October, 2013

Dynamic Free Riding with Irreversible Investments*

Abstract

We study the Markov equilibria of a model of free riding in which n infinitely lived agents choose between private consumption and irreversible contributions to a durable public good. We show that the set of equilibrium steady states converges to a unique point as depreciation converges to zero. For any level of depreciation, moreover, the steady state of the best Markov equilibrium converges to the efficient level as agents become increasingly patient. These results are in stark contrast to what happens in the more commonly studied case in which investments are reversible, where a continuum of very inefficient equilibrium steady states are possible for any level of depreciation, discount factor and size of population.

Marco Battaglini
Department of Economics
Princeton University
Princeton NJ 08544
mbattagl@princeton.edu

Salvatore Nunnari
Department of Political Science
Columbia University
New York, NY 10027
snunnari@columbia.edu

Thomas Palfrey
Division of the Humanities and Social Sciences,
California Institute of Technology,
Pasadena, CA 91125
trp@hss.caltech.edu

*Battaglini gratefully acknowledges financial support from the Alfred P. Sloan Foundation. Palfrey gratefully acknowledges financial support from NSF (SES-0962802). We are grateful to seminar participants at the Einaudi Institute for Economics and Finance, Essex, London School of Economics, Ohio State, Princeton, Toulouse, Warwick, Yale and at the 2012 Southwest Economic Theory Conference, the co-Editor, and four anonymous referees for helpful comments. Steve Matthews provided very helpful detailed comments on an earlier draft. Juan Ortner provided excellent research assistance.

1 Introduction

The most significant kinds of free rider problems are characterized by two key features. First, they are dynamic. Public goods, for example, are often durable: it takes time to accumulate them and they depreciate slowly, projecting their benefits for many years. Similarly, environmental problems depend on variables that slowly evolve over time like capital goods. In all these examples what matters for the agents in the economy is the stock of the individual contributions accumulated over time. Second, they are characterized by irreversibility. Major investments in public goods, such as bridges, roads, and military equipment, are not easily transformed in other forms of consumption; similarly, the effects of pollution are difficult or impossible to reverse. Although there is a large literature dedicated to free rider problems, surprisingly little is known about dynamic problems with irreversibility. How large are the inefficiencies in dynamic environments with irreversibility? How do the distortions with irreversibility compare to the distortions in static models and in dynamic models with reversibility? To date, these questions have been posed only for very specific environments, and never fully answered.

In this paper, we present a simple model of free riding to address these questions. In the model, n infinitely lived agents allocate their income between private consumption and contributions to a public good in every period. The public good is durable and depreciates at a rate d. We study the properties of the Markov equilibria in which the public good is irreversible. We present three main results. First, we show that, for any group size and rate of depreciation, the steady state of the best Markov equilibrium converges to the efficient steady state as the agents' discount factor converges to one. Second, as depreciation converges to zero, the set of steady states converges to a unique point. It follows that, when the discount factor is high and depreciation is low, all equilibrium steady states are close to efficient. Third, however, convergence to the steady state is inefficiently slow. From these results we draw two general conclusions on dynamic free rider games with irreversibility: first, contrary to what happens with the more widely studied case with reversible investments, multiplicity of steady state equilibria is not an issue when depreciation is small; second, the problem with dynamic free riding with irreversibility is not so much an inefficient steady state, but an inefficiently slow accumulation path.

These results are related to two strands of literature. First, they are related to the large literature on dynamic public good games with reversibility. As in our work, this literature has studied the inefficiencies arising in the Markov equilibria of dynamic public goods games. Contrary to our work, however, this literature has focused exclusively on environments with reversible investments.[1] In contrast, our approach is general enough to allow sharp comparisons of

[1] Classic contributions include Levhari and Mirman [1981], Fershtman and Nitzan [1991]. More recent works

equilibrium public good investments with and without reversibility. The comparisons we obtain show clearly that the irreversibility of investments is very important in modelling dynamic free rider problems. With reversibility, for any group size, discount factor or rate of depreciation, there is always a continuum of equilibrium steady states, most of which very inefficient. Irreversibility, therefore, has a *beneficial* effect on welfare because it eliminates the many inefficient equilibria that arise in economies with reversibility.

The second strand of literature to which our paper is related is the research on monotone contribution games, notably Lockwood and Thomas [2002] and Matthews [2013].[2] In these papers, the techniques used to fully characterize subgame perfect equilibria require that players have a dominant strategy of zero contribution in the stage game and the depreciation rate is exactly zero. This literature has been focused on the comparison of the most efficient subgame perfect equilibria in economies with reversibility (when the actions can increase and decrease) and with irreversibility (when the actions cannot decrease), and it arrives at a conclusion that is essentially the opposite of ours. While we find that irreversibility is beneficial, its main finding is that, when the discount factor is sufficiently high, irreversibility always induces less efficient allocations because it limits the effectiveness of trigger strategies in punishing deviations. That conclusion, however, critically depends on the assumption that players have a dominant strategy in the stage game of zero contribution, and that the rate of depreciation of the state variable is exactly zero. As we prove in Section 5.2, when depreciation is positive, even if arbitrarily small, subgame perfect equilibria are not sufficiently restrictive to allow a clear comparison of economies with and without irreversibility.

As a methodological contribution, the paper develops a novel approach to characterize the Markov equilibria that may have more general applicability in the analysis of stochastic games with discrete time. The idea is to construct pure strategies that induce an objective function with a flat top: the flat region makes the players indifferent between different rates of accumulation. This provides additional freedom in choosing the players' reaction functions that is essential in proving existence of a pure strategy Markov equilibrium.

include Dockner and Long [1993], Dutta and Radner [2004], Rubio and Casino [2002]; Battaglini and Coate [2007] and Besley and Persson [2011] who have studied dynamic public good games in which investments are chosen in a collective decision making process.

[2] A number of important papers in the monotone games literature are less directly related. These papers require additional assumptions that make their environments hard to compare to ours. Gale [2001] studies games in which agents care only about the limit contributions as $t \to \infty$. Admati and Perry [1991], Compte and Jehiel [2004] and Marx and Matthews [2000] consider environments in which the benefit of the contribution occurs at the end of the game if a threshold is reached and in which players receive either partial or no benefit from interim contributions. The first two of these papers, moreover, assume that players contribute sequentially, one at a time, so the game form is different.

2 The model

Consider an economy with n agents. There are two goods: a private good x and a public good g. The level of consumption of the private good by agent i in period t is x_t^i, the level of the public good in period t is g_t. An allocation is an infinite nonnegative sequence $z = (x_\infty, g_\infty)$ where $x_\infty = (x_1^1, ..., x_1^n, ..., x_t^1, ..., x_t^n, ...)$ and $g_\infty = (g_1, ..., g_t, ...)$. We refer to $z_t = (x_t, g_t)$ as the allocation in period t. The utility U^j of agent j is a function of $z^j = (x_\infty^j, g_\infty)$, where $x_\infty^j = (x_1^j, ..., x_t^j, ...)$. We assume that U^j can be written as $U^j(z^j) = \sum_{t=1}^{\infty} \delta^{t-1} \left[x_t^j + u(g_t)\right]$, where $u(\cdot)$ is continuously twice differentiable, strictly increasing, and strictly concave on $[0, \infty)$, with $\lim_{g \to 0^+} u'(g) = \infty$ and $\lim_{g \to +\infty} u'(g) = 0$. The future is discounted at a rate δ. There is a linear technology by which the private good can be used to produce public good, with a marginal rate of transformation $p = 1$. The private consumption good is nondurable, the public good is durable, and the stock of the public good depreciates at a rate $d \in [0, 1]$ between periods. Thus, if the level of public good at time $t-1$ is g_{t-1} and the total investment in the public good is I_t, then the level of public good at time t will be $g_t = (1-d)g_{t-1} + I_t$.

In an *Irreversible Investment Economy* (IIE) the public policy in period t is required to satisfy three feasibility conditions: (i) $x_t^j \geq 0 \quad \forall j, \forall t$; (ii) $g_t \geq (1-d)g_{t-1} \quad \forall t$; (iii) $I_t + \sum_{j=1}^{n} x_t^j \leq W$ $\forall t$, where W is the aggregate per period level of resources in the economy. The first condition guarantees that private allocations are nonnegative. The second condition is the irreversibility condition, and is equivalent to $I_t \geq 0 \ \forall t$. In contrast, a Reversible Investment Economy (RIE) is an economy where the second constraint is replaced by $g_t \geq 0$, so I_t can be negative. We compare IIE to RIE in Section 5.1.

It is convenient to distinguish the state variable at t, g_{t-1}, from the policy choice g_t and to reformulate the budget condition. If we denote $y_t = (1-d)g_{t-1} + I_t$ as the new level of public good after investing I_t in the current period when the last period's level of the public good is g_{t-1}, then the public policy in period t can be represented by a vector $(y_t, x_t^1, ..., x_t^n)$. Substituting y_t, the budget balance constraint $I_t + \sum_{j=1}^{n} x_t^j \leq W$ can be rewritten as $\sum_{j=1}^{n} x_t^j + [y_t - (1-d)g_{t-1}] \leq W$. With this notation, we must have $x_t \geq 0, y_t \geq (1-d)g_{t-1}$ in a IIE.

