordinates

$$\{[x_0, x_1, \ldots, x_\ell] \mid x_i \geq 0\}\,.$$

The boundary of $(Y_{\mathcal{A}})_\geq$ is its intersection with the coordinate hyperplanes, which are defined by the vanishing of at least one homogeneous coordinate.

Example 6.3. The surface of Example 4.3 is the image of the toric variety Y_Δ, where Δ is the hexagon of Example 1.4. Figure 2 shows the image of the non-negative part of Y_Δ. The control points are the spheres (dots) and the boundary consists of the thickened lines. The six control points corresponding to the vertices

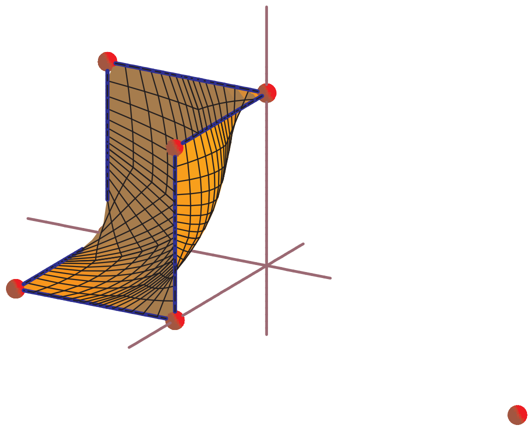


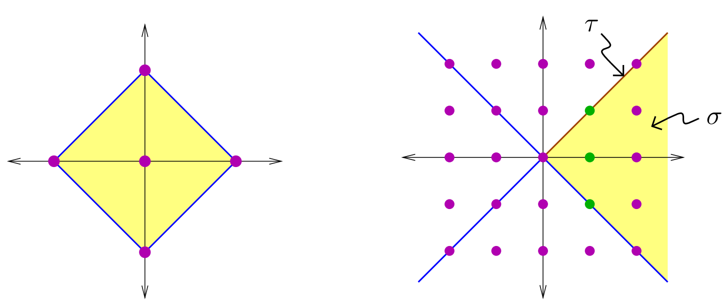
FIGURE 2. Hexagonal toric patch

of the hexagon lie on the non-negative part of Y_Δ. The seventh control point corresponding to the center of the hexagon appears in the lower right. It lies in the octant opposite to the non-negative part of Y_Δ and causes Y_Δ to 'bulge' towards the origin.

7. The Double Pillow

We devote this section to the construction of the toric variety $X_\Sigma(\mathbb{R})$ for a single example, where Σ is the normal fan of the cross polytope $\Diamond \subset \mathbb{R}^2$. As remarked in Section 13 of [**Cox03**], $X_\Sigma \simeq Y_\Diamond$ as $\Diamond$ is 2-dimensional. Krasauskas [**Kra02**] calls the corresponding toric surface the 'pillow with antennae'. We display $\Diamond$ together with its normal fan Σ, with one of its full-dimensional cones shaded.

(7.1)

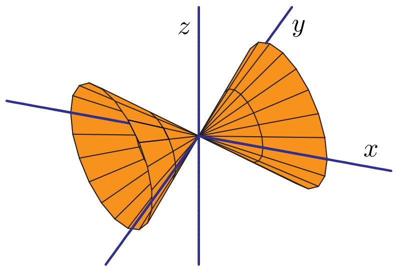


Each full-dimensional cone σ is self-dual and they are all isomorphic. Thus $Y_\Diamond(\mathbb{R})$ $(= X_\Sigma(\mathbb{R}))$ is obtained by gluing together four isomorphic affine toric varieties $U_\sigma(\mathbb{R})$, as σ ranges over the 2-dimensional cones in Σ. A complete picture of the gluing involves the affine varieties $U_\tau(\mathbb{R})$, where τ is one of the rays of Σ. We next describe these two toric varieties $U_\sigma(\mathbb{R})$ and $U_\tau(\mathbb{R})$, for σ a 2-dimensional cone and τ a ray of Σ.

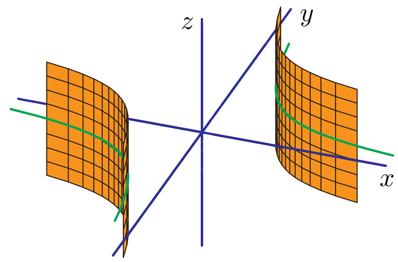
Let σ be the shaded cone in (7.1). Since $\sigma = \sigma^\vee$, we see that $\sigma^\vee \cap \mathbb{Z}^2$ is minimally generated by the vectors $(1,-1), (1,0)$, and $(1,1)$, and so $U_\sigma(\mathbb{R})$ is the closure in $\mathbb{R}^3$ of the image of the map

$$\varphi : (s,t) \longmapsto (st^{-1}, st, s),$$

which is defined by the equation $xy = z^2$ (where (x,y,z) are the coordinates for $\mathbb{R}^3$). This is a right circular cone in $\mathbb{R}^3$, which we display below at left.



