2005

Normality of very even nilpotent varieties in D-2l

E Sommers
University of Massachusetts - Amherst, esommers@math.umass.edu

NORMALITY OF VERY EVEN NILPOTENT VARIETIES IN D_{2l}

ERIC SOMMERS

ABSTRACT. For the classical groups, Kraft and Procesi [4], [5] have resolved the question of which nilpotent orbits have closures which are normal and which are not, with the exception of the very even orbits in D_{2l} which have partition of the form (a^{2k}, b^2) for a, b distinct even natural numbers with $ak + b = 2l$.

In this article, we show that these orbits do have normal closure. We use the technique of [8].

1. SOME LEMMAS IN A_l

We retain the notation of [8]. Throughout, G is a connected simple algebraic group over $\mathbf{C}$, B a Borel subgroup, T a maximal torus in B. The simple roots are denoted by Π, and they correspond to the Borel subgroup opposite to B. Let $\{\omega_i\}$ be the fundamental weights of G corresponding to Π. If $\alpha \in \Pi$, then P_α denotes the parabolic subgroup of semisimple rank one containing B and corresponding to α. If P is a parabolic subgroup of G, we denote by $\mathfrak{u}_P$ the Lie algebra of its unipotent radical.

We recall

Proposition 1.1. [3] *Let V be a rational representation of B and assume that V extends to a representation of the parabolic subgroup P_α where α is a simple root. Let $\lambda \in X^*(T)$ be such that $m = \langle \lambda, \alpha^\vee \rangle \geq -1$. Then there is a G-module isomorphism*

$$H^i(G/B, V \otimes \lambda) = H^{i+1}(G/B, V \otimes \lambda - (m+1)\alpha) \ \textit{for all} \ i \in \mathbb{Z}.$$

In particular, if $m = -1$, then all cohomology groups vanish.

For the rest of this section and the next, let $G = SL_{l+1}(\mathbf{C})$. We index the simple roots $\Pi = \{\alpha_j\}$ so that α_1 is an extremal root and α_j is next to α_{j+1} in the Dynkin diagram of type A_l.

The following lemma follows easily from several applications of the previous proposition.

Lemma 1.2. [7] *Let V be a rational representation of B which extends to a representation of P_{α_j} for $a \leq j \leq b$. Let $\lambda \in X^*(T)$ be such that $\langle \lambda, \alpha_j^\vee \rangle = 0$ for $a < j \leq b$. Set $r = \langle \lambda, \alpha_a^\vee \rangle$ and assume that $a - b - 1 \leq r \leq -1$. Then $H^*(V \otimes \lambda) = 0$.*

A similar statement holds by applying the non-trivial automorphism to the Dynkin diagram of type A_l. We use this lemma to prove

Date: 7/23/03; 9/17/03.

The author was supported in part by NSF grants DMS-0201826 and DMS-9729992.

The author thanks Viktor Ostrik for directing him to this problem.

Lemma 1.3. *Let V be a representation of B which is stable under the parabolic subgroups P_{α_j} for $1 \leq j \leq b$. Let $\lambda \in X^*(T)$ be such that $\langle \lambda, \alpha_a^\vee \rangle = 1$ for some a satisfying $1 \leq a < b$. Assume that $\langle \lambda, \alpha_j^\vee \rangle = 0$ for $1 \leq j \neq a < b$. Set $k = \langle \lambda, \alpha_b^\vee \rangle$. If $-b-1 \leq k \leq -1$ and $k+b-a \neq -1$, then $H^*(V \otimes \lambda) = 0$.*

Proof. If $k+b-a \geq 0$, the result follows directly from Lemma 1.2. On the other hand, if $k+b-a \leq -2$, then as in the proof of Lemma 1.2 in [7],

$$H^i(V \otimes \lambda) = H^{i+b-a}(V \otimes \mu)$$

where

$$\mu = \lambda + (-k-1)\alpha_b + (-k-2)\alpha_{b-1} + \cdots + (-k-b+a)\alpha_{a+1}.$$