The initial stock of public good is $g_0 \geq 0$, exogenously given. Public policies are chosen as in the classic free rider problem, modeled by a voluntary contribution game. In period t, each agent j is endowed with $w_t^j = W/n$ units of private good. We assume that each agent has full property rights over a share of the endowment (W/n) and in each period chooses on its own how to allocate its endowment between an individual contribution to the stock of public good (which is shared by all agents) and private consumption, taking as given the strategies of the other agents. The individual contribution by agent j at time t is denoted i_t^j (so $i_t^j = W/n - x_t^j$) and in an irreversible investment economy we require $i_t^j \in [0, W/n]$ $\forall j$. The total economy-wide increase in the stock of the public good in any period is then given by the sum of the agents' individual contributions.[3]

To study the properties of the dynamic free rider problem described above, we study symmetric Markov perfect equilibria, where all agents use the same strategy, and these strategies are time-independent functions of the state, g.[4] A strategy is a pair $(x(\cdot), i(\cdot))$, where $x(g)$ is an agent's level of consumption and $i(g)$ is an agent's contribution to the stock of public good in state g. Given these strategies, by symmetry, the stock of public good in state g is $y(g) = (1-d)g + ni(g)$. For the remainder of the paper we refer to $y(g)$ as the *investment function*. Associated with any Markov perfect equilibrium of the game is a value function, $v(g)$, which specifies the expected discounted future payoff to an agent when the state is g. An equilibrium is continuous if the investment function, $y(g)$, and the value function, $v(g)$, are both continuous in g. In the remaining of the paper we will focus on continuous equilibria. In the following we refer to equilibria with the properties described above simply as equilibria.

We are interested in studying the long term properties of the allocation. Given an equilibrium $(y(g), v(g))$, an allocation y^o is a steady state if it is a fixed point of the investment function: $y^o = y(y^o)$. A steady state y^o is said to be stable if there is a neighborhood $N_\varepsilon(y^o)$ of y^o such that for any $N_{\varepsilon'}(y^o) \subseteq N_\varepsilon(y^o), g \in N_{\varepsilon'}(y^o)$ implies $y(g) \in N_{\varepsilon'}(y^o)$. Intuitively, starting in a neighborhood of a stable steady state, g remains in a neighborhood of a stable steady state for all future periods. In what follows we will focus only on steady states that are stable and we will refer to stable steady states simply as steady states. We say that convergence to a steady state is gradual if it is not reached in finite time starting from a left neighborhood of it. We say that convergence is monotonic if the state converges monotonically to the steady state.

3 The planner's problem

As a benchmark with which to compare the equilibrium allocations, we first analyze the sequence of public policies that would be chosen by a utilitarian planner. The planner's solution is extremely simple in the environment described in the previous section: this feature will help highlighting the subtlety of the strategic interaction studied in the next two sections.

[3] The state variable in our model can have alternative interpretations. For example, one may think of g as physical public capital, which seems natural to assume is irreversible. Once a bridge is constructed, it cannot be demolished and costlessly transformed back to consumption. But we have in mind a more general interpretation. In some applications g could represent social capital or certain aspects of aggregate human capital (literacy rates, for example). The key aspect of g is its durability.

[4] We study the subgame perfect equilibria of the game in Section 5.2 where we compare the predictions in the two equilibrium concepts.

The planner's problem has a recursive representation in which g is the state variable, and $v_P(g)$, the planner's value function can be represented recursively as:

$$v_P(g) = \max_{y,x} \left\{ \begin{array}{c} \sum_{j=1}^n x^i + nu(y) + \delta v_P(y) \\ s.t \ \ \sum_{j=1}^n x^i + y - (1-d)g \leq W, \ x^i \geq 0 \ \forall i, \ y \geq (1-d)g \end{array} \right\} \tag{1}$$

By standard methods (see Stokey, Lucas, and Prescott [1989]), we can show that a continuous, strictly concave and differentiable $v_P(g)$ that satisfies (1) exists and is unique. The optimal policies have an intuitive characterization. When the accumulated level of public good is low, the marginal benefit of increasing y is high, and the planner finds it optimal to spend as much as possible on building the stock of public good: in this region of the state space $y_P(g) = W + (1-d)g$ and $\sum_{j=1}^n x^i = 0$. When g is high, the planner will be able to reach the level of public good $y_P^*(\delta, d, n)$ that solves the planner's unconstrained problem: i.e. $nu'(y_P^*(\delta, d, n)) + \delta v_P'(y_P^*(\delta, d, n)) = 1$. Applying the envelope theorem, we can show that at the interior solution $y_P^*(\delta, d, n)$ we have $v_P'(y_P^*(\delta, d, n)) = 1 - d$. It follows that $y_P^*(\delta, d, n) = [u']^{-1}\left(\frac{1-\delta(1-d)}{n}\right)$. The investment function has the following simple structure. If $(1-d)g < (y_P^*(\delta, d, n) - W)$, then $y_P^*(\delta, d, n)$ is not feasible: the planner spends W on the public good so $y_P(g) = (1-d)g + W$. If $(1-d)g \in (y_P^*(\delta, d, n) - W, y_P^*(\delta, d, n))$, instead, the planner can choose $y_P(g) = y_P^*(\delta, d, n)$ without violating the constraints. If $(1-d)g > y_P^*(\delta, d, n)$, then the irreversibility constraint is binding and $y_P(g) = (1-d)g$.

This investment function implies that the planner's economy converges to one of two possible steady states. If $W/d \leq y_P^*(\delta, d, n)$, then the rate of depreciation is so high that the planner cannot reach $y_P^*(\delta, d, n)$ (except temporarily if the initial state is sufficiently large). In this case the steady state is $y_P^o = W/d$, and the planner invests all resources in all states on the equilibrium path. If $W/d > y_P^*(\delta, d, n)$, $y_P^*(\delta, d, n)$ is sustainable as a steady state. In this case, in the steady state $y_P^o = y_P^*(\delta, d, n)$, and the (per agent) level of private consumption is positive: $x^* = (W + (1-d)g - y_P^*)/n > 0$. For the rest of the paper, we assume that $W/d > y_P^*(\delta, d, n)$. The analysis in the other case is similar.

4 Equilibrium with irreversibility

In contrast to the planner's solution, in equilibrium no agent can directly choose the stock of public good y: an agent (say j) chooses only his own level of private consumption x and the level of its own contribution to the stock of public good. The agent realizes that in any period, given g and the other agents' level of private consumption, her contribution ultimately determines y. It is therefore as if agent j chooses x and y, subject to three feasibility constraints. The first constraint is a resource constraint that specifies the level of the public good $y = W + (1-d)g - [x + (n-1)x(g)]$. This constraint requires that the stock of public good y equals total resources, $W + (1-d)g$, minus the sum of private consumptions, $x + (n-1)x(g)$. The function $x(g)$ is the *equilibrium* per capita level; naturally, the agent takes the equilibrium level of the other players, $(n-1)x(g)$, as given. The second constraint requires that private consumption x is non negative. The third requires total consumption nx to be no larger than total resources W. Agent j's problem can therefore be written as:

$$\max_{y,x} \left\{ \begin{array}{c} x + u(y) + \delta v(y) \\ \\ s.t \;\; x + y - (1-d)g = W - (n-1)x(g) \\ \\ W - (n-1)x(g) + (1-d)g - y \geq 0, \text{ and } x \leq W/n \end{array} \right\} \tag{2}$$

where $v(g)$ is his equilibrium value function.

In a symmetric equilibrium, all agents consume the same fraction of resources, so agent j can assume that in state g the other agents each consume $x(g) = [W + (1-d)g - y(g)]/n$, where $y(g)$ is the equilibrium investment function. Substituting the first constraint of (2) in the objective function, recognizing that agent j takes the strategies of the other agents as given, and ignoring constant terms, the agent's problem can be written as:

$$\max_{y} \left\{ \begin{array}{c} u(y) - y + \delta v(y) \\ y \leq \frac{W+(1-d)g}{n} + \frac{n-1}{n} y(g), \;\; y \geq \frac{(1-d)g}{n} + \frac{n-1}{n} y(g) \end{array} \right\} \tag{3}$$

To interpret the second constraint, note that it can be written as $y \geq (1-d)g + \frac{n-1}{n}[y(g) - (1-d)g]$: the new level of public good cannot be lower than $(1-d)g$ plus the investments from all the other agents (in a symmetric equilibrium, an individual investment is $[y(g) - (1-d)g]/n$). Similarly, the second constraint can be written as $y \leq (1-d)g + \frac{n-1}{n}[y(g) - (1-d)g] + W/n$: the new level of public good can not be larger than $(1-d)g$, plus the investments from all the other agents, plus the maximal individual contribution W/n.

A continuous symmetric Markov equilibrium is fully described in this environment by two functions: an aggregate investment function $y(g)$, and an associated value function $v(g)$. The aggregate investment function $y(g)$ must solve (14) given $v(g)$. The value function $v(g)$ must be consistent with the agents' strategies, so, for all g:

$$v(g) = \frac{W + (1-d)g - y(g)}{n} + u(y(g)) + \delta v(y(g)) \tag{4}$$

An equilibrium in an Irreversible Investment Economy is a pair of functions, $y(\cdot)$ and $v(\cdot)$, such that for all $g \geq 0$, $y(g)$ solves (14) given the value function $v(\cdot)$, and for all $g \geq 0$, $v(g)$ solves

(4) given $y(g)$. For a given value function, if an equilibrium exists, the problem faced by an agent looks apparently similar to the problem of the planner, but with two important differences. First, in the objective function the agent does not internalize the effect of the public good on the other agents. This is the classic free rider problem, present in static models as well: it induces a suboptimal investment in g. The second difference with respect to the planner's problem is that the agent takes the current and future contributions of the other agents as given. The incentives to invest depend on the agent's expectations about the other agents' current and future contributions, which are captured implicitly by the investment function $y(g)$.

To understand the complications associated with constructing an equilibrium with irreversibility, consider what happens when the irreversibility constraint becomes binding in an economy with zero depreciation. Suppose first that, as in the planner's problem, the value function is strictly concave. In this case the investment function looks very much like the planner's investment function: the agents find it optimal to invest as much as possible until the unconstrained optimum is feasible, say y^o; and then they find it optimal to stay constant at this level until the irreversibility constraint becomes binding: $y(g) = y^o$ for $g \leq y^o$, so y^o is the steady state. The binding irreversibility constraint, however, forces the agent to choose a higher level of public good on the right of y^o: $y(g) = g > y^o$ for $g > y^o$. Because of the free rider problem, y^o is strictly lower than the efficient level: the players recognize that any increase in their investment crowds out the other players' investments in the following periods and so they underinvest. The irreversibility constraint acts as a beneficial commitment device and limits the ability of other players to "eat" additional investments. A marginal increase in g is more valuable on the right than on the left of the steady state y^o. However, this generates a contradiction since y^o would be suboptimal: by marginally increasing g at y^0 the agents obtain a higher utility. In equilibrium we must have that the rate of investment on the right and on the left of the point at which the irreversibility constraint is binding are the same. The investment function, therefore must be tangent to the irreversibility constraint $y = g$, as in the equilibrium represented in Figure 1. Satisfying this smooth pasting condition, however, is impossible with a strictly concave value function because it requires that the agents are willing to invest at a sufficiently high speed on the left of y^0 to guarantee that $y'(g) = 1$ at the point of contact with the irreversibility constraint. The challenge is to construct equilibria with this property.

