Dynamic Strategic Information Transmission

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Abstract

This paper studies strategic information transmission in a finite-horizon environment where, each period, a privately informed expert sends a message and a decision-maker takes an action. Our main contribution is to show that communication in this dynamic environment is drastically different from in a one-shot game. First, full information revelation is possible; we provide a constructive method to build such fully revealing equilibria. Second, we show that complicated communication, where far-away types pool together, allows for dynamic manipulation of beliefs to enable better information release in the future. Finally, we show that conditioning future messages on past actions improves incentives for information revelation.

Keywords: asymmetric information; cheap talk; dynamic strategic communication; full information revelation.

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Biased experts impede information transmission, which has serious consequences in many situations: Worse projects are financed, beneficial reforms are blocked, and firms may fail to reward the most productive employees. The seminal analysis of strategic information transmission by Crawford and Sobel (1982) has had a number of applications, ranging from economics and political science to philosophy and biology.[1] In their paper, a biased and privately-informed expert and a decision-maker interact only once. The conflict of interest between them results in coarse information revelation, and in some cases, in no information revelation at all. There are, however, many environments in which the expert and receiver interact repeatedly and information transmission is dynamic. Many sequential decisions have to take place, and the decision-maker seeks the expert's advice prior to each one of them.

We study strategic information transmission in a dynamic, finite-horizon extension of the Crawford and Sobel setup. Each period, the expert sends a message and the decision-maker takes an action. Only the expert knows the state of the world, which remains constant throughout the game. We maintain all other features of the Crawford and Sobel (1982) environment, in particular the conflict of interest between expert and decision-maker. The goal is to investigate the extent to which conflicts of interest impede information transmission in multi-period interactions.

Our most surprising and difficult-to-establish finding (Theorem 1) shows that full information revelation is possible. We show this result in a challenging environment where the horizon is finite, and both players are fully patient. The construction of the fully revealing equilibrium relies on two key novel features. The first is the use of what we call "separable groups": the expert employs a signaling rule in which far-apart types pool together initially, but eventually find it optimal to separate and reveal the truth. The second feature is to make advice contingent on actions: the expert promises to reveal the truth later, but only if the decision-maker follows his advice now; this initial advice, in turn, is designed to reward the expert for revealing information. In a nutshell, communication in a multi-period interaction is facilitated via an initial signaling rule that manipulates posteriors (in a way that enables precise information release in the future), initial actions which reward the expert for employing this signaling rule, and trigger strategies which reward the decision-maker for choosing these initial actions. Moving from a one-shot to a finitely-repeated game often leaves the qualitative feature of equilibria unchanged; we show here that finite repetition has a significant impact on the equilibrium nature of strategic communication.

We now explain in more detail our construction of a fully revealing equilibrium. We first show that it is possible to divide all states into separable groups. A separable group is a set of states (types) which are sufficiently far apart that each type would rather reveal the truth, than mimic any other type in his group. The expert's initial signaling rule then reveals the separable group

[1] For a survey with applications across disciplines see Sobel (2008).

containing the truth. Therefore, this creates histories after which it is common knowledge that the decision-maker puts probability one on a particular separable group, at which point the types in this group will find it optimal to separate. The division of all types into separable groups is quite delicate, because, given that there is a continuum of types, we need to form a continuum of such groups. The expert anticipates that once he joins a separable group, he will forgo his informational advantage. For the expert to join the separable group containing his true type, we have to make sure that he does not want to mimic a nearby type by joining some other separable group. This is accomplished via our choice of initial actions, which ensure that any future gain to the expert from mimicking some other type is offset by the initial cost. These expert-incentivizing actions are not myopically optimal for the decision-maker, so we employ trigger strategies: the expert (credibly) threatens to babble in the future if the decision-maker fails to choose the actions that he recommends at the beginning. The final part of the proof then shows that we can design the separable groups and initial actions such that the decision-maker would rather follow the expert's initial advice, knowing that he will then eventually learn the exact truth, than choose the myopically optimal action in the initial periods, knowing that he will then never learn more than the separable group containing the truth.

In a follow-up section (Section 4.1), we adapt our construction to a continuous-time setting, obtaining some more attractive results and generalizations. In particular, Theorem 1 proves that full information revelation is possible when the decision-maker and expert are both perfectly patient, but only for some horizons and some priors (held by the decision-maker) over the state space; moreover, the welfare properties of the equilibrium are both difficult to calculate, and in some cases, not very appealing. Proposition 4 shows that with a trivial modification to the timeline, and for the same set of priors covered by Theorem 1, our construction yields also a fully revealing equilibrium for any pair of discount factors, so long as the decision-maker is at least as patient as the expert. Our second main result, Theorem 2, shows that in a continuous-time setting with an impatient expert (positive discount rate), a sufficiently patient decision-maker, and a sufficiently long horizon, our fully revealing equilibrium works for nearly all priors over the state space, and moreover the decision-maker's average payoff loss (compared to a full information setting) goes to zero as he becomes perfectly patient.

We demonstrate several additional differences between dynamic and static communication games. First, we emphasize that fully revealing equilibria cannot have the monotonic partitional structure from Crawford-Sobel (1982): if attention is restricted to monotonic partition equilibria, learning quickly stops. Moreover, we argue that non-monotonic equilibria can be strictly Pareto superior to all dynamic monotonic equilibria. Welfare properties of equilibria also differ in a dynamic setup. Crawford and Sobel (1982) show that, ex ante, both the expert and the decision-maker will (under typical assumptions) prefer equilibria with more partitions. We provide an example that shows that this is not necessarily the case for dynamic equilibria.[2] We also present an example showing that dynamic monotonic partition equilibria can Pareto-dominate all static equilibria, and an example showing that non-monotonic equilibria can strictly Pareto-dominate the best dynamic monotonic equilibrium.

Our work shows that the nature of dynamic strategic communication is quite distinct from its static counterpart. In the static case, because of the conflict of interest between the decision-maker and the expert, nearby expert types have an incentive to pool together, precluding full information revelation. The single-crossing property also implies that at equilibrium, the action is a monotonic step function of the state. These two forces make complex signaling (even though possible) irrelevant. In our dynamic setup, the key difference is that today's communication sets the stage for tomorrow's communication. Complex signaling helps in this setting, because it can generate posteriors that put positive probability only on expert types who are so far apart, they have no incentive to mimic each other; this is what enables fully revealing equilibria.

Related Literature

Crawford and Sobel (1982) is the seminal contribution on strategic information transmission. That paper has inspired an enormous amount of theoretical work and myriads of applications. Here we study a dynamic extension. Much of the previous work on dynamic communication has focused on the role of reputation; see, for example, Sobel (1985), Morris (2001), and Ottaviani and Sorensen (2006), (2006b). Some other dynamic studies allow for multi-round communication protocols, but with a single round of actions. Aumann and Hart (2003) characterize geometrically the set of equilibrium payoffs when a long conversation is possible. In that paper, two players–one informed and one uninformed–play a finite simultaneous move game. The state of the word is finite, and players engage in direct (no mediator) communications, with a potentially infinitely long exchange of messages, before both choosing simultaneously costly actions. In contrast, in our model, only the informed party sends messages, the uninformed party chooses actions, and the state space is infinite. Krishna and Morgan (2004) add a long communication protocol to Crawford and Sobel (1982)'s game, and Goltsman, Hőrner, Pavlov and Squintani (2009) characterize such optimal protocols. Forges and Koessler (2008a, 2008b) allow for a long protocol in a setup where messages can be certifiable. In all those papers, once the communication phase is over, the decision maker chooses one action. In our paper, there are multiple rounds of communication and actions (each expert's message is followed by an action of the decision-maker). The multiple actions correlate

[2] A similar phenomenon occurs when communication is noisy, as shown in an example of the working paper version of Blume, Board, and Kawamura (2007). In their example, a two-step partition Pareto dominates a three-step partition.

incentives in a way that was not possible in these earlier works: the expert is able to condition his advice on the decision-maker's past behavior, and additionally, the decision-maker is able to choose actions which reward the expert appropriately for following a path of advice that ultimately leads to revelation of the true state.