Now $\langle \mu, \alpha_j^\vee \rangle = 0$ for $1 \leq j < a$ and $\langle \mu, \alpha_a^\vee \rangle = k+b-a+1$. By the hypothesis on λ and the present assumption about $k+b-a$, we have

$$-a \leq k+b-a+1 \leq -1.$$

Then Lemma 1.2 yields the desired vanishing. □

2. A THEOREM FOR A_l (REVIEW)

Let P_m denote the maximal proper parabolic subgroup of $G = SL_{l+1}(\mathbf{C})$ containing B corresponding to all the simple roots except α_m. Denote the Lie algebra of the unipotent radical of P_m by $\mathfrak{u}_m$. The action of P_m on $\mathfrak{u}_m$ gives a representation of P_m (and also B). Denote the dual representation by $\mathfrak{u}_m^*$. Set $m' = \min\{m,\ l+1-m\}$. In [7], Lemma 1.2 and Proposition 1.1 were used to prove

Theorem 2.1. [7] *Let r be an integer in the range $2m'-2-l \leq r \leq 0$. Then there is a G-module isomorphism*

$$H^i(G/B, S^n\mathfrak{u}_m^* \otimes r\omega_m) = H^i(G/B, S^{n+rm'}\mathfrak{u}_{l+1-m}^* \otimes -r\omega_{l+1-m}) \text{ for all } i, n \geq 0.$$

3. A THEOREM FOR D_{2l+1}

Theorem 2.1 has an analog in type D_{2l+1}. We label the simple roots of G of type D_{2l+1} as in [6], so α_{2l-1} lies at the branched vertex of the Dynkin diagram. Let P be the maximal proper parabolic subgroup containing B corresponding to all the simple roots except α_{2l}. And let P' be the maximal proper parabolic subgroup containing B corresponding to all the simple roots except α_{2l+1} (so P and P' are interchanged by an outer automorphism of G).

Theorem 3.1. *Let r be an integer in the range $-3 \leq r \leq 0$. Then there is a G-module isomorphism*

$$H^i(G/B, S^n\mathfrak{u}_P^* \otimes r\omega_{2l}) = H^i(G/B, S^{n+rl}\mathfrak{u}_{P'}^* \otimes -r\omega_{2l+1}) \text{ for all } i, n \geq 0.$$

Proof. **Step 1.**

In this step, r may be an arbitrary integer. Consider the intersection $V = \mathfrak{u}_P \cap \mathfrak{u}_{P'}$. We will show in Step 1 that for all i, n

$$H^i(S^n\mathfrak{u}_P^* \otimes r\omega_{2l}) = H^i(S^nV^* \otimes r\omega_{2l}). \tag{1}$$

We begin by taking the Koszul resolution of the short exact sequence

$$0 \to U \to \mathfrak{u}_P^* \to V^* \to 0$$

(this defines U) and tensoring it with $r\omega_{2l}$. This gives

$$0 \to \cdots \to S^{n-j}\mathfrak{u}_P^* \otimes \wedge^j U \otimes r\omega_{2l} \to \cdots \to S^n \mathfrak{u}_P^* \otimes r\omega_{2l} \to S^n V^* \otimes r\omega_{2l} \to 0.$$

We claim that $H^*(S^{n-j}\mathfrak{u}_P^* \otimes \wedge^j U \otimes r\omega_{2l}) = 0$ for $1 \leq j \leq \dim U$ from which Equation 1 will follow. The T-weights of U are those of the form $\alpha_k + \alpha_{k+1} + \cdots + \alpha_{2l}$, where $1 \leq k \leq 2l$. Therefore, if λ is a T-weight of $\wedge^j U$, then λ is of the form

$$(0, \ldots, 0, 1, \ldots, 1, 2, \ldots, 2, \ldots, j-1, \ldots, j-1, j, \ldots, j, 0)$$