In what follows, we proceed in two steps. First, to prove existence of an equilibrium we construct equilibria in which the value function has a flat top, as for example in Figure 1. We then prove that there is no loss of generality in focusing on this particular class in order to study the set of equilibrium steady states when depreciation is sufficiently low.

In an equilibrium in which the objective function has a flat region the investment function may

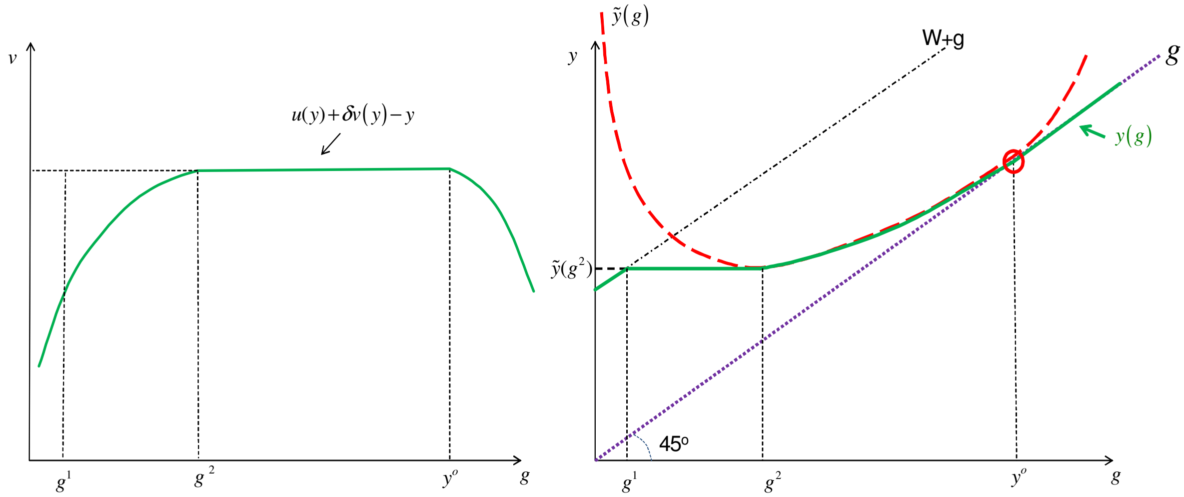

Figure 1: An example of "flat top" equilibrium with $d = 0$.

take a more general form than the planner's solution. Figure 1 represents a typical equilibrium when $d = 0$ to illustrate the logic of the construction. The equilibrium investment function takes the following form:

$$y(g) = \begin{cases} \min\left\{W + g, \widetilde{y}\left(g^2\right)\right\} & g < g^2 \\ \widetilde{y}\left(g\right) & g \in \left[g^2, y^o\right] \\ g & g \geq y^o \end{cases} \tag{5}$$

where equilibrium investment is characterized by two critical levels, g^2, y^o and an investment function $\widetilde{y}(g)$, which is a non decreasing function with values in $[g, W + g]$. To see why $y(g)$ may take the form of (5), consider Figure 1. The right panel of the figure illustrates a canonical equilibrium investment function. The steady state is labeled y^o in the figure, the point at which the (bold) investment function intersects the (dotted) diagonal. The left panel graphs the corresponding objective function, $u(y) - y + \delta v(y)$. For $g < g^1$, the objective function of (14) is strictly increasing in y and resources are insufficient to reach the level that maximizes the unconstrained objective function.[5] In this case it is optimal to invest all resources: $y(g) = W + g$ in $g \leq g^1$. For $g > y^o$, the objective function is decreasing. The agents would like to reduce g, but the irreversibility constraint is binding, so $y(g) = g$. For intermediate levels of $g \in \left[g^1, y^o\right]$, an interior level of investment $y \in \left(g^2, y^o\right)$ is chosen. This is possible because the objective function is flat in this

[5] In Figure 1 it is assumed that we have $W + g > g^2$ for for $g \geq g^1$, so the agent can afford to choose a level of y that maximizes the objective function (i.e. $y \in \left[g^2, g^3\right]$) if and only if $g \geq g^1$.

region: an agent is indifferent between any $y \in [g^2, y^o]$. In the example of Figure 1, agents choose $y(g) = \widetilde{y}(g^2)$ in $g \in [g^1, g^2]$, and the increasing function $\widetilde{y}(g)$ in $y \in [g^2, y^o]$. The key observation here is that since the objective function has a flat region, the agents are willing to choose an *increasing* investment function in $[g^2, y^o]$ that is tangent to the irreversibility constraint.

Can we construct an investment function that makes the value function flat on top and that also satisfies the equilibrium conditions? For an investment curve as in Figure 1 to be an equilibrium, we need to make sure that the agents are indifferent between investing and consuming for all states in $[g^2, y^o]$. If this condition does not hold, the agents do not find it optimal to choose an interior level $y(g)$. The marginal utility of investments is zero if and only if

$$u'(g) + \delta v'(g) - 1 = 0 \ \forall g \in [g^2, y^o] \tag{6}$$

Since the expected value function is (4), in the general case with $d \geq 0$ we have:

$$v'(g) = \frac{1 - d - y'(g)}{n} + u'(y(g))y'(g) + \delta v'(y(g))y'(g) \tag{7}$$

Substituting this formula in (6), we see that the investment function $y(g)$ must solve the following differential equation:

$$\frac{1 - u'(g)}{\delta} = \frac{1 - d - y'(g)}{n} + u'(y(g))y'(g) + \delta v'(y(g))y'(g) \tag{8}$$

This condition is useful only if we eliminate the last (endogenous) term: $\delta v'(y(g))y'(g)$. To see why this is possible, note that $y(g)$ is in $[g^2, y^o]$ for any $g \in (g^2, y^o)$ in the example of Figure 1. In this case, (6) implies $\delta v'(y(g)) = 1 - u'(y(g))$. Substituting this condition in (8) we obtain the following necessary condition:

$$y'(g) = \frac{1 - d - \frac{n(1-u'(g))}{\delta}}{1 - n} \tag{9}$$

Condition (9) shows that there is a unique way to specify the shape of the investment function that is consistent with a "flat" objective function in equilibrium. This necessary condition, however, leaves considerable freedom to construct multiple equilibria: (9) defines a simple differential equation with a solution $\widetilde{y}(g)$ unique up to a constant. The equilibrium $\widetilde{y}(g)$ is pinned down when we impose the smooth pasting condition discussed above. Using (9), it can be verified that $y'(g) = 1$ at $g = [u']^{-1}(1 - \delta)$. The tangency condition $y(y^o) = y^o$ provides the initial condition for the differential equation (9), and so uniquely defines $\widetilde{y}(g)$ in (5). The dashed line in Figure 1 represents this function.

The following proposition proves that an equilibrium exists for any $d \geq 0$ and indeed a continuum of equilibria exist for $d > 0$ (when $d = 0$ the upper and lower bounds of the equilibrium steady states in Proposition 1 coincide).[6]

[6] The details of the proofs of Proposition 1 and all the following results are in the online appendix.

Proposition 1. *For any d, δ, n and $y^o \in \left[[u']^{-1}(1-\delta(1-d)), [u']^{-1}(1-\delta(1-\frac{d}{n}))\right]$, there is an equilibrium with steady state y^o in an irreversible economy. In all these equilibria convergence is monotonic and gradual.*

Perhaps unsurprisingly, for any $d > 0$ and $\delta < 1$ all equilibria in Proposition 1 are inefficient: first, the steady state is below optimum $y_P^*(\delta, d, n)$ (defined in Section 3); second, convergence is inefficiently slow since convergence is in finite time in a planner's solution. Inefficiencies, however, disappear in the best Markov equilibrium as players become patient since the upper bound of the set characterized in Proposition 1 converges to the planner's first best:

Corollary 1. *For any d and n, the steady state in the most efficient equilibrium of an irreversible economy converges to the efficient level as $\delta \to 1$.*

Proposition 1 does not put bounds on the equilibrium set and so on potential inefficiencies in other equilibria. In the next result we show that multiplicity disappears as depreciation converges to zero. Let $\overline{y}_{IR}(\delta, d, n)$ and $\underline{y}_{IR}(\delta, d, n)$ be the infimum and the supremum of the set of equilibrium steady states. We have:

Proposition 2. *For any δ and n, we have that $\left|\overline{y}_{IR}(\delta, d, n) - \underline{y}_{IR}(\delta, d, n)\right| \to 0$ as $d \to 0$. Moreover, there is a $\overline{d} > 0$ such that for $d < \overline{d}$, all equilibrium paths are gradual.*

The first part of Proposition 2 shows that when depreciation is small, there is a small set of feasible equilibrium steady states. Indeed, the set of steady states converges to a singleton as depreciation converges to zero. This is a property entirely due to irreversibility since, as we discuss in the next section, with reversibility there is a continuum of equilibria for any level of depreciation (and for any δ and n as well). The second point shows that equilibria must look alike also in terms of the convergence path when d is sufficiently small: in all of them, convergence is gradual.[7] This also is a property that is due to the irreversibility constraint, since in reversible economies we can have convergence in finite time.

5 Two Comparisons

In this section we discuss two natural comparisons to put the results of Propositions 1 and 2 in perspective. In Section 5.1 we compare the equilibrium sets with and without irreversibility. In Section 5.2 we compare Markov to Subgame Perfect Equilibria.