In our setup, the dynamic nature of communication enables full information revelation. In contrast, full information revelation is not possible in the dynamic setup of Anderlini, Gerardi, and Lagunoff (2008) who consider dynamic strategic communication in a dynastic game and show that if preferences are not fully aligned, "full learning" equilibria do not exist. Renault, Solan, and Vieille (2011) examine dynamic sender-receiver games, in a setup where the information is not fully persistent and the state is finite. They restrict attention to Markov equilibria and characterize the set of equilibrium payoffs. In contrast, we assume fully persistent information, a finite horizon, and a state drawn from a continuum, and focus mainly on non-Markovian equilibria.[3]

Our model bears some similarities to models of static strategic communication with multiple receivers. In those models the expert cares also about a sequence of actions, but in contrast to our model, the actions are chosen by different individuals. An important difference is that in our model, the receiver cares about the entire vector of actions chosen; in those models, each receiver cares only about his own action. This enables our use of trigger strategies, which we find is a necessary feature of equilibria with eventual full information revelation. Still, some of the properties of the equilibria that we obtain also appear in the models with multiple receivers. For example, our non-monotonic example presented in Section 3 resembles Example 2 of Goltsman and Pavlov (2008). It is also similar to Example 2 in Krishna and Morgan (2004).[4],[5]

Full information revelation is possible in other variations of the Crawford and Sobel (1982) setup: When the decision maker consults two experts as in Battaglini (2002), Eso and Fong (2008) and Ambrus and Lu (2010);[6] when information is completely or partially certifiable, as in Mathis (2008); and when there are lying costs and the state is unbounded as in Kartik, Ottaviani and Squintani (2007). In the case of multiple experts, playing one against the other is the main force that supports truthful revelation. In the case of an unbounded state, lying costs become large and support the truth. In the case of certifiable information, one can exploit the fact that messages are

[3]Ivanov (2011) allows for a dynamic communication protocol in a setup where the expert is also initially uninformed and the decision maker controls the quality of information available to the expert. He builds on our construction and employs separable groups, but in a much different informational setting. (The decision-maker gives the uninformed expert a device that initially reveals (to the expert only) the separable group containing the truth, and contains a built-in threat to only reveal the exact state if the expert reports this information truthfully).

[4]Equilibria can be non-monotonic also in environments where the decision maker consults two experts as in Krishna and Morgan (2001).

[5]For the expert, inducing a sequence of actions is payoff-equivalent to inducing a probability distribution over actions, so our analysis for the expert would go through also in such settings. Our set-up is however more flexible on the decision-maker's side, since conditioning advice on past actions (trigger strategies) is impossible in mediated static setups.

[6]But full information is not possible in the setup of Levy and Razin (2007).

state-contingent to induce truth telling. All these forces are very different from the forces behind our fully revealing construction.

1 Motivating Example: An Impatient Financial Advisor

One of the most stark results of the static strategic communication game is that there is no equilibrium with full information revelation. Although the state can take a continuum of values, the expert sends at most finitely many signals to the decision-maker, implying that a substantial amount of information is not transmitted.

In this example, we show how to construct a fully revealing equilibrium when the expert is myopic, using just two stages. There are two essential ingredients of this example. First, the expert types who pool together in the first period are far enough apart that they can be separated in the second period: that is, each possible first-period message is sent by a separable group of types. Second, each separable group induces the same optimal (for the decision-maker) first-period action. This implies that the expert does not care which group he joins (since a myopic expert cares only about the 1st-period action, which is constant across groups).

Example 1 *Fully revealing equilibrium with an impatient expert ($\delta_E = 0$).*

Suppose there is an expert (financial advisor) and a decision-maker (employee). The expert knows the true state of the world θ, which is drawn from a uniform distribution on $[0, 1]$ and remains constant over time. The players' payoffs in period $t \in \{1, 2\}$ depend on both the state, θ, and on the action chosen by the decision-maker, y_t. More precisely, payoffs in period t are given by

$$u_t^E(y_t, \theta, b) = -(y_t - \theta - b)^2 \text{ and } u_t^{DM}(y, \theta) = -(y_t - \theta)^2 . \tag{1}$$

where $b > 0$ is the expert's "bias". The expert is myopic, with $\delta_E = 0$; the construction works for any discount factor for the decision-maker.

The expert employs the following signaling rule. In period 1, expert types $\{\frac{1}{8}-\varepsilon; \frac{3}{8}+\varepsilon, \frac{4}{8}+\varepsilon, 1-\varepsilon\}$ pool together and send the message m_ε, for all $\varepsilon \in [0, \frac{1}{8}]$. For all state pairs $\{\frac{1}{8} + \tilde{\varepsilon}, \frac{7}{8} - \tilde{\varepsilon}\}$ with $\tilde{\varepsilon} \in (0, \frac{1}{4})$, the expert sends a message $m_{\tilde{\varepsilon}}$. That is, we have two types of separable groups, indexed by ε and $\tilde{\varepsilon}$. Given this signaling rule, the best response of the decision-maker in period 1 is to choose:

$$\begin{aligned} y_1(m_\varepsilon) &= \frac{\frac{1}{8} - \varepsilon + \frac{3}{8} + \varepsilon + \frac{4}{8} + \varepsilon + 1 - \varepsilon}{4} = 0.5 \text{ for all } \varepsilon \in \left[0, \frac{1}{8}\right], \\ y_1(m_{\tilde{\varepsilon}}) &= \frac{\frac{1}{8} + \tilde{\varepsilon} + \frac{7}{8} - \tilde{\varepsilon}}{2} = 0.5 \text{ for all } \tilde{\varepsilon} \in \left(0, \frac{1}{4}\right). \end{aligned}$$

In period 2, the expert reveals the truth, and so the decision-maker chooses an action equal to the true state. After any out-of-equilibrium initial message, the decision maker assigns equal probability to all states, leading to action $y_1^{out} = 0.5$. After any out-of-equilibrium second-period message, the decision maker assigns probability 1 to the lowest type in his information set (prior to the off-path message), and accordingly chooses an action equal to this type.

We now argue that this is an equilibrium for any $b < \frac{1}{16}$:

First, notice that all messages (even out-of-equilibrium ones) induce the same action in period 1. Hence, the expert is indifferent between all possible first-period messages if he puts zero weight on the future. So, in particular, a myopic expert will find it optimal to send the "right" message, following the strategy outlined above. Now consider, for example, the history following an initial message m_ε. The decision-maker's posterior beliefs assign probability $\frac{1}{4}$ to each of the types in $\left\{\frac{1}{8} - \varepsilon, \frac{3}{8} + \varepsilon, \frac{4}{8} + \varepsilon, 1 - \varepsilon\right\}$. The expert's strategy at this stage is to tell the truth: so, if he sends a message that he is type $k \in \left\{\frac{1}{8} - \varepsilon, \frac{3}{8} + \varepsilon, \frac{4}{8} + \varepsilon, 1 - \varepsilon\right\}$, then the decision-maker will believe that k is the true state, and accordingly will choose action k; if the expert deviates to some off-path message, then the decision-maker will assign probability 1 to the lowest type in his information set, $\frac{1}{8} - \varepsilon$, and accordingly choose action $\frac{1}{8} - \varepsilon$. Therefore, to prove that the expert has no incentive to deviate, we need only show that each expert type $k \in \left\{\frac{1}{8} - \varepsilon, \frac{3}{8} + \varepsilon, \frac{4}{8} + \varepsilon, 1 - \varepsilon\right\}$ would rather tell the truth, than mimic any of the other types in his group. Type k prefers action k to k' (so will not mimic type k') whenever

$$-\left(k - k - b\right)^2 \geq -\left(k' - k - b\right)^2 \Leftrightarrow \left(k' - k\right)\left(k' - k - 2b\right) \geq 0$$

i.e., whenever $k' < k$, or whenever $k' > k + 2b$. So in particular, to make sure that no type in $\left\{\frac{1}{8} - \varepsilon, \frac{3}{8} + \varepsilon, \frac{4}{8} + \varepsilon, 1 - \varepsilon\right\}$ wishes to mimic any other type in this group, it is sufficient to make sure that every pair of types are at least $2b$ apart. Since the closest-together types in the group, $\frac{3}{8} + \varepsilon$ and $\frac{4}{8} + \varepsilon$, are separated by $\frac{1}{8}$, we conclude that the group is separable whenever $\frac{1}{8} > 2b \Leftrightarrow b < \frac{1}{16}$. And similarly after messages $m_{\widetilde{\varepsilon}}$.