in the basis of simple roots. If this expression contains a subsequence of the form $m, m, m+1$, then λ will have inner product -1 with the simple coroot corresponding to the middle m. Hence $H^*(Q \otimes \lambda) = 0$ where Q is any P-representation by Proposition 1.1. The same result holds if there are any 0's in the initial part of the expression. Therefore, we are reduced to considering those λ of the form

$$(1, 2, 3, \ldots, j-1, j, j, \ldots, j, 0).$$

Such a λ satisfies $\langle \lambda, \alpha_{2l+1}^\vee \rangle = -j$ with the exception of the case $j = 2l$, where instead $\langle \lambda, \alpha_{2l+1}^\vee \rangle = -j + 1 = -2l + 1$. In the latter case $H^*(Q \otimes \lambda) = 0$ by Lemma 1.2 applied to to the parabolic subgroup with Levi factor of type A_{2l} consisting of all simple roots except α_{2l}. For the cases where $j < 2l$, we can apply Lemma 1.3, also for the A_{2l} consisting of all simple roots except α_{2l}. In that case, $a = j$, $b = 2l$, $k = -j$ and so $k + b - a = 2l - 2j$, which, being an even number, is never -1. Also, clearly $-b - 1 \leq k \leq -1$. Thus we conclude that for all weights λ appearing in $\wedge^j U$, we have $H^*(Q \otimes \lambda) = 0$ for any P-representation Q. Hence for $Q := S^{n-j}\mathfrak{u}_P^* \otimes r\omega_{2l}$, it follows that $H^*(Q \otimes \wedge^j U) = 0$ by the usual filtration argument.

Step 2.

Let V_1 be the B-stable subspace of $\mathfrak{u}$ consisting of the direct sum of all root spaces $\mathfrak{g}_\alpha$ where $-\alpha$ is bigger than or equal to the root

$$(0, \ldots, 0, 1, 2, 1, 1)$$

in the usual partial ordering on roots. Let V_2 be the B-stable subspace of $\mathfrak{u}$ consisting of the direct sum of all root spaces $\mathfrak{g}_\alpha$ where $-\alpha$ is bigger than or equal to the root

$$(0, 0, \ldots, 0, 1, 2, 2, 1, 1).$$

Let μ be a weight of the form $r\,\omega_{2l} + s\,\omega_{2l+1}$ where r, s are integers. Assume that $-3 \leq r \leq -1$ and that $s = 0$ if $r = -3$. In this step we show for all $n \geq 0$ that

$$H^*(S^n V_1^* \otimes \mu) = 0. \tag{2}$$

Take the Koszul resolution of

$$0 \to U_2 \to V_1^* \to V_2^* \to 0$$

(this defines U_2) and tensor it with μ. We will show that

$$H^*(S^n V_2^* \otimes \mu) = 0$$

and

$$H^*(S^{n-j}V_1^* \otimes \wedge^j U_2 \otimes \mu) = 0$$

for $1 \leq j \leq 2l-2$ and then Equation (2) will follow (the dimension of U_2 is $2l-2$ as shown below).

The subspace V_2^* is stable under the minimal parabolic subgroups P_{α_m} for $m = 2l-1, 2l$, and $2l+1$. It follows from the assumption on μ that $H^*(S^n V_2^* \otimes \mu) = 0$ by Lemma 1.2 applied to the A_3 determined by the simple roots α_m for $m = 2l-1, 2l$, and $2l+1$.