[7] The property that the convergence path is gradual has been highlighted as a general feature of subgame perfect equilibria with irreversibility by Lockwood and Thomas [2002]. As we prove in Proposition 3, their result requires an assumption that d is exactly zero. When $d > 0$, graduality of the equilibrium path is not a general feature of subgame perfect equilibria.

5.1 Irreversible vs. reversible economies

In economies with reversibility (RIE) the agents can choose a negative level of investment and so the public good can be reduced below $(1-d)g_t$. The fesibility constraints defining the economy become: (i) $x_t^j \geq 0 \ \forall j, \forall t$; (ii) $y_t \geq 0$; (iii) and $\sum_{j=1}^{n} x_t^j + [y_t - (1-d)g_{t-1}] \leq W \ \forall t$. In Battaglini et al. [2012] we characterize the symmetric equilibria of the dynamic free rider game in these economies assuming that the agents withdraw up to $1/n$ of the accumulated stock. Let $\overline{y}_R(\delta, d, n)$ and $\underline{y}_R(\delta, d, n)$ be the upper and lower bounds of the set of steady states in a RIE. In Battaglini et al [2012] we show that $\overline{y}_R(\delta, d, n) = [u']^{-1} (1 - \delta(1 - d/n))$ and $\underline{y}_R(\delta, d, n) \leq [u']^{-1} (1 - \delta(1-d)/n)$. This allows a neat comparison with the equilibria in an irreverible economy. Let E_R and E_{IR} be the sets of equilibrium steady states respectively with and without reversibility. The following result is an immediate implication of Propositions 1-2:

Proposition 3. *For any δ and n, there is a $\overline{d} > 0$ such that $E_{IR} \subset E_R$ for all $d < \overline{d}$. Moreover, $\left|\underline{y}_{IR}(\delta, d, n) - \overline{y}_R(\delta, d, n)\right| \to 0$ as $d \to 0$.*

Proposition 3 makes clear that irreversibility has two effects. First, it reduces the set of equilibrium steady states, that with irreversibility is *strictly* included in the set with reversibility if depreciation is sufficiently small. Second, and more importantly, the irreversibility constraint eliminates the inefficient equilibria. When the economy is reversible, there are always very inefficient equilibria, with steady states not larger than $[u']^{-1} (1 - \delta(1-d)/n)$, that are worse than the steady states reached by an agent in autarky (that is, alone by herself). On the other hand, as $d \to 0$, the lower bound of the steady states with irreversibility converges to the upper bound of the set with reversibility. In addition to these two effects, Battaglini et al. [2012] show that with reversibility there is always an (inefficient) equilibrium in which convergence to the steady state is not gradual. This shows that graduality is a feature associated to irreversibility.

The fact that reversibility affects so much the equilibrium set may appear surprising. In a planner's solution the irreversibility constraint is irrelevant: it affects neither the steady state (that is unique), nor the convergence path.[8] Even in economies with irreversibility, the irreversibility constraint is typically never binding on the equilibrium path starting from a low level of g (see Figure 1 for example). The reason why irreversibility is so important in a dynamic free rider game is precisely the fact that equilibrium investments are inefficiently low and the irreversibility constraint may limit the inefficiency by acting as a *commitment device*. The intuition is as follows. In the equilibria of a dynamic free rider problem, both with and without irreversibility, the agents

[8] Assuming the initial state g_0 is smaller than the steady state, on the convergent path the stock of public good is never reduced: it keeps increasing until the steady state is reached, and then it stops; the irreversibility constraint is, thus, never binding on the equilibrium path.

holds back their individual contributions for fear that they will crowd out the contributions of other players, or even be appropriated by other agents in future periods. In economies with irreversibility, however, the irreversibility constraint may limit the ability of the agents to appropriate the accumulated public good. In general, the irreversibility constraint is binding only for states that are so high that they are not reached on the equilibrium path; still, the fact that in these states free riding will be limited affects the entire equilibrium investment function. In states just below the point in which the constraint is binding, the agents know that the constraint will not allow the other agents to reduce the public good when it passes the threshold. These incentives induce higher investments and a higher value function, with a ripple effect on the entire investment function.

5.2 Subgame perfect equilibria vs. Markov equilibrium

Previous to our work, the effects of irreversibility have been studied in the literature on monotone games, in particular by Lockwood and Thomas [2002] and Matthews [2013].[9] Both papers make two key assumptions: first, a zero contribution is a dominant strategy for all players, and so the game can be reduced to a repeated version of a prisoners' dilemma game;[10] second, the state variable g can only stay constant or increase because depreciation is exactly zero. Under these conditions Lockwood and Thomas [2002] have characterized the most efficient subgame perfect equilibrium (SPE) and Matthews [2013] has characterized all SPE.[11] This literature arrives at a conclusion that is opposite to ours: while we show irreversibility has a positive welfare effect, they show it has a negative effect. They prove that SPE in economies with irreversibility are more inefficient than in economies with reversibility, at least when agents are sufficiently patient. This finding is shown by proving an *Anti-Folk Theorem*: with reversibility, the efficient allocation is achievable in a SPE for a sufficiently high discount factor (but less than one); with irreversibility, SPE are inefficient *for all* $\delta < 1$.[12] This literature has also stressed the conclusion that the most efficient equilibrium path is characterized by gradualism (and hence inefficiently slow) if

[9] See footnote 2 for other important contributions in the monotone games literature that are not as closely related to our paper.

[10] In our model we assume standard preferences $u(g)$ satisfying the Inada condition ($u'(g) \to \infty$ as $g \to 0$ and $u'(g) \to 0$ as $g \to \infty$). In this case the dominant strategy assumption is never satisfied since the players find it optimal to make strictly positive contributions, no matter what the other players do.

[11] Matthews [2012] presents the most general analysis of monotone games to date. Although Matthews does not allow for depreciation, his model assumes very general specifications for the players' preferences and makes weak assumptions on the timing of contributions. For this general version of the model, Matthews can characterize a necessary condition for a SPE; however, for a characterization of the SPE, this requires the same assumptions as in Lockwood and Thomas [2002].

[12] The intuition behind this result is that irreversibility limits the ability of agents to punish each other: in the worst continuation equilibrium players stop making contributions, but they can not eat or destroy the accumulated state g.

and only if the investment is irreversible. The central new finding in this literature is the fact that *any comparison at all* can be made focusing on the large set of subgame perfect equilibria. This is interesting because the set of SPE is not generally very informative in games with perfect information for reasons similar to standard folk theorems for repeated games.

With this background, we next compare the most efficient SPE paths in RIE and IIE under the assumptions of our model, allowing a positive (but possibly arbitrarily small) rate of depreciation. We say that an investment path is a *SPE path* if it coincides with the equilibrium path of a SPE.

Proposition 4. *For any $d > 0$ and $n > 1$, there is a $\overline{\delta} < 1$ such that the most efficient SPE path in a RIE and the most efficient SPE path in a IIE both coincide with the Pareto efficient investment path for all $\delta > \overline{\delta}$. Hence, neither the most efficient SPE path in a RIE nor the most efficient SPE path in a IIE are characterized by gradualism for all $\delta > \overline{\delta}$.*

Proposition 4 makes clear that the results on the effects of the irreversibility constraint obtained in the literature on monotone games critically depend on the assumption that depreciation is exactly zero: they can not be interpreted as results describing economies with a small, perhaps arbitrarily small, degree of depreciation, because when depreciation is not zero, results are *qualitatively* different.[13] Perhaps more seriously, Proposition 4 suggests that when $d > 0$ a comparison between the most efficient SPE in RIE and IIE with patient players is no longer insightful, since the efficient allocation is an equilibrium in both environments.

To see why the rate of depreciation is so important, consider for simplicity the environment in Lockwood and Thomas [2002], where a zero contribution is a dominant strategy (an assumption that is not used in the proof of Proposition 4).[14] When $d = 0$, the worst punishment for an agent is that all other agents stop making contributions, which is always the worst equilibrium by assumption (since zero contribution is a dominant strategy). But with $d = 0$ this punishment becomes increasingly irrelevant as g approaches the efficient steady state: in the worst case, a deviation is punished by an allocation that remains forever close to the efficient allocation. This is the reason why irreversibility makes it impossible to obtain an efficient allocation in Lockwood and Thomas [2002]. The environment is very different if d is positive, even if arbitrarily small. In this case, after a deviation, the state would gradually decline and it would eventually approach zero. This convergence may certainly be slow when d is small: but as $\delta \to 1$ only the long run

[13] Lockwood and Thomas [2002]'s result can be extended to the case of arbitrarily small amounts of depreciation, but only if the order of limits is reversed, i.e., by fixing δ and taking d to zero Thus, it is always possible to find a sequence d^n, δ^n converging to $\{0, 1\}$ such that the efficient allocation path is a SPE path for any n both in a RIE and in a IIE: it follows that, for low d and high δ, a sharp comparison between the two is not possible.

[14] For consistency, Proposition 4 is stated under the assumptions described in Section 2 according to which players do not have a dominant strategy. The strategy of our proof can be immediately adapted (in a simplified form) to the environment considered by Lockwood and Thomas [2002].

matters, so the punishment for a deviation is sufficiently high to induce all players to make the efficient contribution.

6 Conclusions

In this paper we have studied a simple model of free riding in which n infinitely lived agents choose between private consumption and contributions to a durable public good. We study the properties of the Markov equilibria in which contributions to the public good are irreversible.

We present three main results. First, we show that, no matter what the depreciation rate is, the steady state of the best Markov equilibrium converges to the efficient level as the agents' discount factor converges to one. Second, as depreciation converges to zero, the set of steady states converges to a unique point. It follows that, when the discount factor is high and depreciation is low, all equilibrium steady states are approximately efficient. Third, however, convergence to the steady state is always inefficiently slow. From these results we conclude that, with irreversible investments, multiplicity of steady state equilibria is not problematic when depreciation is small; and that the problem with dynamic free riding is not so much an inefficient steady state, but an inefficiently slow accumulation path.