This construction does not apply with a more patient expert ($\delta_E > 0$), because it does not provide a forward-looking expert with incentives to join the "right" separable group. For example, consider type $\frac{3}{8}$, and suppose that $b = \frac{1}{16}$. The truthful strategy is to reveal group $\{\frac{1}{8}, \frac{3}{8}, \frac{4}{8}, 1\}$ in period 1, and then tell the truth in period 2, inducing actions $(y_1, y_2) = \left(\frac{1}{2}, \frac{3}{8}\right)$. However such strategy cannot be part of an equilibrium if $\delta_E > 0$. The best deviation for type $\theta = \frac{3}{8}$ is to mimic type $\frac{3}{8} + \frac{1}{16}$ – initially claiming to be part of the group $\left\{\frac{1}{8} - \frac{1}{16}, \frac{3}{8} + \frac{1}{16}, \frac{4}{8} + \frac{1}{16}, \frac{7}{8} - \frac{1}{16}\right\}$, and then subsequently claiming that the true state is $\frac{3}{8} + \frac{1}{16}$ – thereby inducing actions $(y_1, y_2) = \left(\frac{1}{2}, \frac{3}{8} + \frac{1}{16}\right)$. This deviation then leads to no change in the first-period action, but the 2nd-period action is now equal to type $\frac{3}{8}$'s bliss point, $\frac{3}{8} + \frac{1}{16}$. When $\delta_E > 0$ we need to provide the expert with better incentives to join the "right" separable group: since θ prefers $\theta + b$'s action in the future, he must prefer his own action now. This is much more complex, but in Section 4, we show how to construct such separation-inducing actions.

2 The Model

We extend the classic model of Crawford and Sobel (1982) to a dynamic setting. There are two players, an expert (E) and a decision-maker (DM), who interact for finitely many periods. The expert knows the true state of the world $\theta \in [0,1]$, which is constant over time and is distributed according to the c.d.f. F, with associated density f. Both players care about their discounted payoff sum: when the state is θ and the decision-maker chooses actions $y^T = (y_1, .., y_T)$ in periods $1, 2, ..., T$, payoffs are given by:

$$\begin{aligned} \text{expert:} \quad & U^E(y^T, \theta, b) = \Sigma_{t=1}^T \delta_E^{t-1} u^E(y_t, \theta, b) \\ \text{DM:} \quad & U^{DM}(y^T, \theta) = \Sigma_{t=1}^T \delta_{DM}^{t-1} u^{DM}(y_t, \theta) \end{aligned}$$

where $b > 0$ is the expert's "bias" and reflects a conflict of interest between the players, and δ_E, δ_{DM} are the players' discount factors. We assume that $u^E(y_t, \theta)$ and $u^{DM}(y_t, \theta, b)$ satisfy the conditions imposed by Crawford and Sobel (1982): for $i = DM, E$, $u^i(\cdot)$ is twice continuously differentiable, $u_1^i(y, \theta) = 0$ for some y and $u_{11}^i(\cdot) < 0$ ($\forall \theta$, so that u^i has a unique maximizer y for each pair (θ, b)), and $u_{12}^i(\cdot) > 0$ (so that the best action from an informed player's perspective is strictly increasing in θ). Most of our main results will make the more specific assumption that preferences are quadratic, as given by (1).

At the beginning of each period t, the expert sends a (possibly random) message m_t to the decision-maker. The decision-maker then updates his beliefs about the state, and chooses an action $y_t \in \mathbb{R}$ that affects both players' payoffs. Let $y^{DM}(\theta)$ and $y^E(\theta)$ denote, respectively, the decision-maker's and the expert's most preferred actions in state θ; we assume that for all θ, $y^{DM}(\theta) \neq y^E(\theta)$, so that there is a conflict of interest between the players regardless of the state.

The decision maker observes his payoffs only at the end of the game. (If the decision-maker could observe his payoff each period, the problem would be trivial, as he could simply invert his payoff to determine the true state θ. As usual, we could alternatively assume stochastic payoffs, with sufficient noise that the decision-maker is unable to learn anything about the state from observing his payoff realizations).

A *strategy profile* $\sigma = (\sigma_i)_{i=E,DM}$ specifies a strategy for each player. Let h_t denote a history that contains all the reports submitted by the expert, $m^{t-1} = (m_1, ..., m_{t-1})$, and all actions chosen by the decision-maker, $y^{t-1} = (y_1, ..., y_{t-1})$, up to stage t. The set of all feasible histories at t is denoted by H_t. A behavioral strategy for the expert, σ_E, consists of a sequence of signaling rules that map $[0,1] \times H_t$ to a probability distribution over reports $\mathcal{M}$. Let $q(m\,|\theta, h_t)$ denote the probability that the expert sends message m at history h_t when his type is θ. A strategy for the decision-maker, σ_{DM}, is a sequence of maps from H_t to actions. We use $y_t(m\,|h_t) \in \mathbb{R}$ to denote the action that the decision maker chooses at h_t given a report m. A *belief system*, μ, maps H_t to the set of probability distributions over $[0,1]$. Let $\mu(\theta|h_t)$ denote the decision-maker's beliefs about the experts's type after a history h_t, $t = 1, ..., T$. We seek assessments (σ, μ) that form *Perfect Bayesian Equilibria (PBE),* and suppress dependence on h_t when there is no risk of confusion.

In the paper we use the following terminology:

Definition 1 *An equilibrium is called* ***babbling*** *if for all* m *with* $q(m\,|\theta, h_t) > 0$, *all* $\theta \in [0,1]$, *all* h_t *and* t, *we have that* $y_t(m\,|h_t) = \hat{y}$.

In other words, we call an equilibrium babbling if the same action is induced, with probability one, for all states $\theta \in [0,1]$ and all $t \in T$.

Definition 2 *We call a signaling rule* q ***uniform*** *if* $q(m\,|\theta, h_t)$ *is uniform, with support on* $[\theta_i, \theta_{i+1}]$ *if* $\theta \in [\theta_i, \theta_{i+1}]$ *(so that intervals of types pool together, following Theorem 1 of Crawford and Sobel (1982))*

Definition 3 *A* ***partition equilibrium*** *is an equilibrium in which, for all* t *and at every history* h_t, *the expert employs only uniform signaling rules.*

Definition 4 *An equilibrium is* ***fully revealing*** *if there exists* $\hat{T} \leq T$ *such that for all* $\theta \in [0,1]$, *and all histories along the equilibrium path, the expert reveals the true state with probability one by period* $\hat{T}$, *and accordingly* $y_t(\theta) = y^{DM}(\theta)\ \forall t \geq \hat{T}$.

We first briefly summarize the findings of the one-shot strategic information transmission game of Crawford and Sobel (1982). We then briefly study the properties of uniform signaling rules in our dynamic setup, before moving on to consider more complex communication rules.

2.1 Uniform Signaling: The Canonical Static Communication

Crawford and Sobel (1982) show that in a one-shot strategic information transmission game, all equilibria are equivalent to partition equilibria: the expert follows a pure strategy in which intervals of types pool together (by sending the same message), inducing actions which are increasing step functions of the state. Communication is then coarse: even though the state θ takes on a continuum of values, only finitely many actions are induced at equilibrium.

The reasons behind this result can be summarized as follows. Fix an equilibrium of the one-shot game, and let $y(\theta)$ denote an action induced when the state is θ. The conflict of interest between the expert and the decision-maker implies that at most finitely many actions can be induced at equilibrium. Together with the single-crossing condition and the fact that $u^E(\cdot)$ is strictly concave in y, this implies that equilibrium actions are an increasing step function of the state. Importantly, Crawford and Sobel (1982) show that, without loss of generality, the actions induced at equilibrium can be taken to arise from uniform signaling rules. This result follows from the observation that all messages inducing the same action y can be replaced by a single message. Therefore, more complex signaling rules play no role in the static setup.