Now the T-weights of U_2 are

$$\alpha_k + \alpha_{k+1} + \cdots + \alpha_{2l-2} + 2\alpha_{2l-1} + \alpha_{2l} + \alpha_{2l+1}$$

where $1 \leq k \leq 2l-2$. If λ is a weight of $\wedge^j U_2$, then λ is of the form

$$(0, \ldots, 0, 1, \ldots, 1, 2, \ldots, j-1, j, \ldots, j, 2j, j, j)$$

in the basis of simple roots. As in the previous step, if there are any 0's present or if any of the integers between 1 and $j-1$ inclusive are repeated, then

$$H^*(Q \otimes \lambda) = 0$$

where $Q := S^{n-j}V_1^* \otimes \mu$ since Q is stable under the action of the parabolic subgroups P_{α_k} for $1 \leq k \leq 2l-2$. Hence we are reduced to considering those λ of the form

$$(1, 2, 3, \ldots, j-2, j-1, j, \ldots, j, 2j, j, j)$$

for $1 \leq j \leq 2l-2$. Such a λ satisfies $\langle \lambda, \alpha_{2l-2}^\vee \rangle = -j$ with the exception of $j = 2l-2$ where $\langle \lambda, \alpha_{2l-2}^\vee \rangle = -2l+3$. In the latter case $H^*(Q \otimes \lambda) = 0$ by Lemma 1.2 applied to the A_{2l-2} consisting of the first $2l-2$ simple roots. For the cases where $j < 2l-2$, we can apply Lemma 1.3, also for the A_{2l-2} consisting of the first $2l-2$ simple roots. In that case, $a = j$, $b = 2l-2$, $k = -j$ and so $k + b - a = 2l-2j-2$, which is never -1. Also, clearly $-b-1 \leq k \leq -1$. We therefore also have $H^*(Q \otimes \lambda) = 0$.

Consequently, if we filter $\wedge^j U_2$ by B-submodules such that the quotients are one-dimensional, we deduce that

$$H^*(S^{n-j}V_1^* \otimes \wedge^j U_2 \otimes \mu) = 0$$

for $1 \leq j \leq 2l-2$. Hence Equation (2) follows.

Step 3.

In this step, we show that for all i, n

(3) $$H^i(S^n V^* \otimes \mu) = H^i(S^{n-l}V^* \otimes \mu + \omega_{2l} + \omega_{2l+1})$$

for μ as in Step 2.

We take the Koszul resolution of the short exact sequence

$$0 \to U_1 \to V^* \to V_1^* \to 0$$

(this defines U_1) and tensor it with μ arriving at

(4) $$0 \to S^{n-2l+1}V^* \otimes \wedge^{2l-1}U_1 \otimes \mu \to \cdots \to S^{n-j}V^* \otimes \wedge^j U_1 \otimes \mu \to \cdots \to S^n V^* \otimes \mu \to S^n V_1^* \otimes \mu \to 0$$

We first show that $H^*(S^{n-j}V^* \otimes \mu \otimes \lambda) = 0$ for any λ appearing in $\wedge^j U_1$ for $j \neq 0, l$. The weights of U_1 are

$$\alpha_k + \alpha_{k+1} + \cdots + \alpha_{2l-1} + \alpha_{2l} + \alpha_{2l+1}$$

where $1 \leq k \leq 2l-1$. If λ is a weight of $\wedge^j U_1$, then as in the previous steps we are quickly reduced to those λ of the form

$$(1,2,3,\dots,j-2,j-1,j,\dots,j,j,j,j)$$

for $1 \leq j \leq 2l-1$. Such a λ satisfies $\langle \lambda, \alpha_{2l-1}^\vee \rangle = -j$ with the exception of $j = 2l-1$ where $\langle \lambda, \alpha_{2l-2}^\vee \rangle = -2l+2$. The latter vanishing follows from Lemma 1.2 applied to to the A_{2l-1} consisting of the first $2l-1$ simple roots. For the cases where $j < 2l-1$, we can apply Lemma 1.3, also for the A_{2l-1} consisting of the first $2l-1$ simple roots. In that case, $a = j$, $b = 2l-1$, $k = -j$ and so $k+b-a = 2l-2j-1$, which is -1 only when $j = l$. Therefore, we deduce that