Finally, we compare these results with the more commonly studied case with reversible investment. We show that irreversibility has two effects. First, it reduces the set of equilibrium steady states, that with irreversibility is strictly included in the set with reversibility if depreciation is sufficiently small. Second, and more importantly, the irreversibility constraint eliminates the inefficient equilibria which, with reversible investments, are possible for any level of depreciation, discount factor and group size.

References

Admati A. and M. Perry (1991), "Joint Projects Without Commitment," *Review of Economic Studies*, 58, 259-276.

Battaglini, M. and S. Coate, (2007), "Inefficiency in Legislative Policy-Making: A Dynamic Analysis," *American Economic Review*, 97(1), 118-149.

Battaglini, M., S. Nunnari, and T. Palfrey (2012), "The Free Rider Problem: a Dynamic Analysis," NBER Working Paper No. 17926.

Besley T. and T. Persson (2011), *Pillars of Prosperity*, 2010 Yrjö Jahnsson Lectures, Princeton University Press.

Compte O. and P. Jehiel (2004), "Gradualism in Bargaining and Contribution Games," *Review of Economic Studies*, 71, 975-1000.

Dockner E. J, and N. V. Long (1993), "International Pollution Control: Cooperative vs Noncooperative Strategies," *Journal of Environmental Economics and Management*, 24: 13–29.

Dutta, Prajit K. and Radner, Roy (2004): "Self-enforcing climate-change treaties," *Proc. Nat. Acad. Sci. U.S.*, 101, 4746-51.

Fershtman, C. and S. Nitzan (1991), "Dynamic voluntary provision of public goods," *European Economic Review*, 35, 1057–1067.

Gale, D. (2001), "Monotone Games with Positive Spillovers," *Games and Economic Behavior*, 37, 295-320.

Levhari, D. and L. J. Mirman (1980), "The Great Fish War: An Example Using Nash–Cournot Solution," *Bell Journal of Economics*, 11, 322–334.

Lockwood B. and J. Thomas (2002), "Gradualism and Irreversibility," *Review of Economic Studies*, 69, 339-356.

Marx, L. and S. Matthews (2000), "Dynamic Voluntary Contribution to a Public Project," *Review of Economic Studies*, 67: 327-358.

Matthews, S. (2013), "Achievable Outcomes of Dynamic Contribution Games," *Theoretical Economics*, 8(2): 365-403.

Rubio. S. J. and B. Casino (2002), "A Note on Cooperative vs Noncooperative Strategies in International Pollution Control," *Resource and Energy Economics*, 24: 251–261.

Stokey, N., R. Lucas and E. Prescott, (1989), *Recursive Methods in Economic Dynamics*, Cambridge, MA: Harvard University Press.

7 Appendix

7.1 Proof of Proposition 1

Define $y^*(\delta, d, n) = [u']^{-1}(1 - \delta(1-d)/n)$ and $y^{**}(\delta, d, n) = [u']^{-1}(1 - \delta(1 - d/n))$: these are the points at which (9) is, respectively, zero and one. Define $\overline{y}(d, \delta) = [u']^{-1}(1 - \delta(1-d))$: this is the point at which (9) is equal to $1-d$. Note that $y^*(\delta, d, n) < \overline{y}(d, \delta)$ and $\overline{y}(d, \delta) < y^{**}(\delta, d, n)$. Moreover, since we are assuming that the planner interior solution is feasible ($y^*_P(\delta, d, n) < W/d$), we have $y^{**}(\delta, d, n) < W/d$. To construct an equilibrium with steady state $y^o \in [\overline{y}(\delta, d), y^{**}(\delta, d, n)]$ we proceed in 3 steps.

Step 1. We first construct the strategies associated to a generic y^o. For a generic $y^o \in [\overline{y}(\delta, d), y^{**}(\delta, d, n)]$, let $\widetilde{y}(g\,|y^o)$ be the solution of the differential equation (9) when we require the initial condition: $\widetilde{y}(y^o\,|y^o) = y^o$. Given y^o, moreover, let us define the two thresholds $g^3(y^o) = y^o/(1-d)$ and $g^2(y^o) = \max\{\min_{g\geq 0}\{g\,|\widetilde{y}(g\,|y^o) \leq W + (1-d)g\}, y^*(\delta, d, n)\}$. In words, the second threshold is the largest point between the point at which $\widetilde{y}(g\,|y^o)$ crosses from below $W + (1-d)g$, and $y^*(\delta, d, n)$ (see Figure 1 in the paper for an example). It is easy to verify that, by construction, $g^3(y^o) \geq \overline{y}(\delta, d)$; moreover, $\widetilde{y}(g\,|y^o) \in ((1-d)g, W + (1-d)g)$ with $\widetilde{y}'(g\,|y^o) \in [0, 1]$ and $\widetilde{y}''(g\,|y^o) \geq 0$ in $\left[g^2(y^o), y^o\right]$. For any $y^o \in [\overline{y}(\delta, d), y^{**}(\delta, d, n)]$, we now define the investment function as follows:

$$y(g\,|y^o) = \begin{cases} \min\left\{W + (1-d)g, \widetilde{y}\left(g^2(y^o)\,|y^o\right)\right\} & g \leq g^2(y^o) \\ \widetilde{y}(g\,|y^o) & g^2(y^o) < g \leq y^o \\ y^o & y^o < g \leq g^3(y^o) \\ (1-d)\,g & g > g^3(y^o) \end{cases} \tag{10}$$

Note that when depreciation is zero, then $g^3(y^o) = y^o$ and $y'(g\,|y^o) = 1$ at $g = y^o$: so (10) coincides exactly with the investment function illustrated in Figure 1 in the paper. For future reference, define $g^1(y^o) = \max\left\{0, \left(\widetilde{y}\left(g^2(y)\,|y^o\right) - W\right)/(1-d)\right\}$. This is the point at which $W + (1-d)g = \widetilde{y}\left(g^2(y^o)\,|y^o\right)$, if positive. Since $\widetilde{y}\left(g^2(y)\,|y^o\right) < W + (1-d)g^2(y^o)$, $g^1(y^o) \in \left[0, g^2(y^o)\right]$. We have:

Lemma A.1. *$y(g\,|y^o) \in \left[g^2(y^o), y^o\right]$ for $g \in \left[g^2(y^o), y^o\right]$.*

Proof. Because $y(g\,|y^o)$ is monotonic non-decreasing in $g \in \left[g^2(y^o), y^o\right]$, for any $g \in \left[g^2(y^o), y^o\right]$ we have $y(g\,|y^o) \in \left[y\left(g^2(y^o)\,|y^o\right), y^o\right]$. Since $y(g\,|y^o)$ has slope lower than one in $\left[g^2(y^o), y^o\right]$ and $y(y^o\,|y^o) = y^o$ for $y^o \geq g^2(y^o)$, we must have $y\left(g^2(y^o)\,|y^o\right) \geq g^2(y^o)$, so $y(g\,|y^o) \geq g^2(y^o)$ *for* $g \in \left[g^2(y^o), y^o\right]$. Similarly, $y(y^o\,|y^o) = y^o$ implies $y(g\,|y^o) \leq y^o$ *for* $g \in \left[g^2(y^o), y^o\right]$. ■

Step 2. We now construct the value functions corresponding to each steady state y^o. For

$g \in \left[g^2(y^o), y^o\right]$ define the value function recursively as

$$v(g|y^o) = \frac{W + (1-d)g - y\left(g\,|y^o\,\right)}{n} + u(y\left(g\,|y^o\,\right)) + \delta v(y\left(g\,|y^o\,\right)). \tag{11}$$

By Theorem 3.3 in Stokey, Lucas, and Prescott (1989), the right hand side of (11) is a contraction: it defines a unique, continuous and differentiable value function $v(g\,|y^o\,)$ for this interval of g. (Differentiability follows from the differentiability of $y\left(g\,|y^o\,\right)$). Note that $y\left(g\,|y^o\,\right) = \widetilde{y}\left(g\,|y^o\,\right)$ for any $g \in \left[g^2(y^o), y^o\right]$ and, by Lemma A.1, $\widetilde{y}\left(g\,|y^o\,\right) \in \left[g^2(y^o), y^o\right]$ *for* $g \in \left[g^2(y^o), y^o\right]$. From the definition of $\widetilde{y}\left(g\,|y^o\,\right)$ and the discussion in Section 4 in the paper, it follows that $u'(g)+\delta v'(g;y^o) = 1$ for any $g \in \left[g^2(y^o), y^o\right]$. In the rest of the state space we define the value function recursively. In $\left[g^1(y^o), g^2(y^o)\right]$, if $g^1(y^o) < g^2(y^o)$, the value function is defined as:

$$v(g\,|y^o\,) = \frac{W + (1-d)g - y\left(g^2(y^o)\,|y^o\right)}{n} + u(y\left(g^2(y^o)\,|y^o\,\right)) + \delta v(y\left(g^2(y^o)\,|y^o\,\right)) \tag{12}$$

where $v(y\left(g^2(y^o)\,|y^o\,\right))$ is well defined since $y\left(g^2(y^o)\,|y^o\,\right) \in \left[g^2(y^o), y^o\right]$.

Lemma A.2. *For* $g \in \left[g^1(y^o), y^o\right]$, $u(g) + \delta v(g\,|y^o\,)$ *is concave with slope larger or equal than* 1.