2.2 Uniform Signaling: A Special Kind of Dynamic Communication

We now focus on simple partitional communication protocols (uniform signaling) and study their properties in our dynamic setup. We show two results. The first result is that with monotonic partition equilibria, the decision maker never learns the truth:

Proposition 1 *For all horizons T, there exists no fully revealing monotonic partition equilibrium.*

This result follows almost immediately from Crawford and Sobel (1982). A short sketch of the argument is as follows. Suppose, by contradiction, that there exists a fully revealing monotonic partition equilibrium. Then, there exists a period $\hat{T} \leq T$ in which the last subdivision occurs, with $y_t(\theta) = y^{DM}(\theta)$ for all $t \geq \hat{T}$. Then, the incentive constraint at time $\widehat{T}$ for type θ to not mimic type $\theta + \varepsilon$ is

$$\left(1 + \delta_E + \delta_E^2 + ... + \delta_E^{T-\hat{T}-1}\right) u^E\left(y^{DM}(\theta), \theta, b\right) \geq \left(1 + \delta_E + \delta_E^2 + ... + \delta_E^{T-\hat{T}-1}\right) u^E\left(y^{DM}(\theta + \varepsilon), \theta, b\right)$$

and similarly for type $\theta + \varepsilon$. These conditions are equivalent to the static equilibrium conditions in Crawford and Sobel (1982), who proved that they imply that at most finitely many actions can be induced at equilibrium, a contradiction to full revelation.

We now proceed to show that if all static equilibria are babbling, then all dynamic monotonic partition equilibria are equivalent to babbling:

Proposition 2 *If all static equilibria are equivalent to the babbling equilibrium, then all dynamic monotonic partition equilibria are equivalent to babbling.*

Proof: see Appendix A.

We now move on to show that dynamic monotonic partition equilibria can Pareto-dominate all equilibria of the one-shot game. In Appendix B, we construct a two-period example in which $\delta_E = \delta_{DM} = 1$, the state θ is uniformly distributed on $[0, 1]$, and preferences are given by (1), with $b = \frac{1}{12}$. In the most informative static equilibrium, the state space is partitioned into two pieces,

$\left[0, \frac{1}{3}\right] \cup \left[\frac{1}{3}, 1\right]$, inducing actions $\frac{1}{6}$ and $\frac{4}{6}$. On the other hand, there exists a partition equilibrium of the two-period game in which the state space is divided ultimately into three sub-intervals, $[0, 0.25] \cup [0.25, 0.45833] \cup [0.458\,33, 1]$, and which is (ex-ante) strictly Pareto superior to repetition of the static equilibrium.

However, in dynamic settings it is also possible that equilibria with more partitions may be Pareto-inferior to those with less (we present an example of such an equilibrium in Appendix C). This happens because a larger ultimate number of partitions may require extensive pooling earlier on, inducing overall lower welfare. This finding is in contrast to Crawford and Sobel (1982), who show that under their Condition M (essentially a unique equilibrium for each partition size N), equilibria can be easily Pareto ranked: both the expert and the decision-maker prefer (ex ante) the equilibrium with the highest number of partitions.[7] Our findings suggest that Pareto comparisons in dynamic cases are less straightforward, even if we restrict attention to monotonic partition equilibria.

We proceed to study the role of complex signaling in our dynamic game.

3 An Example with Complex Signaling and Dynamic Information Revelation

In this section, we present an example in which the expert employs a complex signaling rule, which induces non-monotonic actions. In this example, the bias is so severe that in a static setting, all equilibria would be babbling. We show that even in these extreme bias situations, some information can be revealed with just two rounds. This equilibrium has the feature that the decision maker learns the state quite precisely when the news is either horrific or terrific, but remains agnostic for intermediate levels. Finally we show that for a range of biases, this non-monotonic equilibrium is Pareto superior to all monotonic ones.

Example 2 *Dynamic equilibria can be non-monotonic*

Consider a two period game where $\delta_E = \delta_{DM} = 1$, types are uniformly distributed on $[0, 1]$ and preferences are given by (1). We will construct an equilibrium with the following "piano teacher" interpretation: a child's parent (the decision-maker) wants the amount of money he spends on lessons to correspond to the child's true talent θ, whereas the piano teacher (expert) wants to inflate this number. In our equilibrium, parents of children who are at either the bottom or top extreme of the talent scale get the same initial message, "you have an interesting child" ($m_{1(1)}$

[7] The equilibrium with the largest number of partitions is the only equilibrium that satisfies the "no incentive to separate" (NITS) condition (Chen, Kartik and Sobel (2008)).

below), and then find out in the second period whether "interesting" means great ($m_{2(3)}$) or awful ($m_{2(1)}$); parents of average children are told just that in both periods. More precisely, let the expert use the following signaling rule:

In period 1, expert types in $[0, \underline{\theta}) \cup (\bar{\theta}, 1]$ send message $m_{1(1)}$ with probability 1, and types in $[\underline{\theta}, \bar{\theta}]$ send message $m_{1(2)}$ with probability 1. In period 2, the expert adopts the following signaling rule: types in $[0, \underline{\theta})$ send message $m_{2(1)}$, types in $[\underline{\theta}, \bar{\theta}]$ send a message $m_{2(2)}$, and types in $(\bar{\theta}, 1]$ send $m_{2(3)}$ (all with probability 1). With this signaling rule, the optimal actions for the decision-maker in period 1 are $y_{1(1)} = \frac{\underline{\theta}^2 - \bar{\theta}^2 + 1}{2(\underline{\theta} - \bar{\theta} + 1)}$, $y_{1(2)} = \frac{\underline{\theta} + \bar{\theta}}{2}$; in period 2, they are $y_{2(1)} = \frac{\underline{\theta}}{2}$, $y_{2(2)} = \frac{\underline{\theta} + \bar{\theta}}{2}$, $y_{2(3)} = \frac{1 + \bar{\theta}}{2}$. After any out-of-equilibrium message, the decision-maker assigns equal probability to all states in $[\underline{\theta}, \bar{\theta}]$, and so will choose action $y^{out} = \frac{\underline{\theta} + \bar{\theta}}{2}$. With these out-of-equilibrium beliefs, no expert type has any incentive to send an out-of-equilibrium message.

In order for this to be an equilibrium, type $\underline{\theta}$ must be indifferent between message sequences $A \equiv (m_{1(1)}, m_{2(1)})$ and $B \equiv (m_{1(2)}, m_{2(2)})$:

$$-\left(\frac{\underline{\theta}^2 - \bar{\theta}^2 + 1}{2(\underline{\theta} - \bar{\theta} + 1)} - \underline{\theta} - b\right)^2 - \left(\frac{\underline{\theta}}{2} - \underline{\theta} - b\right)^2 = -2\left(\frac{\underline{\theta} + \bar{\theta}}{2} - \underline{\theta} - b\right)^2 \tag{2}$$

and type $\bar{\theta}$ must be indifferent between message sequences B and $C \equiv (m_{1(1)}, m_{2(3)})$:

$$-\left(\frac{\underline{\theta}^2 - \bar{\theta}^2 + 1}{2(\underline{\theta} - \bar{\theta} + 1)} - \bar{\theta} - b\right)^2 - \left(\frac{1 + \bar{\theta}}{2} - \bar{\theta} - b\right)^2 = -2\left(\frac{\underline{\theta} + \bar{\theta}}{2} - \bar{\theta} - b\right)^2. \tag{3}$$

At $t = 2$ it must also be the case that type $\underline{\theta}$ prefers $m_{2(1)}$ to $m_{2(3)}$, and the reverse for type $\bar{\theta}$: that is $-(\frac{\underline{\theta}}{2} - \underline{\theta} - b)^2 \geq -(\frac{1+\bar{\theta}}{2} - \underline{\theta} - b)^2$ and $-(\frac{1+\bar{\theta}}{2} - \bar{\theta} - b)^2 \geq -(\frac{\underline{\theta}}{2} - \bar{\theta} - b)^2$. The global incentive compatibility constraints, requiring that all types $\theta < \underline{\theta}$ prefer sequence A to B and that all types $\theta > \bar{\theta}$ prefer C to B, reduce to a requirement that the average induced action be monotonic, which is implied by indifference constraints (2) , (3) .[8]

A solution of the system of equations (2) and (3) gives an equilibrium if $0 \leq \underline{\theta} < \bar{\theta} \leq 1$. We solved this system numerically, and found that the highest bias for which it works is $b = 0.256$. Here, the partition cutoffs in our equilibrium are given by $\underline{\theta} = 0.0581$, $\bar{\theta} = 0.9823$. The corresponding optimal actions for period 1 are $y_{1(1)} = 0.253$, $y_{1(2)} = 0.52$, and for period 2 they are $y_{2(1)} = 0.029$, $y_{2(2)} = 0.52$, $y_{2(3)} = 0.991$. Note that while the first period action is non-monotonic, the average action $\bar{y} = \frac{y_1 + y_2}{2}$ is still weakly increasing in the state. Ex ante payoffs are -0.275 for the expert,