$$H^*(S^{n-j}V^* \otimes \wedge^j U_1 \otimes \mu) = 0$$

when $j \neq 0, l$. And furthermore,

$$H^i(S^{n-l}V^* \otimes \wedge^l U_1 \otimes \mu) = H^i(S^{n-l}V^* \otimes \lambda \otimes \mu),$$

where $\lambda = (1,2,3,\dots,l-1,l,\dots,l,l,l)$. Now $S^{n-l}V^* \otimes \mu$ is stable under P_{α_m} for $1 \leq m \leq 2l-1$. Hence $l-1$ applications of Proposition 1.1 yields

$$H^i(S^{n-l}V^* \otimes \lambda \otimes \mu) = H^{i+l-1}(S^{n-l}V^* \otimes \mu + \omega_{2l} + \omega_{2l+1}).$$

By breaking Equation (4) into short exact sequences and taking cohomology on G/B, we conclude that

$$H^i(S^n V^* \otimes \mu) = H^i(S^{n-l}V^* \otimes \mu + \omega_{2l} + \omega_{2l+1}),$$

where we are using

$$H^*(S^n V_1^* \otimes \mu) = 0$$

from Step 2.

Step 4. We obtain the theorem by using Step 3 repeatedly, starting with $\mu = r\omega_m$ with r in the prescribed range of the statement of the theorem. After $-r$ steps we arrive at

$$H^i(S^n V^* \otimes r\omega_{2l}) = H^i(S^{n+rl}V^* \otimes -r\omega_{2l+1}),$$

for all i, n. The proof is completed by using Step 1 and the symmetric version of Equation 1 (obtained by applying an outer automorphism of G) which gives

$$H^i(S^{n+rl}V^* \otimes -r\omega_{2l+1}) = H^i(S^{n+rl}\mathfrak{u}_{P'}^* \otimes -r\omega_{2l+1})$$

for all i, n.

□

In what follows, we will use Theorem 2.1 in the more general situation of Section 4 in [8]. Similarly we can apply Theorem 3.1 in an analogous general situation. Namely, suppose G is of general type and P is a parabolic subgroup of G containing B with Levi factor L containing a simple factor of type A_{2l}. Furthermore, suppose this simple factor belongs to a Levi subgroup L' of G of type D_{2l+1} and $[L, L'] \subset L'$. Then the analog in G of Theorem 3.1 holds just as the analog of Theorem 2.1 does in Proposition 6 in [8].

4. MAIN THEOREM

For the rest of the paper G is connected of type D_{2l}. We want to show that both nilpotent orbits in $\mathfrak{g}$ with partition (a^{2k}, b^2) for a, b distinct even natural numbers with $ak + b = 2l$ (see [2]) have normal closure. Let $\mathcal{O}$ denote one of these two orbits.

Following the idea of [8], we find a nilpotent orbit $\mathcal{O}'$ which we already know has normal closure and which contains $\mathcal{O}$ in its closure. If we can show that the regular functions on $\mathcal{O}$ are naturally a quotient of the regular functions on $\mathcal{O}'$, then it follows that $\mathcal{O}$ also has normal closure. To that end we consider the nilpotent orbit $\mathcal{O}'$ in $\mathfrak{g}$ with partition $\lambda = (a^{2k}, b+1, b-1)$.

Lemma 4.1. *The closure of $\mathcal{O}'$ is normal.*

Proof. The only minimal degenerations of $\mathcal{O}'$ in $\mathfrak{g}$ are the two orbits with partition $\mu = (a^{2k}, b^2)$ (which together are one orbit for the full orthogonal group of rank $2l$). Hence by [5] the singularity of the closure of $\mathcal{O}'$ along the union of these two orbits is smoothly equivalent to the singularity of the closure of the orbit with partition (2) along the orbit with partition $(1, 1)$ in type A_1 (we remove the first $2k$ rows from λ and μ, and then remove the first $b - 1$ columns from the resulting partitions). Hence this is a singularity of type A_1 and so by [5], $\mathcal{O}'$ has normal closure. ☐