Proof. If $g^1(y^o) = g^2(y^o)$, the result is immediate. Assume therefore, $g^1(y^o) < g^2(y^o)$. In this case $g^2(y^o) = y^*(\delta, d, n)$. For any $g \in \left[g^1(y^o), g^2(y^o)\right]$, $y(g;y^o) = y\left(y^*(\delta, d, n)\,|y^o\right)$. So we have $v'(g\,|y^o\,) = (1-d)/n$ implying: $u'(g) + \delta v'(g\,|y^o\,) = u'(g) + \delta(1-d)/n > 1$ since $g \le g^2(y^o) = y^*(\delta, d, n)$. ■

Consider $g < g^1(y^o)$. In $\left[g_{-1}, g^1(y^o)\right]$ the value function is defined as:

$$v(g\,|y^o\,) = u(W + (1-d)g) + \delta v(W + (1-d)g\,|y^o\,) \tag{13}$$

where $g_{-1} = \max\left\{0, \left[g^1(y^o) - W\right]/(1-d)\right\}$. Assume that we have defined the value function in $g \in \left[g_{-t}, g_{-(t-1)}\right]$ as v_{-t}, for all t such that $g_{-(t-1)} > 0$. Then we can define $v_{-(t+1)}$ as (13) in $\left[g_{-(t+1)}, g_{-t}\right]$ with $g_{-(t+1)} = \left[g_{-t} - W\right]/(1-d)$.

Lemma A.3. *For* $g \in [0, y^o]$, $u(g) + \delta v(g\,|y^o\,)$ *is concave with slope greater than or equal than* 1.

Proof. We prove this by induction on t. Consider now the interval $\left[\left[g^1(y^o) - W\right]/(1-d), g^1(y^o)\right]$. In this range we have $v'(g\,|y^o\,) = \left[u'(W + (1-d)g) + \delta v'(W + (1-d)g\,|y^o\,)\right](1-d) \ge 1-d$, since $W + (1-d)g \in \left[g^1(y^o), y^o\right]$. It follows that for $g \in \left[\left[g^1(y^o) - W\right]/(1-d), g^1(y^o)\right]$: $u'(g) + \delta v'(g\,|y^o\,) \ge u'(g) + \delta(1-d) > 1$. Where the last inequality follows from the fact that $g \le g^1(y^o) < \overline{y}(\delta, d)$. We conclude that $u'(g) + \delta v'_{-1}(g\,|y^o\,)$ is concave, it has derivative larger than 1. Assume that we have shown that for $g \in \left[g_{-t}, g^3(y^o)\right]$, $u(g) + \delta v_{-t}(g\,|y^o\,)$ is concave and $u'(g) + \delta v'_{-t}(g\,|y^o\,) > 1$. Consider in $g \in \left[g_{-(t+1)}, g_{-t}\right]$. We have:

$$v'(g\,|y^o\,) = \left[u'(W + (1-d)g) + \delta v'(W + (1-d)g\,|y^o\,)\right](1-d) \ge 1-d$$

since $W + (1-d)g \geq \left[g_{-t}, g^3(y^o)\right]$. So $u'(g) + \delta v'(g\,|y^o\,) \geq u'(g) + \delta(1-d) \geq 1$. By the same argument as above, moreover, v is concave at g_{-t}. We conclude that for any $g \leq g^1$, $u(g) + \delta v(g\,|y^o\,)$ is concave and it has slope larger than 1. ■

For $g \in \left(y^o, g^3(y^o)\right]$ we define the value function as: $v(g\,|y^o\,) = \frac{W+(1-d)g-y^o}{n} + u(y^0) + \delta v(y^o\,|y^o\,)$.

Lemma A.4. *For $g \leq g^3(y^o)$, $u(g) + \delta v(g\,|y^o\,)$ is concave with slope less than or equal than 1.*

Proof. For $g \in \left(y^o, g^3(y^o)\right]$, $v'(g\,|y^o\,) = (1-d)/n$. Since $g \geq y^o \geq y^*(\delta, d, n)$, we have $u'(g) + \delta v'(g\,|y^o\,) = u'(g) + \delta\,(1-d)/n < 1$. Previous lemmas imply $u(g) + \delta v(g|y^o)$ is concave and has slope greater than or equal than 1 for $g \leq g^3(y^o)$. ■

Finally consider $g > g^3(y^o)$.

Lemma A.5. *For any $g \geq g^3(y^o)$, $u(g) + \delta v(g\,|y^o\,)$ has slope less than or equal than 1.*

Proof. In $g > g^3(y^o)$, we must have $(1-d)g \in \left[y^o, g^3(y^o)\right]$. From the proof of Lemma A.5 we know that $u'(g) + \delta v'(g) < 1$ for $g \in \left[y^o, g^3(y^o)\right]$, so we have:

$$v'(g) = (1-d)\left[u'((1-d)g) + \delta v'((1-d)g)\right] < 1-d$$

for $g > g^3(y^o)$. This fact implies that $u'(g) + \delta v'(g) < u'(g) + \delta\,(1-d)$ for any $g > g^3(y^o)$. Since $g^3(y^o) > \overline{y}(\delta, d)$ we have $u'(g) + \delta\,(1-d) < u'(\overline{y}(\delta, d)) + \delta\,(1-d) = 1$ for $g > g^3(y^o)$. It follows that $v^*(g)$ is has slope lower than 1 in $g > g^3(y^o)$. ■

From Lemmata A1-A5 we conclude that $u(g) + \delta v(g\,|y^o\,)$ has a global maximum at any $g \in \left[g^3(y^o), y^o\right]$.

Step 3. Define $x(g\,|y^o\,) = \left[W + (1-d)g - y(g\,|y^o\,)\right]/n$ and $i(g\,|y^o\,) = \left[y(g\,|y^o\,) - (1-d)g\right]/n$ as the levels of per capita private consumption and investment, respectively. Note that by construction, $x(g\,|y^o\,) \in [0, W/n]$. We now establish that $y(g\,|y^o\,)$, $x(g\,|y^o\,)$ and the associated value function $v\,(g\,|y^o\,)$ defined in the previous steps constitute an equilibrium. The fact that $v\,(g\,|y^o\,)$ describes the expected continuation value to an agent follows by construction. To see that $y(g\,|y^o\,)$ is an optimal reaction function given $v(g\,|y^o\,)$, note that an agent solves the following problem:

$$\max_y \left\{ \begin{array}{c} u(y) - y + \delta v(y) \\ y \leq \frac{W+(1-d)g}{n} + \frac{n-1}{n}y(g),\; y \geq \frac{(1-d)g}{n} + \frac{n-1}{n}y(g) \end{array} \right\} \tag{14}$$

where $y(g) = y(g\,|y^o\,)$. The investment function $y(g\,|y^o\,)$ satisfies the constraints of this problem if $y(g\,|y^o\,) \leq \frac{W+(1-d)g}{n} + \frac{n-1}{n}y(g\,|y^o\,)$, so if $y(g\,|y^o\,) \leq W + (1-d)g$; and if $y(g\,|y^o\,) \geq \frac{(1-d)g}{n} + \frac{n-1}{n}y(g\,|y^o\,)$, so if $y(g\,|y^o\,) \geq (1-d)\,g$. Both conditions are automatically satisfied by construction. If $g < g^1(y^o)$, we have $u'(y) + \delta v'(y) \geq 1$ for all $y \in [(1-d)g, W + (1-d)g]$, so $y(g\,|y^o\,) =$

$W + (1-d)g$ is optimal. If $g \geq g^3(y^o)$, $u'(y) + \delta v'(y) < 1$ for all $y \in [(1-d)g, W + (1-d)g]$, so $y(g\,|y^o) = (1-d)g$. In $g \in \left(g^1(y^o), g^3(y^o)\right]$ a point maximizing $u(y) + \delta v(y)$ is feasible and chosen, so again $y(g\,|y^o)$ is an optimal choice. ■

7.2 Proof of Proposition 2

Consider a sequence $d^m \to 0$. For each d^m there is at least an associated equilibrium $y_m(g)$, $v_m(g)$ with steady state y_m^o. To prove the result we proceed in two steps. In Section 7.2.1 we prove that for any $\xi > 0$, there is a $\widetilde{m}$ such that for $m > \widetilde{m}$, $\underline{y}_{IR}(\delta, d^m, n) \geq [u']^{-1}(1-\delta) - \xi$. In Section 7.2.2 we prove that the steady state of any equilibrium can not be larger than $[u']^{-1}(1-\delta(1-d/n))$. Since, as shown in Proposition 1, $[u']^{-1}(1-\delta(1-d/n))$ is an equilibrium steady state for any $d \geq 0$ and it converges to $[u']^{-1}(1-\delta)$, we must have $\left|\overline{y}_{IR}(\delta, d, n) - \underline{y}_{IR}(\delta, d, n)\right| \to 0$ as $d \to 0$. In Lemmata A.6 and A.7 presented in Section 7.2.2 we show that $y'(g) \in (0,1)$ in a left neighborhood of the steady state y^o if $y^o > [u']^{-1}(1-\delta(1-d)/n)$. Since all equilibrium steady states converge to $[u']^{-1}(1-\delta) > [u']^{-1}(1-\delta/n)$, this implies that that convergence of g to the steady state is gradual in all equilibria if d is sufficiently small.

7.2.1 The lower bound

We prove the result by contradiction. Suppose to the contrary there is a sequence of steady states y_m^0, with associated equilibrium investment and value functions $y_m(g)$, $v_m(g)$, and an $\xi > 0$ such that $y_m^0 < \overline{y}(0) - \xi$ for any arbitrarily large m, where $\overline{y}(d) = [u']^{-1}(1-\delta(1-d))$. Define $y_m^0(g) = y_m(g)$, and $y_m^j(g) = y_m(y_m^{j-1}(g))$ and consider a marginal deviation from the steady state from y_m^0 to $y_m^0 + \Delta$. By the irreversibility constraint we have $y_m(g) \geq (1-d^m)\,g$. Using this property and the fact that y_m^0 is a steady state, so $y_m^j(y_m^0) = y_m^0$, we have:

$$y_m(y_m^0 + \Delta) - y_m(y_m^0) \geq (1-d^m)\left(y_m^0 + \Delta\right) - y_m^0 = (1-d^m)\,\Delta - d^m y_m^0$$

This implies that, as $m \to \infty$, for any given Δ: $\left[y_m(y_m^0 + \Delta) - y_m^0\right]/\Delta \geq 1 + o_1(d^m)$ where $o_1(d^m) \to 0$ as $m \to 0$. We now show with an inductive argument that a similar property holds for all iterations $y_m^j(y_m^0)$. Assume we have shown that: $\left[y_m^{j-1}(y_m^0 + \Delta) - y_m^0\right]/\Delta \geq 1 + o_{j-1}(d^m)$ where $o_{j-1}(d^m) \to 0$ as $m \to 0$. We must have: $y_m(y_m^{j-1}(y_m^0+\Delta)) - y_m^j(y_m^0) \geq (1-d^m)\,y_m^{j-1}(y_m^0 + \Delta) - y_m^0$. We therefore have: $y_m(y_m^{j-1}(y_m^0 + \Delta)) - y_m^0 \geq y_m^{j-1}(y_m^0 + \Delta) - y_m^0 - d^m y_m^{j-1}(y_m^0 + \Delta)$ so we have:

$$\frac{y_m(y_m^{j-1}(y_m^0 + \Delta)) - y_m^0}{\Delta} \geq \frac{y_m^{j-1}(y_m^0 + \Delta) - y_m^0}{\Delta} - \frac{d^m y_m^{j-1}(y_m^0 + \Delta)}{\Delta} \geq 1 + o_j(d^m) \tag{15}$$

where $o_j(d^m) = o_{j-1}(d^m) - \frac{d^m y_m^{j-1}(y_m^0+\Delta)}{\Delta}$, so $o_j(d^m) \to 0$ as $m \to 0$.