[8]Rearranging (3) , the LHS is greater than the RHS for type θ (so he prefers C to B) iff $(\theta - \bar{\theta})\left(\frac{y_{1(1)} + y_{2(3)}}{2} - y_{1(2)}\right) > 0$, so we need $\frac{y_{1(1)} + y_{2(3)}}{2} > y_{1(2)}$ for this to hold $\forall \theta > \bar{\theta}$. This is implied by (3) : adding $2\left(\frac{y_{1(1)} + y_{2(3)}}{2} - \bar{\theta} - b\right)^2$ to both sides and factoring yields $\left(y_{1(2)} - \bar{\theta} - b\right)^2 - \left(\frac{y_{1(1)} + y_{2(3)}}{2} - \bar{\theta} - b\right)^2 = \left(\frac{y_{1(1)} - y_{2(3)}}{2}\right)^2 \geq 0$, so we need $\left|y_{1(2)} - \bar{\theta} - b\right| \geq \left|\frac{y_{1(1)} + y_{2(3)}}{2} - \bar{\theta} - b\right|$; since $y_{1(2)} < \bar{\theta} + b$, this implies $y_{1(2)} \leq \frac{y_{1(1)} + y_{2(3)}}{2}$, as desired. And similarly at $\underline{\theta}$.

and -0.144 for the decision-maker.

Recall that in a one-shot game with quadratic preferences, the only equilibrium is the babbling one whenever $b > \frac{1}{4}$. Proposition 2 implies that at $b = 0.256$, if we restricted attention to *monotonic* partition equilibria, we would again find only a babbling equilibrium, in which the decision-maker chooses action $y^B = 0.5$ in both periods: this yields ex-ante payoffs of -0.298 to the expert and -0.167 to the decision-maker, strictly worse than in our above construction.

Our example therefore illustrates how non-monotonic equilibria can both increase the amount of information revelation, and can also strictly Pareto-dominate all monotonic partition equilibria. By pooling together the best and the worst states in period 1, the expert is willing to reveal in period 2 whether the state is very good or very bad. It also has the following immediate implication:

Proposition 3 *There exist non-monotonic equilibria that are Pareto superior to all monotonic partition equilibria.*

We now move on to our first main result, showing that our dynamic setup correlates the incentives of the expert and decision-maker in such a way that *full information revelation* is possible.

4 Learning the Truth when the Expert is Patient

When the expert is patient rather than myopic, getting him to reveal the truth is much more complicated, as we previewed in Section 1. In this section, we construct a fully revealing equilibrium for the quadratic preferences specified in (1). The equilibrium relies on two main tools: separable groups, and trigger strategies. These tools have no leverage in single-round communications, but are powerful in dynamic communications.

The equilibrium works as follows. In each period, the expert recommends an action to the decision-maker. Initially, each action is recommended by finitely many (at most four) expert types, who then subdivide themselves further into separable groups of two with an interim recommendation. If the decision-maker chooses all initial actions recommended by the expert, then the expert rewards him by revealing the truth in the final stage of the game, recommending an action $y(\theta) = \theta$. If the decision-maker rejects the expert's early advice, then the expert babbles for the rest of the game, and so the decision-maker never learns more than the separable group containing the truth.

We provide here an outline of how we construct fully revealing equilibria for quadratic preferences.[9] We then state our first main theorem, with full proof details in Appendix D.

[9] The general ideas and techniques we present here could be used also in constructing fully revealing equilibria for more general preferences. However, the details of the construction are closely tied to the players' preferences (for example, obtaining an explicit solution to the system of differential equations referenced in (4), which for a general function $u^E(y, \theta, b) \equiv W(\theta, b)$ would become $2\alpha_a W_1(u_1(\theta), \theta)\, u_1'(\theta) + 2(1-\alpha_a) W_1(u_2(\theta), \theta)\, u_2'(\theta) + (T-2) W_1\left(y^{DM}(\theta), \theta\right) \frac{dy^{DM}(\theta)}{d\theta} = 0$). For tractability, our main results therefore use the widely-applied quadratic loss functions.

Equilibrium Outline: Separable Groups

Rather than having intervals of types pool together, we construct pairs of far-away types ("partners") who pool together in the initial periods. The advantage is that once the expert joins one of these separable groups, revealing the two possible true states to the decision-maker, we no longer need to worry about him mimicking nearby types: his only options are to tell the truth, or to mimic his partner. Of course, an important part of the proof is to ensure that each expert type wants to join the "right" separable group. For a myopic expert, this is straightforward: if the expert cares only about the first-period action, then to make him join the right group, it is sufficient that the first-period action be constant across groups. (In fact, the myopic expert result relied only on separable groups, without the need for trigger strategies: we were able to group types such that the (constant) action recommended by each group was equal to the average type within the group, i.e. so that it coincided with the decision-maker's myopically optimal choice). For a patient expert, the construction is significantly more involved, as it must take dynamic incentives into account. In particular, the actions induced in the initial stages cannot be flat: if this were the case, then an expert who cares about the future would simply join whichever separable group leads to the best future action. Hence, for a patient expert, we need to construct initial action functions which provide appropriate incentives: if type θ knows that some type θ' will get a more favorable action in the revelation phase, then type θ's group must induce an initial action which is more favorable to type θ than that induced by (θ')'s group.

Equilibrium Outline: Strategies

Before we proceed with the sketch, it is useful to simplify notation and work with a scaled type space by dividing all actions and types by b. When we say that "type $\theta \in [0, \frac{1}{b}]$ recommends $u(\theta)$ in period 1, for disutility $(u(\theta) - \theta - 1)^2$", we mean that (in the unscaled type space) "type θb recommends action $u(\theta)b$, for disutility $(u(\theta)b - \theta b - b)^2 = b^2 (u(\theta) - \theta - 1)^2$".

We first partition the scaled type space $[0, \frac{1}{b}]$ into four intervals, with endpoints $[0, \theta_1, \theta_2, \theta_3, \frac{1}{b}]$. The separable groups are as follows: at time $t = 0$, each type $\theta \in [0, \theta_1]$ pools with a partner $g(\theta) \in [\theta_2, \theta_3]$ to send a sequence of recommended actions $(u_1(\theta), u_2(\theta))$, and then reveal the truth at time $t = 2$ iff the decision-maker followed both initial recommendations. Each type $\theta \in [\theta_1, \theta_2]$ initially pools with a partner $h(\theta) \in [\theta_3, \theta_4]$ to recommend a sequence of actions $(v_1(\theta), v_2(\theta))$, then revealing the truth at time $T - \tau$ (iff the expert followed their advice). For the purpose of this outline, take the endpoints $\theta_1, \theta_2, \theta_3$ as given, along with the partner functions $g : [0, \theta_1] \to [\theta_2, \theta_3]$; $h : [\theta_1, \theta_2] \to [\theta_3, \theta_4]$, and recommendation functions u_1, u_2, v_1, v_2. In the appendix, we derive the parameters and functions that work, and provide the full details of how to construct fully revealing equilibria.

We now describe the strategy for the expert and for the decision maker. For notational purposes it is useful to further subdivide the expert types into three groups: I, II, and III.

At time $t = 0$, there are then three groups of experts. Group I consists of types $\theta^I \in [\theta_1, \theta_2]$ with their partners $h\left(\theta^I\right) \in \left[\theta_3, \frac{1}{b}\right]$. Group II consists of all types $\theta^{II} \in [0, \theta_1]$ whose initial recommendation coincides with that of a Group I pair, together with their partners $g(\theta^{II}) \in [\theta_2, \theta_3]$. Group III consists of all remaining types $\theta^{III} \in [0, \theta_1]$ and their partners $g(\theta^{III}) \in [\theta_2, \theta_3]$. In other words, we divided the types in intervals $[0, \theta_1] \cup [\theta_2, \theta_3]$ into two groups, II and III, according to whether or not their initial messages coincide with that of a group I pair.