Lemma 4.2. *The orbit $\mathcal{O}'$ is a Richardson orbit for any parabolic with Levi factor of type*

$$\overbrace{A_{2k-1} \times \cdots \times A_{2k-1}}^{\frac{a-b}{2}-1} \times A_{2k} \times A_{2k} \times \overbrace{A_{2k+1} \times \cdots \times A_{2k+1}}^{\frac{b}{2}-1}.$$

Any parabolic with Levi factor of type

$$\overbrace{A_{2k-1} \times \cdots \times A_{2k-1}}^{\frac{a-b}{2}} \times \overbrace{A_{2k+1} \times \cdots \times A_{2k+1}}^{\frac{b}{2}}$$

has Richardson orbit one or the other of the two nilpotent orbits with partition (a^{2k}, b^2).

Proof. Both statements follow from Section 7 in [2]. ☐

It will be convenient to represent parabolic subgroups containing B by the simple roots of G which are **not** simple roots of their Levi factors. Thus we can speak of such a parabolic subgroup as a subset of the numbers 1 to $2l + 1$, with each number i corresponding to the simple root α_i.

Set $d = a - b$ and let P' be the parabolic represented by

$$\{2k + 1, 4k + 2, 6k + 2, \ldots, kd + 2, k(d + 2) + 2, k(d + 4) + 4, k(d + 6) + 6, \ldots, 2l - 2k - 2, 2l\}$$

and let P'' be represented by

$$\{2k + 1, 4k + 2, 6k + 2, \ldots, kd + 2, k(d + 2) + 2, k(d + 4) + 4, k(d + 6) + 6, \ldots, 2l - 2k - 2, 2l - 1\},$$

so P' are P'' are interchanged by an outer automorphism of D_{2l+1}. By the previous lemma $\mathcal{O}'$ is Richardson for both P' and P''. Let P be the parabolic represented by

$$\{2k, 4k + 2, 6k + 2, \ldots, kd + 2, k(d + 2) + 2, k(d + 4) + 4, k(d + 6) + 6, \ldots, 2l - 2k - 2, 2l\}.$$

Then without loss of generality we can take $\mathcal{O}$ to be the Richardson orbit for P (again by the previous lemma).

Theorem 4.3. *There is a short exact sequence*

$$0 \to H^0(S^{n-2l-k(a-4)-1}\mathfrak{u}^*_{P''} \otimes \nu) \to H^0(S^n\mathfrak{u}^*_{P'}) \to H^0(S^n\mathfrak{u}^*_P) \to 0, \tag{5}$$

where $\nu = \omega_{4k+2}$ *if* $a > 4$ *and* $\nu = 2\omega_{2l-1}$ *if* $a = 4$ *(and hence* $b = 2$*).*

Proof. We use two elements from the proof of Theorem 2.1 in [7]. Let P_1 be the parabolic represented by

$$\{2k+2, 4k+2, 6k+2, \ldots, kd+2, k(d+2)+2, k(d+4)+4, k(d+6)+6, \ldots, 2l-2k-2, 2l\}$$

and set $V = \mathfrak{u}_P \cap \mathfrak{u}_{P_1}$. Then Step 1 of the proof of Theorem 2.1 (for a group of type A_{4k+1} applied to the first $4k+1$ simple roots of G) yields the isomorphism $H^i(S^n\mathfrak{u}^*_P) = H^i(S^nV^*)$ for all i, n. And Step 3 of the proof Theorem 2.1 yields the long exact sequence

$$\cdots \to H^i(S^{n-2k-1}\mathfrak{u}^*_{P'} \otimes \mu) \to H^i(S^n\mathfrak{u}^*_{P'}) \to H^i(S^nV^*) \to H^{i+1}(S^{n-2k-1}\mathfrak{u}^*_{P'} \otimes \mu) \to \ldots$$

where μ equals

$$(1, 2, 3, \ldots, 2k, 2k+1, 2k, \ldots, 2, 1, \overbrace{0, 0, \ldots, 0}^{2l-4k-1}).$$

This is obtained by taking the Koszul resolution of

$$0 \to U \to \mathfrak{u}^*_{P'} \to V^* \to 0$$

(this defines U) and simplifying the terms.