$U_\sigma(\mathbb{R})$



$U_\tau(\mathbb{R})$

Let τ be the ray generated by $(1,1)$, which is a face of σ. Then $\tau^\vee$ is the half-space $\{(u,v) \in \mathbb{R}^2 \mid u+v \geq 0\}$, which is the union of both 2-dimensional cones in Σ containing τ. Since $\tau^\vee \cap \mathbb{Z}^2$ has generators $(1,-1), (-1,1)$, and $(1,0)$, we see that $U_\tau(\mathbb{R})$ is the closure in $\mathbb{R}^3$ of the image of the map

$$\varphi : (s,t) \longmapsto (st^{-1}, s^{-1}t, s),$$

which has equation $xy = 1$. This is the cylinder with base the hyperbola $xy = 1$, which is shown above at right.

We describe the gluing. We know that $U_\tau(\mathbb{R}) \subset U_\sigma(\mathbb{R})$ and they both contain the torus $(\mathbb{R}^*)^2$. This common torus is their intersection with the complement of the coordinate planes, $xyz \neq 0$, and their boundaries are their intersections with the coordinate planes. The boundary of the cylinder is the curve $z = 0$ and $xy = 1$, which is defined by $s = 0$ and displayed on the picture of $U_\tau(\mathbb{R})$. Also, $t \neq 0$ on the cylinder. The boundary of the cone is the union of the x and y axes. Since $t^2 = y/x$ on the cone, the locus where $t = 0$ is the x axis. Thus $U_\tau(\mathbb{R})$ is naturally identified with the complement of the x axis in $U_\sigma(\mathbb{R})$ where the curve $z = 0, xy = 1$ in $U_\tau(\mathbb{R})$ is identified with the y-axis in $U_\sigma(\mathbb{R})$.

If τ' is the other ray defining σ, then $U_{\tau'}(\mathbb{R})$ ($\simeq U_\tau(\mathbb{R})$) is identified with the complement of the y axis in $U_\sigma(\mathbb{R})$. A convincing understanding of this gluing procedure is obtained by considering the rational surface Z in $\mathbb{RP}^3$ which is the image of the toric variety $Y_\Diamond(\mathbb{RP}^3)$ under the projection map given by the points $(1,\pm1,0,0)$ and $(1,0,\pm1,0)$ associated to the vertices $(\pm1,0)$ and $(0,\pm1)$ of $\Diamond$, and $(0,0,0,1)$ associated to its center. We display this surface in Figure 3.

This surface has the implicit equation

$$(x^2 - y^2)^2 - 2x^2w^2 - 2y^2w^2 - 16z^2w^2 + w^4 = 0.$$

and its dense torus has parametrization

$$[w,x,y,z] = [s + t + \tfrac{1}{s} + \tfrac{1}{t},\ s - \tfrac{1}{s},\ t - \tfrac{1}{t},\ 1].$$

It has curves of self-intersection along the lines $x = \pm y$ in the plane at infinity ($w = 0$). As the self-intersection is at infinity, this affine surface is a good illustration of the toric variety $Y_\Diamond(\mathbb{R})$, and so we refer to this picture to describe $Y_\Diamond(\mathbb{R})$.

This surface contains 4 lines $x \pm y = \pm 1$ and their complement is the dense torus in $Y_\Diamond(\mathbb{R})$. The complement of any three lines is the piece $U_\tau(\mathbb{R})$ corresponding to a

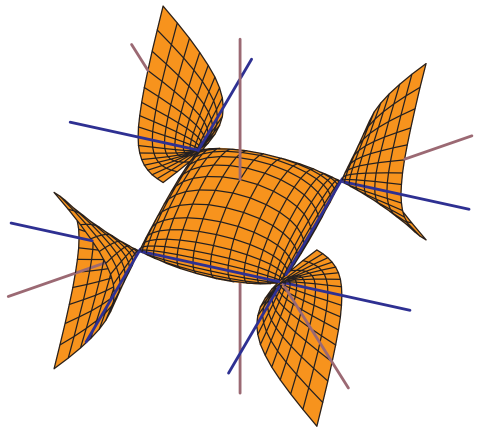


Figure 3. The Double Pillow.

ray τ. Each of the four singular points is a singular point of one cone $U_\sigma(\mathbb{R})$, which is obtained by removing the two lines not meeting that singular point. Finally, the action of the group $\{(\pm 1, \pm 1)\}$ on $Y_\Diamond(\mathbb{R})$ may also be seen from this picture. Each singular point is fixed by this group. The element $(-1,-1)$ sends $z \mapsto -z$, interchanging the top and bottom halves of each piece, while the elements $(1,-1)$ and $(-1,1)$ interchange the central 'pillow' with the rest of $Y_\Diamond(\mathbb{R})$. In this way, we see that $Y_\Diamond(\mathbb{R})$ is a 'double pillow'.