The timeline of the expert's advice is as follows:

↓	↓	↓	↓	↓
Phase 1 ($t = 0$)	**Phase 2** ($t = 2\alpha_0$)	**Phase 3** ($t = 2\alpha_a$)	**Phase 4** ($t = 2$)	**Phase 5** ($T - \tau$)
Group $I : v_1$	Group I :	Groups II, III :	Groups II, III :	Group I :
Groups $II, III : u_1$	switch to v_2	switch to u_2	reveal the truth	reveal the truth

Figure 1: Timeline

where $0 < \alpha_0 \leq \alpha_a < 1$ are specified in the appendix (see Section D.2.1, paragraph following (35)).

In words: in the initial phase, types in Group I recommend v_1: $v_1\left(\theta^I\right) = v_1\left(h\left(\theta^I\right)\right)$, while types in Groups II and III recommend the action u_1: $u_1\left(\theta^{II}\right) = u_1\left(g\left(\theta^{II}\right)\right)$. Importantly, the recommendations for the types in Groups I and II coincide (while Group III recommendations do not coincide with those of any Group I pair): for every θ^I in Group I, there exists θ^{II} in Group II with $v_1\left(\theta^I\right) = u_1\left(\theta^{II}\right)$. This is why, for ease of exposition, we have a subdivision into groups; upon receiving a recommendation $v_1(\theta^I) = u_1(\theta^{II})$, the decision-maker believes that it could have come from any of the types in $\left\{\theta^I, h\left(\theta^I\right), \theta^{II}, g\left(\theta^{II}\right)\right\}$ (see footnote 19 in Section D.2.1 for why this is needed). At time $t = 2\alpha_0$, group I pairs $\{\theta^I, h(\theta^I)\}$ change their recommendation to $v_2(\theta^I) = v_2(h(\theta^I))$, while Groups II and III continue to follow the recommendation function u_1 (that is, they do not yet change their advice). Thus, at this stage, the decision-maker learns whether he is facing a Group I, II or III pair. At time $t = 2\alpha_a \geq 2\alpha_0$, group II and III pairs switch to the recommendation function u_2, where α_a may depend on the specific pair.[10] Group I continues to follow the recommendation function v_2, revealing no further information at this stage. At time $t = 2$, group II and III pairs separate: each type θ^{II} or θ^{III} in $[0, \theta_1]$ sends a message equal to his type (thus revealing the truth for the final $T - 2$ periods), and similarly their partners $g(\theta^{II}), g(\theta^{III})$ send messages equal to their own types. At time $T - \tau > 2$, Group I pairs $\{\theta^I, h(\theta^I)\}$ separate, with type θ^I recommending action θ^I and type $h(\theta^I)$ recommending $h(\theta^I)$ for the final τ periods. It should be noted that the times at which the decision-maker is instructed to change

[10] In Proposition D3 in the appendix, we describe Group II, III types and their recommendations parametrically, as functions of a variable a. Then, in Lemma D7.1 (iv), we choose α_a to ensure the desired overlap of the u_1, v_1 recommendation functions.

his action $(2\alpha_0, 2\alpha_a, T-\tau)$ are not necessarily integers in our construction. In the continuous time setting we study in Section 4.1, this clearly poses no problem; if there are integer constraints, it is straightforward to "scale up" all the time parameters (see footnote 21).

The decision-maker's strategy is to follow all on-path recommendations. An off-path recommendation at time $t=0$ is treated as a mistake coming from the pair $\{0, g(0)\}$, and subsequent off-path recommendations are simply ignored as errors (full details at the start of Section D.1 in the Appendix).

To summarize: In the initial phase, separable groups are formed. Each expert type sends a recommendation sequence of the form $\left(\underbrace{v_1(\theta^I)}_{2\alpha_0 \text{ periods}}, \underbrace{v_2(\theta^I)}_{T-\tau-2\alpha_0 \text{ periods}}\right)$ or $\left(\underbrace{u_1(\theta^i)}_{2\alpha_a \text{ periods}}, \underbrace{u_2(\theta^i)}_{2(1-\alpha_a) \text{ periods}}\right)$, with $i \in \{II, III\}$, and such that for all $\theta^I \in [\theta_1, \theta_2]$ there exists $\theta^{II} \in [0, \theta_1]$ with $v_1(\theta^I) = u_1(\theta^{II})$. During these phases, the decision-maker is able to infer the separable group containing the expert's true type, but, rather than choosing the corresponding myopically optimal action, he chooses the actions u_1, u_2, v_1, v_2 recommended by the expert. These action functions are constructed to provide the expert with incentives to join the *right* separable group at time 0. The final phases are the revelation phases: the separable groups themselves separate, revealing the exact truth to the decision-maker, provided that he has followed all of the expert's previous advice; any deviation results in babbling by the expert during the revelation phase.

Incentivizing the Expert

Finally, we briefly explain the construction of the functions (u_1, u_2) and (v_1, v_2), and the corresponding partner functions g, h (and endpoints $\theta_1, \theta_2, \theta_3$), which are given parametrically in the Appendix (see equations $(14), (15)$ in Section D, with further details in Section D.3.1). For the expert, three sets of constraints must be satisfied:

Expert Local IC:

The first set of constraints can be thought of as local incentive compatibility constraints–that is, those applying within each type θ's interval $[\theta_i, \theta_{i+1}]$. These (dynamic) incentive compatibility constraints ensure that, say, the expert of type $\theta \in [0, \theta_1]$ prefers to induce actions $u_1(\theta)$ (for $2\alpha_a$ periods), $u_2(\theta)$ (for $2(1-\alpha_a)$ periods), and then reveal his type θ for the final $T-2$ periods, than e.g. to follow the sequence $\big(u_1(\theta'), u_2(\theta'), \theta'\big)$ prescribed for some other type θ' in the same interval $[0, \theta_1]$ (and analogously within each of the other three intervals). For types $\theta \in [0, \theta_1]$, this boils down to a requirement that u_1, u_2 satisfy the following differential equation,

$$2\alpha_a u_1'(\theta)\,(u_1(\theta) - \theta - 1) + 2(1-\alpha_a)u_2'(\theta)\,(u_2(\theta) - \theta - 1) = T - 2 \tag{4}$$

and that the "average" action, $2\alpha_a u_1(\theta)+2(1-\alpha_a)u_2(\theta)+(T-2)\theta$, be weakly increasing in θ. We provide a more detailed explanation and solution of this equation in the Appendix, Section D.3.1, and derive similar equations for the other three intervals.

Note that a longer revelation phase (that is, an increase in the RHS term $(T-2)$ in (4)) requires a correspondingly larger distortion in the action functions u_1, u_2 : if the expert anticipates a lengthy phase in which the DM's action will match the true state (whereas the expert's bliss point is to the right of the truth), then it becomes more difficult in the initial phase to provide him with incentives not to mimic the advice of types to his right. This is why a longer horizon does not trivially imply better welfare properties.

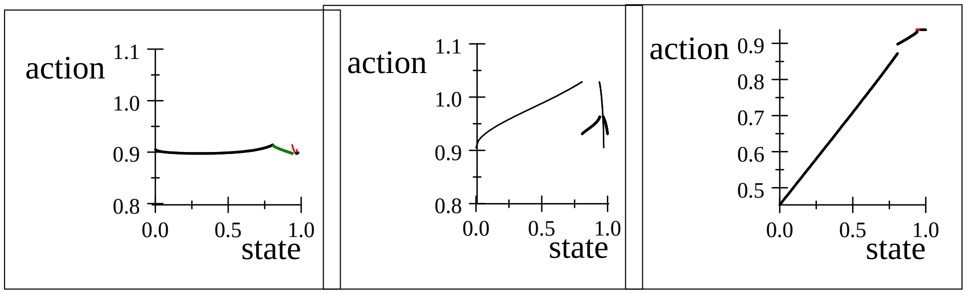
Expert Global IC:

The next set of constraints for the expert can be thought of as "global" incentive compatibility constraints, ensuring that no expert type wishes to mimic any type in any other interval. In the appendix, we show that this boils down to two additional requirements: each endpoint type $\theta_1, \theta_2, \theta_3$ must be indifferent between the two equilibrium sequences prescribed for his type (for example, type θ_1 must be indifferent between sequences $\left(\underbrace{u_1(\theta_1)}_{2\alpha_a}, \underbrace{u_2(\theta_1)}_{2(1-\alpha_a)}, \underbrace{\theta_1}_{T-2}\right)$ and $\left(\underbrace{v_1(\theta_1)}_{2\alpha_0}, \underbrace{u_2(\theta_1)}_{T-\tau-2\alpha_0}, \underbrace{\theta_1}_{\tau}\right)$), and the "average" action must be either continuous or jump up at each endpoint (see Appendix Lemma D.3.2, with further details in Section D.3.1).