The remainder of the proof involves showing that

$$H^i(S^{n-2k-1}\mathfrak{u}^*_{P'} \otimes \mu) = H^i(S^{n-2l-k(a-4)-1}\mathfrak{u}^*_{P''} \otimes \nu)$$

for all i, n.

This is carried out by using Theorem 2.1 numerous times (for $r = -1$ and the l in that theorem equal to either $4k$ or $4k+1$ and $m' = 2k$ or $2k+1$, respectively) and Theorem 3.1 once (for $r = -2$ and the l in that theorem equal to k).

After $\frac{a-b-2}{2}$ applications of Theorem 2.1 with $r = -1$, l there equal to $4k$, and $m' = 2k$, we have

$$H^i(S^{n-2k-1}\mathfrak{u}^*_{P'} \otimes \mu) = H^i(S^{n-k(a-b)-1}Q_1^* \otimes \mu_1)$$

where μ_1 equals

$$(1, 2, 3, \ldots, 2k, \overbrace{2k+1, \ldots, 2k+1}^{k(a-b-2)+1}, 2k, \ldots, 2, 1, \overbrace{0, 0, \ldots, 0}^{k(b-2)+b-1}),$$

and Q_1 is the Lie algebra of the unipotent radical of

$$\{2k+1, 4k+1, 6k+1, \ldots, k(d-2)+1, kd+1, k(d+2)+2, k(d+4)+4, k(d+6)+6, \ldots, 2l-2k-2, 2l\}.$$

Next, we apply Theorem 2.1 $\frac{b-2}{2}$ more times with $r = -1$, l there equal to $4k+1$, and $m' = 2k+1$, to obtain

$$H^i(S^{n-k(a-b)-1}Q_1^* \otimes \mu_1) = H^i(S^{n-ka+2k-b/2}Q_2^* \otimes \mu_2)$$

where μ_2 equals

$$(1, 2, 3, \ldots, 2k, \overbrace{2k+1, \ldots, 2k+1}^{2l-4k-1}, 2k, \ldots, 2, 1, 0),$$

and Q_2 is the Lie algebra of the unipotent radical of

$\{2k{+}1, 4k{+}1, 6k{+}1, \ldots, k(d{-}2){+}1, kd{+}1, k(d{+}2){+}3, k(d{+}4){+}5, k(d{+}6){+}7, \ldots, 2l{-}2k{-}1, 2l\}$.

Next, we use Theorem 3.1 with $r = -2$ for the case D_{2k+1} applied to the simple roots α_i of G with $2l - 2k \leq i \leq 2l$. This yields

$$H^i(S^{n-ka+2k-b/2}Q_2^* \otimes \mu_2) = H^i(S^{n-ka-b/2}Q_3^* \otimes \mu_3)$$

where μ_3 equals

$$(1, 2, 3, \ldots, 2k, \overbrace{2k+1, \ldots, 2k+1}^{2l-4k-1}, 2k+2, 2k+3, 2k+4, \ldots, 4k, 2k+1, 2k),$$

and Q_3 is the Lie algebra of the unipotent radical of

$\{2k{+}1, 4k{+}1, 6k{+}1, \ldots, k(d{-}2){+}1, kd{+}1, k(d{+}2){+}3, k(d{+}4){+}5, k(d{+}6){+}7, \ldots, 2l{-}2k{-}1, 2l{-}1\}$.

If $2l - 4k - 1 = 1$, which is the case if and only if $a = 4$ and $b = 2$, we have $\mu_3 = 2\omega_{2l-1}$ and the latter parabolic subgroup is P''.