We can write the value function after the deviation to $y_m^0 + \Delta$ as:

$$V(y_m^0 + \Delta) = \sum_{j=0}^{\infty} \delta^{j-1} \left[\frac{W + (1 - d^m)\, y_m^{j-1}(y_m^0 + \Delta) - y_m^j(y_m^0 + \Delta)}{n} + u(y_m^j(y_m^0 + \Delta)) \right]$$

For any given function $f(x)$, define $\Delta f(x) = f(x + \Delta) - f(x)$. We can write:

$$\begin{aligned} \Delta V(y_m^0)/\Delta &= \sum_{j=0}^{\infty} \delta^{j-1} \left[\begin{array}{c} \frac{(1-d^m)\Delta y_m^{j-1}(y_m^0)/\Delta - \Delta y_m^j(y_m^0)/\Delta}{n} \\ + \left[u'(y_m^0) + o\,(\Delta)\right] \Delta y_m^j(y_m^0)/\Delta \end{array} \right] \\ &\geq \sum_{j=0}^{\infty} \delta^{j-1} \left[\begin{array}{c} \frac{(1-d^m)(1+o_{j-1}(d^m)) - (1+o_j(d^m))}{n} \\ + \left[u'(y_m^0) + o\,(\Delta)\right] (1 + o_j\,(d^m)) \end{array} \right] \end{aligned} \tag{16}$$

where $o\,(\Delta) \to 0$ as $\Delta \to 0$. In the first equality we use the fact that if we choose Δ small, since $y_m(g)$ is continuous, $\Delta y_m^j(y_m^0)$ is small as well. This implies that

$$\left(u(y_m^j(y_m^0 + \Delta)) - u(y_m^j(y_m^0))\right) / \left[y_m^j(y_m^0 + \Delta) - y_m^j(y_m^0)\right]$$

converges to $u'(y_m^j(y_m^0))$ as $\Delta \to 0$. The inequality in 16 follows from (15). Given Δ, as $m \to \infty$, we therefore have $\lim_{m\to\infty} \Delta V(y_m^0)/\Delta \geq \frac{u'(y_m^0)+o(\Delta)}{1-\delta}$. We conclude that for any $\varepsilon > 0$, there must be a Δ_ε such that for any $\Delta \in (0, \Delta_\varepsilon)$ there is a m_Δ guaranteeing that $\Delta V(y_m^0)/\Delta \geq \frac{u'(y_m^0)}{1-\delta} - \varepsilon$ for $m > m_\Delta$. After a marginal deviation to $y_m^0 + \Delta$, therefore, the change in agent's objective function is:

$$u'(y_m^0) + \delta \Delta V(y_m^0)/\Delta - 1 \geq \frac{u'(y_m^0)}{1-\delta} - \delta\varepsilon - 1$$

for m sufficiently large. A necessary condition for the un-profitability of a deviation from y_m^0 to $y_m^0 + \Delta$ is therefore: $y_m^0 \geq [u']^{-1} (1 - \delta + \delta\varepsilon\,(1 - \delta))$. Since ε can be taken to be arbitrarily small, for an arbitrarily large m, this condition implies $y_m^0 \geq \overline{y}(0) - \xi/2$, which contradicts $y_m^0 < \overline{y}(0) - \xi$. We conclude that $\underline{y}_{IR}(\delta, d, n) \to \overline{y}(0)$ as $d \to 0$.

7.2.2 The upper bound

Suppose to the contrary that there is stable steady state at $y^o > [u']^{-1} (1 - \delta\,(1 - d/n))$. We must have $y^o \in \left([u']^{-1} (1 - \delta\,(1 - d/n)), W/d\right]$, since it is not feasible for a steady state to be larger than W/d. Consider a left neighborhood of y^o, $N_\varepsilon(y^o) = (y^o - \varepsilon, y^o)$. The value function can be written in $g \in N_\varepsilon(y^o)$ as:

$$v(g) = u(y(g)) + \delta v(y(g)) - y(g) + \frac{W + (1-d)g}{n} + (1 - 1/n)\, y(g) \tag{17}$$

where $y(g)$ is the equilibrium strategy associated to y^o. In $N_\varepsilon(y^o)$ the constraint $y \geq \frac{1-d}{n} g + \frac{n-1}{n} y(g)$ cannot be binding (else we would have $y(g) = (1-d)g$, but this is not possible in a neighborhood of $y^o > 0$). We consider two cases.

Case 1. Suppose first that $y^o < W/d$. We must therefore have that $y(g) < W + (1-d)g$ in $N_\varepsilon(y^o)$, so the constraint $y \le \frac{W+(1-d)g}{n} + \frac{n-1}{n}y$ is not binding. The solution is in the interior of the constraint set of (14), and the objective function $u(y(g)) + \delta v(y(g)) - y(g)$ is constant for $g \in N_\varepsilon(y^o)$.

Lemma A.6. *If $y^o > [u']^{-1}(1 - \delta(1-d)/n)$, then there is a left neighborhood $N_\varepsilon(y^o)$ in which $y(g)$ is not constant.*

Proof. Suppose to the contrary that, for any $N_\varepsilon(y^o)$, there is an interval in $N_\varepsilon(y^o)$ in which $y(g)$ is constant. Using the expression for $v(g)$ presented above, we must have $v'(g) = (1-d)/n$ for any g in this interval. Since $N_\varepsilon(y^o)$ is arbitrary, then we must have a sequence $g^m \to y^o$ such that $v'(g^m) = (1-d)/n \ \forall m$. We can therefore write:

$$\begin{aligned}
\lim_{\Delta\to 0} \frac{v(y^o) - v(y^o - \Delta)}{\Delta} &= \lim_{\Delta\to 0}\lim_{m\to\infty} \frac{v(g^m) - v(g^m - \Delta)}{\Delta} \\
&= \lim_{m\to\infty}\lim_{\Delta\to 0} \frac{v(g^m) - v(g^m - \Delta)}{\Delta} = \frac{1-d}{n}
\end{aligned}$$

where the second equality follows from the continuity of $v(g)$. This implies that $v^-(y^o)$, left derivative of $v(g)$ at y^o, is well defined and equal to $\frac{1-d}{n}$. Consider now a marginal reduction of g at y^o. The change in utility is (as $\Delta \to 0$):

$$\begin{aligned}
\Delta U(y^o) &= u(y^o - \Delta) - u(y^o) + \delta\left[v(y^o - \Delta) - v(y^o)\right] + \Delta \\
&= \left[1 - \left(u'(y^o) + \delta\frac{1-d}{n}\right)\right]\Delta
\end{aligned}$$

In order to have $\Delta U(y^o) \le 0$, we must have $u'(y^o) + \delta(1-d)/n \ge 1$. This implies $y^o \le [u']^{-1}(1 - \delta(1-d)/n)$, a contradiction. Therefore, if there is stable steady state at $y^o > [u']^{-1}(1 - \delta(1-d)/n)$, then $y(g)$ is not constant in $N_\varepsilon(y^o)$. ■

Lemma A.6 implies that there is a left neighborhood $N_\varepsilon(y^o)$ in which $u(g) + \delta v(g) - g$ is constant if $y^o > [u']^{-1}(1 - \delta(1-d)/n)$ (since otherwise $y(g)$ would be constant). Moreover, since y^o is a stable steady state and $y(g)$ is strictly increasing, $g \in N_{\varepsilon'}(y^o)$ implies $y(g) \in N_{\varepsilon'}(y^o))$ for any open left neighborhood $N_{\varepsilon'}(y^o) = (y^o - \varepsilon', y^o) \subset N_\varepsilon(y^o)$. These observations imply:

Lemma A.7. *If $y^o > [u']^{-1}(1 - \delta(1-d)/n)$, then there is a left neighborhood $N_\varepsilon(y^o)$ in which*

$$y'(g) = \frac{n}{n-1}\left(\frac{1-u'(g)}{\delta} - \frac{1-d}{n}\right) \tag{18}$$

Proof. There is a $N_\varepsilon(y^o)$ and a constant K such that $\delta v(g) = K + g - u(g)$ for $g \in N_\varepsilon(y^o))$. Hence $v(g)$ is differentiable in $N_\varepsilon(y^o)$. Moreover, $y(g) \in N_\varepsilon(y^o)$ for all $g \in N_\varepsilon(y^o)$. Hence $u(y(g)) + \delta v(y(g)) - y(g)$ is constant in $g \in N_\varepsilon(y^o)$ as well. These observations and the definition of $v(g)$ imply that $v'(g) = \frac{1-d}{n} + \left(1 - \frac{1}{n}\right)y'(g)$ in $N_\varepsilon(y^o)$. Given that $u'(g) + \delta v'(g) = 1$ in