Expert Separation:

The final constraint requires that each pair of types indeed be "separable", that is, sufficiently far apart that each type would rather tell the truth than mimic his partner. In our rescaled type space with quadratic preferences, this requires choosing partner functions g, h satisfying $|g(\theta)-\theta| \geq 2$ and $|h(\theta)-\theta| \geq 2$, which we do in the appendix (Section D.1.2). It turns out to be very tricky to satisfy the global incentive compatibility constraints together with the local constraints: it in fact requires a minimum of two distinct actions prior to the revelation phase (this is why e.g. Group III pairs must change their recommendation from u_1 to u_2 at time $2\alpha_a$, even though doing so reveals no further information), and that the type space be partitioned into a minimum of four intervals. Moreover, for any partition into four intervals, there is in fact only *one* partner function $g:[0,\theta_1] \rightarrow [\theta_2, \theta_3]$ that works, and we believe that there is no partition which would allow for expert-incentivizing action functions which are myopically optimal from the decision-maker's perspective. This is why our construction relies on trigger strategies: the expert *only* reveals the truth if the decision-maker follows all of his advice.

We graph the equilibrium actions u_1, v_1 in the left-most graph, actions u_2, v_2 in the middle graph, and the average action in the right-most graph, for $b=\frac{1}{60.885}$ and $T=4$:



Actions u_1, v_1 Actions u_2, v_2 Average Action

Incentivizing the decision-maker:

Suppose that the expert recommends an action $u_1(\theta)$, which the decision-maker believes could only have come from types $\theta, g(\theta)$. If the decision-maker follows the recommendation, then he expects the expert to switch his recommendation to $u_2(\theta)$ at time $2\alpha_a$, and then recommend the true state θ for the final $T-2$ periods. If the decision-maker assigns probabilities $p_\theta, 1-p_\theta$ to types $\theta, g(\theta)$, then this yields an expected disutility of

$$p_\theta \left(2\alpha_a \left(u_1(\theta)-\theta\right)^2 + 2(1-\alpha_a)\left(u_2(\theta)-\theta\right)^2\right) + (1-p_\theta)\left(2\alpha_a \left(u_1(\theta)-g(\theta)\right)^2 + 2(1-\alpha_a)\left(u_2(\theta)-g(\theta)\right)^2\right)$$

(noting that disutility in the final $T-2$ periods is zero). The problem is that the initial recommendations $u_1(\theta), u_2(\theta)$ do not coincide with the decision-maker's myopically optimal action, $y^*(\theta) \equiv p_\theta\theta + (1-p_\theta)g(\theta)$. We therefore employ *trigger strategies*: the expert only reveals the truth in the final stage if the decision-maker follows his recommendations at the beginning of the game. If the decision-maker ever rejects his advice, then the expert babbles for the rest of the game, and so the decision-maker's disutility is at best

$$T \cdot \left[p_\theta \cdot \left(\underbrace{p_\theta\theta + (1-p_\theta)g(\theta)}_{y^*(\theta)} - \theta \right)^2 + (1-p_\theta) \cdot \left(\underbrace{p_\theta\theta + (1-p_\theta)g(\theta)}_{y^*(\theta)} - g(\theta) \right)^2 \right]$$

So, for the equilibrium to work for the decision-maker, we need to make sure that the benefit to learning the exact state, rather than just the separable group containing it, is large enough to compensate him for the cost of following the expert's initial recommendations, rather than deviating to the myopically optimal actions. This is what limits the priors for which our construction works, and imposes the upper bound $b \cong \frac{1}{61}$ on the bias (see Appendix D.3.2, end of first paragraph). The construction works for the expert $\forall b < \frac{1}{16}$ (see appendix, end of proof of Proposition D2 in Section D.1.2).

Beliefs

We assume that the decision-maker is Bayesian: if he believes that the expert's first-period messages are given by a function $M : [0,1] \to \mathbb{R}$, with the property that

$$M(x) = M(p(x))$$

for all x in some interval $[\underline{x}, \overline{x}]$ and $p : [\underline{x}, \overline{x}] \to [0,1] \backslash [\underline{x}, \overline{x}]$ some continuous differentiable function (i.e., types x and $p(x)$ are "partners" who follow the same messaging strategy), then, after receiving the message $m = M(x) = M(p(x))$, the decision-maker's beliefs satisfy

$$\frac{\Pr(x|m)}{\Pr(p(x)|m)} = \lim_{\Delta \to 0} \frac{F(x+\Delta) - F(x-\Delta)}{F\left(p(x+\Delta)\right) - F\left(p(x-\Delta)\right)} = \frac{f(x)}{f(p(x))} \left| \frac{1}{p'(x)} \right| \tag{5}$$

This says that the likelihood of type x relative to $p(x)$ is equal to the unconditional likelihood ratio (determined by the prior F), times a term which depends on the shape of the p-function, in particular due to its influence on the size of the interval of p-types compared to their partner interval, $[\underline{x}, \overline{x}]$.[11]

We now state our main result:

Theorem 1 *Suppose that $\delta_E = \delta_{DM} = 1$ and that the preferences of the expert and of the decision maker are given by* (1) *. For any bias $b \leq \frac{1}{61}$, there is an infinite set of priors F, and a horizon T^*, for which a fully revealing equilibrium exists whenever $T \geq T^*$.*

The details of the construction can be found in Appendix D.

Substantively, this Theorem establishes an unexpected finding: even with a forward-looking expert and an infinite state space, there exist equilibria in which the truth is revealed in finite time. We initially expected to prove the opposite result. Technically, the construction involves several innovative ideas that we expect to be useful in analyzing many dynamic games with persistent asymmetric information.

[11] To understand this formula, consider an example in which F is uniform and $p(\cdot)$ is linear, say $p(x) = \alpha + \beta x$. In this case, the interval $[p(\underline{x}), p(\overline{x})]$ is β times as large as the interval $[\underline{x}, \overline{x}]$, so intuitively, it is as if the message sent by type x is sent by β "copies" of type $p(x)$: therefore, the decision maker's beliefs assign β times as much weight to type $p(x)$ as to type x, which is precisely what our formula says. Beliefs are assigned analogously after period 1.

Discussion

The true state is revealed at either time 2 or time $T - \tau$, where $T - \tau \leq 4$ (specified at start of Appendix D.2.1). Thus, the decision-maker chooses his best possible action, equal to the true state, in all but the first few periods. It is tempting to conclude that a long horizon means an equilibrium approaching the first-best, but unfortunately this is not true when the decision-maker and expert are equally patient. As explained after equation (4) , a long horizon also makes it difficult to incentivize the expert, requiring a proportionally larger distortion in the initial recommendation functions, and thereby imposing a proportionally larger cost to the decision-maker (from having to follow such bad early advice in order to learn the truth). We do, however, show in the next section that if the decision-maker is more patient than the expert, our fully revealing equilibrium has more attractive welfare properties, and works for a much larger range of prior beliefs: If the expert doesn't care much about the future, it becomes easy to incentivize him to join the right separable group, which, in turn, implies little need to distort the initial recommendations, and therefore little cost to the decision-maker from following bad advice in the first couple of periods. The benefit to following this advice–knowing the exact optimal action in all but the first few periods–will then outweigh this cost for a patient decision-maker. (Section 1 illustrated this in the extreme case $\delta_E = 0$, where the decision-maker learned the exact truth with no distortion in the expert's initial advice).