The non-negative part of $Y_\Diamond(\mathbb{R})$ is also readily determined from this picture. The upper part of the middle pillow is the part of $Y_\Diamond(\mathbb{R})$ parametrized by $\mathbb{R}^2_>$, and so its closure is just a square, but with singular corners obtained by cutting a cone into two pieces along a plane of symmetry. In fact, the orthogonal projection to the xy plane identifies this non-negative part with the cross polytope $\Diamond$. From the symmetry of this surface, we see that $Y_\Diamond(\mathbb{R})$ is obtained by gluing four copies of cross polytope $\Diamond$ together along their edges to form two pillows attached at their corners. (The four 'antennae' are actually the truncated corners of the second pillow—projective geometry can play tricks on our affine intuition.)

8. Linear Precision and the Algebraic Moment Map

We observed that the non-negative part of the toric variety $Y_\Diamond$ can be identified with $\Diamond$. The non-negative part of any projective toric variety $Y_\mathcal{A}$ admits an identification with the convex hull Δ of $\mathcal{A}$. One way to realize this identification is through the moment map and algebraic moment map of a toric variety $Y_\mathcal{A} \to \Delta$.

Definition 8.1. Let $Y_\mathcal{A} \subset \mathbb{P}^\ell$ be a projective toric variety given by a collection of exponent vectors $\mathcal{A} \subset \mathbb{R}^n$ with convex hull Δ. The torus $(\mathbb{C}^*)^n$ acts on $\mathbb{P}^\ell$ and on $Y_\mathcal{A}$ via the map $\varphi_\mathcal{A}$. To such an action, symplectic geometry associates a *moment map* $\mu_\mathcal{A} \colon \mathcal{A} \to \mathbb{R}^n$,

$$\begin{array}{rcccl} \mu_\mathcal{A} & : & Y_\mathcal{A} & \longrightarrow & \mathbb{R}^n \\ & & \mathbf{x} & \longmapsto & \dfrac{1}{\sum_{\mathbf{m}\in\mathcal{A}} |x_\mathbf{m}(\mathbf{x})|^2} \displaystyle\sum_{\mathbf{m}\in\mathcal{A}} |x_\mathbf{m}(\mathbf{x})|^2 \mathbf{m}\,. \end{array} \tag{8.2}$$

While the restriction of coordinate function $x_{\mathbf{m}}$ on $\mathbb{P}^\ell$ to $Y_{\mathcal{A}}$ is not a well-defined function, the collection of these coordinate functions is well-defined up to a common scalar factor. It is a basic theorem of symplectic geometry that the image of the moment map is the polytope Δ and the restriction of $\mu_{\mathcal{A}}$ to the non-negative part of $(Y_{\mathcal{A}})_{\geq}$ is a homeomorphism.

More useful to us is the following variant of $\mu_{\mathcal{A}}$, where we do not square the coordinate functions,

$$\begin{array}{rcccl} \alpha_{\mathcal{A}} & : & Y_{\mathcal{A}} & \longrightarrow & \mathbb{R}^n \\ & & \mathbf{x} & \longmapsto & \dfrac{1}{\sum_{\mathbf{m}\in\mathcal{A}} |x_{\mathbf{m}}(\mathbf{x})|} \displaystyle\sum_{\mathbf{m}\in\mathcal{A}} |x_{\mathbf{m}}(\mathbf{x})|\mathbf{m}\ . \end{array} \tag{8.3}$$

This map is very similar to the moment map, and thus is often confused with the moment map†

Remark 8.4. Suppose that $\mathcal{A} = \{\mathbf{m}_0, \mathbf{m}_1, \ldots, \mathbf{m}_\ell\} \subset \mathbb{R}^n$. We claim that on $(Y_{\mathcal{A}})_{\geq}$, the map (8.3) coincides with the linear projection $\pi_{\mathcal{A}} \colon \mathbb{P}^\ell - \to \mathbb{P}^n$ defined by the points in the lift $\mathcal{A}^+$ of $\mathcal{A}$:

$$(1, \mathbf{m}_0), (1, \mathbf{m}_1), \ldots, (1, \mathbf{m}_\ell) \ \in\ \mathbb{R}^{1+n}\ .$$