On the other hand, if $a > 4$, we continue by using Theorem 2.1 another $\frac{b-2}{2}$ times followed by another $\frac{a-b-2}{2}$ times (in reverse of how we have just used it). The result is that

$$H^i(S^{n-ka-b/2-2}Q_3^* \otimes \mu_3) = H^i(S^{n-2ka+4k-b-1}Q_4^* \otimes \mu_4)$$

where μ_4 equals

$$(1, 2, 3, \ldots, 4k+1, \overbrace{4k+2, \ldots, 4k+2}^{2l-4k-3}, 2k+1, 2k+1),$$

and Q_4 is the Lie algebra of the unipotent radical of

$\{2k{+}1, 4k{+}2, 6k{+}2, \ldots, k(d{-}2){+}2, kd{+}2, k(d{+}2){+}2, k(d{+}4){+}4, k(d{+}6){+}6, \ldots, 2l{-}2k{-}2, 2l{-}1\}$.

The latter parabolic is exactly P'' and $\mu_4 = \omega_{4k+2}$. Furthermore, $n - 2ka + 4k - b - 1 = n - 2l - ka + 4k - 1$ since $ak + b = 2l$.

Hence when $a = 4$ or $a > 4$, we have shown that

$$H^i(S^{n-2k-1}\mathfrak{u}_{P'}^* \otimes \mu) = H^i(S^{n-2l-k(a-4)-1}\mathfrak{u}_{P''}^* \otimes \nu)$$

for all i, n. We finish the proof by observing that ν extends to a character of P'' and it is dominant. Hence $H^i(S^{n-2l-k(a-4)-1}\mathfrak{u}_{P''}^* \otimes \nu) = 0$ for $i > 0$ as in [1]. Similarly, $H^i(S^n\mathfrak{u}_{P'}^*) = 0$ and $H^i(S^n\mathfrak{u}_P^*) = 0$ for $i > 0$ and the proof is complete. □

Corollary 4.4. *The closure of $\mathcal{O}$ is normal.*

Proof. We only need to note that the functions of degree n on $\mathcal{O}'$ (and also its closure since the closure is normal) as a G-module are isomorphic to $H^0(S^n\mathfrak{u}_{P'}^*)$. This follows since $\mathcal{O}'$ has trivial G-equivariant fundamental group when G is adjoint (see [2]). Hence the moment map determined by P' must be birational. Thus the short exact sequence of the theorem together with the discussion in Section 3 of [8] yields the result. □

REFERENCES

[1] A. Broer, *Normality of Some Nilpotent Varieties and Cohomology of Line Bundles on the Cotangent Bundle of the Flag Variety*, Lie Theory and Geometry (Boston), Progr. Math., 123, Birkhäuser, Boston, 1994, pp. 1–19.
[2] D. H. Collingwood and W. M. McGovern, *Nilpotent Orbits in Semisimple Lie Algebras*, Van Nostrand Reinhold Mathematics Series, Van Nostrand Reinhold Co., New York, 1993.
[3] M. Demazure, *A very simple proof of Bott's Theorem*, Invent. Math. **33** (1976), 271–272.
[4] H. Kraft and C. Procesi, *Closures of conjugacy classes of matrices are normal*, Invent. Math. **53** (1979), no. 3, 227–247.
[5] ______, *On the geometry of conjugacy classes in classical groups*, Comment. Math. Helv. **57** (1982), no. 4, 539–602.
[6] A. L. Onishchik and E. B. Vinberg, *Lie Groups and Algebraic Groups*, Springer-Verlag, Berlin, 1990.
[7] E. Sommers, *Functions on nilpotent orbits and their covers*, in preparation.
[8] ______, *Normality of nilpotent varieties in E_6*, J. Algebra (2003).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MASSACHUSETTS–AMHERST, AMHERST, MA 01003

INSTITUTE FOR ADVANCED STUDY, PRINCETON, NJ 08540
E-mail address: esommers@math.umass.edu