$g \in N_\varepsilon(y^o)$, we must have: $u'(g) + \delta v'(g) = u'(g) + \delta\left[\frac{1-d}{n} + \left(1 - \frac{1}{n}\right) y'(g)\right] = 1$ which implies (18) for any $g \in N_\varepsilon(y^o)$. $\blacksquare$

Let g^m be a sequence in $N_\varepsilon(y^o)$ such that $g^m \to y^o$. We must have

$$
\begin{aligned}
y^-(y^o) &= \lim_{\Delta \to 0} \frac{y(y^o) - y(y^o - \Delta)}{\Delta} = \\
&= \lim_{\Delta \to 0} \lim_{m \to \infty} \frac{y(g^m) - y(g^m - \Delta)}{\Delta} = \\
&= \lim_{m \to \infty} \lim_{\Delta \to 0} \frac{y(g^m) - y(g^m - \Delta)}{\Delta} = \frac{n}{n-1}\left(\frac{1 - u'(y^o)}{\delta} - \frac{1-d}{n}\right) \qquad (19)
\end{aligned}
$$

where $y^-(y^o)$ is the left derivative of $y(g)$ at y^o, the second equality follows from continuity and the last equality follows from Lemma A.7 and the fact that under the starting assumption we have $y^o > [u']^{-1}(1 - \delta(1 - d/n)) > [u']^{-1}(1 - \delta(1-d)/n)$. Consider a state $(y^o - \Delta)$. For y^o to be stable we need that for any small Δ:

$$
y(y^o - \Delta) \geq y^o - \Delta = y(y^o) + (y^o - \Delta) - y^o
$$

where the equality follows from the fact that $y(y^o) = y^o$. As $\Delta \to 0$, this implies $y^-(y^o) \leq 1$ in $N_\varepsilon(y^o)$. By (19), we must therefore have: $\frac{n}{n-1}\left(\frac{1-u'(y^o)}{\delta} - \frac{1-d}{n}\right) \leq 1$. This implies: $y^o \leq [u']^{-1}(1 - \delta(1-d)/n)$, a contradiction.

Case 2. Assume now that $y^o = W/d$ and consider first the case in which it is a strict local maximum of the objective function $u(y) + \delta v(y) - y$. In this case in a left neighborhood $N_\varepsilon(y^o)$, we have that the upper bound $y \leq \frac{W + (1-d)g}{n} + \frac{n-1}{n} y(g)$ is binding: implying $y(g) = W + (1-d)g$ in $N_\varepsilon(y^o)$. We must therefore have a sequence of points $g^m \to y^o$ such that $g^m = y(g^{m-1})$ and $y(g^m) = W + (1-d)g^m \; \forall m$. Given this, we can write:

$$
\begin{aligned}
v(g^m) &= u(g^{m+1}) + \delta v(g^{m+1}) = u(g^{m+1}) + \delta\left[u(g^{m+2}) + \delta v(g^{m+2})\right] \\
&= \sum\nolimits_{j=0}^{\infty} \delta^j u(W + (1-d)g^{m+j})
\end{aligned}
$$

We therefore must have that $v(g^m)$ is differentiable and $\delta v'(g^m) = \sum\nolimits_{j=0}^{\infty} [\delta(1-d)]^{j+1} u'(W + (1-d)g^{m+j})$. Since $u'(g^m) + \delta v'(g^m) \geq 1$, we have $u'(g^m) + \sum\nolimits_{j=0}^{\infty} [\delta(1-d)]^{j+1} u'(W + (1-d)g^{m+j}) \geq 1$ for all m. Consider the limit as $m \to \infty$. Since $u'(g)$ is continuous and $g^m \to y^o$, we have:

$$
\begin{aligned}
1 &\leq \lim_{m \to \infty} \left[u'(g^m) + \sum\nolimits_{j=0}^{\infty} [\delta(1-d)]^{j+1} u'(W + (1-d)g^{m+j})\right] \\
&= u'(y^o) + \sum\nolimits_{j=0}^{\infty} [\delta(1-d)]^{j+1} u'(y^o) = \frac{u'(y^o)}{1 - \delta(1-d)}
\end{aligned}
$$

This implies $y^o \leq [u']^{-1}(1 - \delta(1-d)) < [u']^{-1}(1 - \delta(1 - d/n))$, a contradiction. Assume now that $y^o = W/d$, but it is not a strict maximum of $u(y) + \delta v(y) - y$ in any left neighborhood. It must be that $u(y) + \delta v(y) - y$ is constant in some left neighborhood $N_\varepsilon(y^o)$. If this were not the case, then in any left neighborhood we would have an interval in which $y(g)$ is constant, but this is impossible by Lemma A.6. But then if $u(y) + \delta v(y) - y$ is constant in some $N_\varepsilon(y^o)$, the same argument as in Step 1 implies a contradiction. ■

7.3 Proof of Proposition 4

We first show that there is a $\delta_1 < 1$, such that for $\delta > \delta_1$ the efficient path is a SPE path in a irreversible investment economy. To this goal, we first define the equilibrium strategies and establish some key properties. Let $y^M(g; d, \delta)$, $v^M(g; d, \delta)$ be, respectively, the investment function and the value function of the Markov equilibrium with the lowest steady state characterized in Proposition 2 when the discount factor is δ and the rate of depreciation is d. Let $g^M(d, \delta) = [u']^{-1}(1 - \delta(1-d)/n)$ be the associated steady state. It is easy to see that, for any d and n, $g^M(d, \delta) < y_P^*(\delta, d, n)$ for all $\delta \in [0, 1]$. Define $y_j^M(g; d, \delta)$ recursively with $y_0^M(g; d, \delta) = g$ and $y_j^M(g; d, \delta) = y^M(y_{j-1}^M(g; d, \delta); d, \delta)$. For any g, $y_j^M(g; d, \delta) \to g^M(d, \delta)$ as $j \to \infty$. It follows that $\lim_{\delta \to 1} \left[(1 - \delta)\, v^M(g; d, \delta)\right] = \left(W - dg^M(d, 1)\right)/n + u(g^M(d, 1))$. Let $y^P(g; d, \delta)$ be the efficient investment function characterized in Section 3 with steady state $g^P(d, \delta) = y_P^*(\delta, d, n)$, and let $v^P(g; d, \delta)$ be the associated expected utility for a player. Similarly, it is easy to see that $\lim_{\delta \to 1} \left[(1 - \delta)\, v^P(g; d, \delta)\right] = \left(W - dg^P(d, 1)\right)/n + u(g^P(d, 1))$, where $y^P(g; d, \delta)$ be the efficient investment function characterized in Section 3 with steady state $g^P(d, \delta) = y_P^*(\delta, d, n))$. It follows that $\lim_{\delta \to 1} \left[(1 - \delta)\, v^P(g; d, \delta)\right] > \lim_{\delta \to 1} \left[(1 - \delta)\, v^M(g; d, \delta)\right]$.

Associated to an aggregate investment function $y^l(g; d, \delta)$, $l = \{M, P\}$, we have the individual contribution function: $i^l(g; d, \delta) = \left[y^l(g; d, \delta) - (1 - d)g\right]/n$. To construct the equilibrium, consider the following trigger strategies. If $g_\tau = y_\tau^P(g_0; d, \delta)$ for all $\tau \leq t$, then $i^t(g_t; d, \delta) = i^P(g; d, \delta)$, where $i_j^t(g_t)$ is the investment at time t of an agent. If $\exists \tau \leq t$ such that $g_\tau \neq y_\tau^P(g_0; d, \delta)$, then $i^t(g_t) = i^M(g; d, \delta)$. Note that, by construction, deviations are not profitable after a τ such that $g_\tau \neq y_\tau^P(g_0; d, \delta)$. For the remaining histories note that the average utility of a deviating agent must converge to $(1 - \delta)\, v^M(g; d, \delta) < (1 - \delta)\, v^P(g; d, \delta)$, so there must be a $\delta_1 < 1$, such that for $\delta > \delta_1$ no deviation is profitable.

The result that we also have a $\delta_2 < 1$, such that for $\delta > \delta_2$ the efficient path is a SPE path in a reversible investment economy can be proven analogously. From Battaglini et al. [2012], we know that there is a Markov equilibrium $\widetilde{y}^M(g; d, \delta)$, $\widetilde{v}^M(g; d, \delta)$ with steady state $\widetilde{g}^M(d, \delta) \leq [u']^{-1}(1 - \delta(1 - d)/n)$, and so strictly lower than the steady state $g^P(d, 1)$ of the planner's solution for all $\delta \in [0, 1]$. Proceeding exactly as above we can see that $\lim_{\delta \to 1} \left[(1 - \delta)\, v^P(g; d, \delta)\right] > \lim_{\delta \to 1} \left[(1 - \delta)\, \widetilde{v}^M(g; d, \delta)\right]$. Associated to an aggregate investment function $\widetilde{y}^M(g; d, \delta)$ we define as above the individual contribution function: $\widetilde{i}^M(g; d, \delta) = \left[\widetilde{y}^M(g; d, \delta) - (1 - d)g\right]/n$. To construct the equilibrium, consider the following trigger strategies. If $g_\tau = y_\tau^P(g_0; d, \delta)$ for all $\tau \leq t$, then $i^t(g_t; d, \delta) = i^P(g; d, \delta)$, where $i^t(g_t)$ is the investment at time t of an agent. If $\exists \tau \leq t$ such that $g_\tau \neq y_\tau^P(g_0; d, \delta)$, then $i^t(g_t) = \widetilde{i}^M(g; d, \delta)$. Note that, by construction, deviations are not profitable after a τ such that $g_\tau \neq y_\tau^P(g_0; d, \delta)$. For the remaining histories note that the average utility of a deviating agent must converge to $(1-\delta)\, v^M(g; d, \delta) < (1-\delta)\, v^P(g; d, \delta)$, so there must be a $\delta_2 < 1$, such that for $\delta > \delta_2$ no deviation is profitable. Given this, the statement of the proposition follows immediately by defining $\overline{\delta} = \max(\delta_1, \delta_2)$. ■