Indeed, we have

$$\pi_{\mathcal{A}}(\mathbf{x}) \ =\ \pi_{\mathcal{A}}([x_0, x_1, \ldots, x_\ell]) \ =\ \sum_{i=0}^{\ell} x_i[1, \mathbf{m}_i] \ =\ \left[\textstyle\sum_i x_i,\ \sum_i x_i \mathbf{m}_i\right]\ .$$

If $\mathbf{x}$ lies in the non-negative part of the projective toric variety $Y_{\mathcal{A}}$, then each coordinate x_i of $\mathbf{x}$ is non-negative with $x_i = x_{\mathbf{m}_i}$. Since $\sum_i x_i > 0$, this shows that $\pi_{\mathcal{A}}(\mathbf{x}) = [1, \alpha_{\mathcal{A}}(\mathbf{x})]$, and thus the map (8.3) coincides with the projection $\pi_{\mathcal{A}}$ on the non-negative part of $Y_{\mathcal{A}}$.

It is for these reasons that we call the linear projection $\pi_{\mathcal{A}}$ the *algebraic moment map.*

This algebraic moment map shares an important property of the moment map.

Theorem 8.5. *The non-negative part $(Y_{\mathcal{A}})_{\geq}$ of the toric variety $Y_{\mathcal{A}}$ is homeomorphic to the convex hull Δ of $\mathcal{A}$ under the algebraic moment map.*

The nature of this homeomorphism is subtle. If the polytope Δ is smooth (that is, the shortest integer vectors normal to the faces that meet at a vertex always form a basis for $\mathbb{Z}^n$), then every point of Δ has a neighborhood in Δ homeomorphic to $\mathbb{R}^k \times \mathbb{R}^{n-k}_{\geq}$, and so we call Δ a *manifold with corners*. In general, a polytope Δ is a manifold with 'singular corners'. It is this structure that is preserved by the homeomorphism of Theorem 8.5. (For more on the algebraic moment map and the structure of $X_{\geq}$ as a manifold with singular corners, see Section 4 of Fulton's book on toric varieties [**Ful93**], where he call $\alpha_{\mathcal{A}}$ the moment map.)

Theorem 8.5 explains why toric patches are of interest in geometric modeling. Since the non-negative part of a projective toric surface is homeomorphic to a polygon, any rational surface parametrized by that toric surface has a non-negative part that is the image of that polygon. In this way, we can obtain multi-sided surface patches from toric surfaces. This theorem not only explains the geometry

†This confusion occurred in the published version of this manuscript.

of such toric patches, but we use it to gain insight into parametrizations of toric patches by the corresponding polytopes.

Let Δ be the convex hull of a set of exponent vectors $\mathcal{A}$. By Theorem 8.5, $(Y_{\mathcal{A}})_{\geq}$ is homeomorphic to Δ, and so there exists a parametrization of $(Y_{\mathcal{A}})_{\geq}$ by Δ preserving their structures as manifolds with 'singular corners'. From the point of view of algebraic geometry, the most natural such parametrization is the inverse of the algebraic moment map $\alpha_{\mathcal{A}}^{-1}\colon \Delta \to (Y_{\mathcal{A}})_{\geq}$. This is also the most natural from the point of view of geometric modeling.

THEOREM 8.6. *The coordinate functions of $\alpha_{\mathcal{A}}^{-1}$ have linear precision.*

A collection of liinearly independent functions $\{f_{\mathbf{m}} \mid \mathbf{m} \in \mathcal{A}\}$ defined on the convex hull $\Delta \subset \mathbb{R}^n$ of $\mathcal{A}$ has *linear precision* if, for any affine function Λ defined on $\mathbb{R}^n$,

$$\Lambda(\mathbf{u}) \;=\; \sum_{\mathbf{m}\in\mathcal{A}} \Lambda(\mathbf{m}) f_{\mathbf{m}}(\mathbf{u}) \qquad \text{for all } \mathbf{u} \in \Delta\,. \tag{8.7}$$

Theorem 8.6 follows immediately from this definition. The functions $\{f_{\mathbf{m}} \mid \mathbf{m} \in \mathcal{A}\}$ define a map $\mathbf{f}$ from Δ to $\mathbb{P}^\ell$ in the natural coordinates of $\mathbb{P}^\ell$ indexed by the exponent vectors in $\mathcal{A}$. Then the right hand side of (8.7) is the result of applying the linear function Λ to the composition

$$\Delta \xrightarrow{\ \mathbf{f}\ } (Y_{\mathcal{A}})_{\geq} \xrightarrow{\ \pi_{\mathcal{A}}\ } \Delta\,,$$

where $\pi_{\mathcal{A}}$ is the linear projection of Remark 8.4, which restricts to give the moment map on $(Y_{\mathcal{A}})_{\geq}$. Then linear precision of the components of $\mathbf{f}$ is simply the statement that $\mathbf{f}$ is the inverse of the algebraic moment map.

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