Remark 1 *If we look at situations where the decision-maker cares only about the ultimate decision, it is easy to see that our construction works for any prior (for all $b < \frac{1}{16}$), and yields the best possible outcome for the decision-maker.*

Remark 2 *If the decision-maker is not Bayesian and simply assigns equal probability to each type within a separable group, there is a simple modification of our construction that yields a fully revealing equilibrium for* any *prior on $[0, 1]$ if $b \leq \frac{1}{61}$.*

4.1 Information Revelation in Continuous Time

The equilibrium we constructed to prove Theorem 1 can be easily modified to yield a fully revealing equilibrium in a continuous-time setting with arbitrary discount rates, so long as the decision-maker is at least as patient as the expert. In particular, suppose that actions and recommendations may be made at any time up until the end of the game, and that the decision-maker and the expert discount the future at (respectively) rates r^{DM}, r^E. We then obtain the following result:

Proposition 4 *Suppose that preferences are given by (1). For any bias $b \leq \frac{1}{61}$ and prior F for which Theorem 1 holds, any horizon $\widehat{T}$, any expert discount rate $r^E > 0$, and any decision-maker discount rate $r^{DM} \leq r^E$, a fully revealing equilibrium exists.*

Proof:

Leave all action functions and specifications from the proof of Theorem 1 unchanged, except for the timeline shown in Figure 1: now, let Group I pairs recommend v_1 up to time $t_1(\alpha_0)$, then v_2 up to time t_4, and then reveal the truth, and let Group II, III pairs now recommend u_1 up to time $t_2(\alpha_a)$, u_2 up to time t_3, then reveal the truth, where

$$t_1(\alpha_0) = \frac{\ln\left(1 - 2\phi\alpha_0 r^E\right)}{-r^E}, t_2(\alpha_a) = \frac{\ln\left(1 - 2\phi\alpha_a r^E\right)}{-r^E}, t_3 = \frac{\ln\left(1 - 2\phi r^E\right)}{-r^E}, t_4 = \frac{\ln\left(1 - (T-\tau)\,\phi r^E\right)}{-r^E} \tag{6}$$

with $\phi = \frac{1-e^{-r^E \widehat{T}}}{T r^E}$ ($\widehat{T}$ is the (freely specified) horizon in the statement of the Proposition, and the T is the horizon used in our original construction, see appendix Section D.2.1).

By construction, this simply multiplies the expert's payoffs from our original construction by a constant, ϕ : For example, the disutility to expert type θ from following the strategy of a Group II or III pair – say, recommending $u_1(\theta')$ up to time $t_2(\alpha_a)$, $u_2(\theta')$ up to time t_3, and θ' up to time $\widehat{T}$ - is

$$\begin{aligned}
& \int_0^{t_2(\alpha_a)} e^{-r^E t}\left(u_1(\theta') - \theta - b\right)^2 dt + \int_{t_2(\alpha_a)}^{t_3} e^{-r^E t}\left(u_2(\theta') - \theta - b\right)^2 + \int_{t_3}^{\widehat{T}} e^{-r^E t}\left(\theta' - \theta - b\right)^2 \\
= \;& \phi\left[2\alpha_a\left(u_1(\theta') - \theta - b\right)^2 + 2(1-\alpha_a)\left(u_2(\theta') - \theta - b\right)^2 + (T-2)\left(\theta' - \theta - b\right)^2\right]
\end{aligned}$$

(the second line simply evaluates the integrals using (6) : for example, $\int_0^{t_1(\alpha_0)} e^{-r^E t} dt = \frac{1-e^{-r^E t_1(\alpha_0)}}{r^E} = 2\phi\alpha_0$). This is precisely ϕ times the payoff, from our original construction, to an expert of type $\theta \in [0, \theta_1]$ from following the strategy prescribed for type $\theta' \in [0, \theta_1]$ (see (25) and (29) in appendix, Section D.1.4). Similarly, the disutility to expert type θ from following the strategy of a Group I pair – say, recommending $v_1(\theta')$ up to time $t_1(\alpha_0)$, $v_2(\theta')$ up to time t_4, and θ' up to time $\widehat{T}$ – is exactly ϕ times the payoff, from our original construction, to a perfectly patient expert from recommending $v_1(\theta')$ up to time $2\alpha_0$, $v_2(\theta')$ up to time $T - \tau - 2\alpha_0$, and the truth up to time T. So, since the expert's payoffs are exactly the same as before, for each possible true type θ and each possible type θ' he could choose to mimic, it follows that if the expert finds it optimal to tell the truth in our original construction (with discrete time and discount factor $\delta_E = 1$), then an expert with continuous-time discount rate r^E will likewise find it optimal to tell the truth, given our modified timeline.

For the DM: if $r^{DM} = r^E$, then we likewise obtain that in continuous time, with discount rate r^{DM} and our modified timeline, all payoffs are identical to those in our construction used to prove (1) . If $r^{DM} < r^E$, so that the DM is more patient than the expert, then things only become easier. As discussed in the appendix (Observation D4 of Section D.2), we need only show that the DM cannot gain by deviating at time $t = 2\alpha_0$ (which is now time $t = t_1(\alpha_0)$ with our modified timeline) if he receives a recommendation $v_2(\theta^I)$ from a Group I pair $\{\theta^I, h(\theta^I)\}$, or at time $t = 0$ (when he may get either a recommendation $u_1(\theta^{II}) = v_1(\theta^I)$ which could have been sent by any of the 4 types in $\{\theta^I, h(\theta^I), \theta^{II}, g(\theta^{II})\}$, or a recommendation $u_1(\theta^{III})$, which could only come from a Group III pair $\{\theta^{III}, g(\theta^{III})\}$. Let's first consider deviations at time $t = t_1(\alpha_0)$: in our modified timeline, if he assigns probabilities $p, 1-p$ to the types $\theta^I, h(\theta^I)$ in his information set, then he expects to earn flow disutility $\left(p\left(v_2(\theta^I) - \theta^I\right)^2 + (1-p)\left(v_2(\theta^I) - h(\theta^I)\right)^2\right)$ from time $t_1(\alpha_0)$ to time t_4, at which point the expert should reveal the truth (so disutility drops to zero for the rest of the game). If he instead deviates to the best myopically optimal action, $x^* \equiv p\theta^I + (1-p)h(\theta^I)$, then from time $t_1(\alpha_0)$ to $\widehat{T}$ he'll earn expected flow disutility

$$p\left(x^* - \theta^I\right)^2 + (1-p)\left(x^* - h(\theta^I)\right)^2 = p(1-p)\left(h(\theta^I) - \theta^I\right)^2$$

We need the equilibrium disutility to be smaller than the disutility from deviating, which rearranges to the following condition:

$$\left(\frac{\int_{t_1(\alpha_0)}^{t_4} e^{-r^{DM}t}dt}{\int_{t_1(\alpha_0)}^{\widehat{T}} e^{-r^{DM}t}dt}\right)\left(\frac{p\left(v_2(\theta^I) - \theta^I\right)^2 + (1-p)\left(v_2(\theta^I) - h(\theta^I)\right)^2}{p(1-p)\left(h(\theta^I) - \theta^I\right)^2}\right) \leq 1 \tag{7}$$

Similarly, at a time $t = 0$ information set of the form $\{\theta^{III}, g(\theta^{III})\}$, the gain to deviating is negative whenever the following expression is ≤ 1:

$$\begin{aligned}
&\frac{\int_0^{\widehat{T}} e^{-r^{DM}t} E[\text{flow disutility from following expert's advice} \mid \theta \in \{\theta^{III}, g(\theta^{III})\}]dt}{\int_0^{\widehat{T}} e^{-r^{DM}t} E[\text{flow disutility from best deviation} | \theta \in \{\theta^{III}, g(\theta^{III})\}]dt} \\
= &\frac{\int_0^{t_2(\alpha_a)} e^{-r^{DM}t}dt}{\int_0^{\widehat{T}} e^{-r^{DM}t}dt}\left(\frac{E[\text{flow disutility from } u_1 \mid \theta \in \{\theta^{III}, g(\theta^{III})\}]}{E[\text{flow disutility from best deviation} | \theta \in \{\theta^{III}, g(\theta^{III})\}]}\right) \\
&+ \frac{\int_{t_2(\alpha_a)}^{t_3} e^{-r^{DM}t}dt}{\int_0^{\widehat{T}} e^{-r^{DM}t}dt}\left(\frac{E[\text{flow disutility from action } u_2 \mid \theta \in \{\theta^{III}, g(\theta^{III})\}]}{E[\text{flow disutility from best deviation} \mid \theta \in \{\theta^{III}, g(\theta^{III})\}]}\right)
\end{aligned} \tag{8}$$

And at a time $t = 0$ information set containing both a Group I pair and a Group II pair, $\{\theta^I, h(\theta^I), \theta^{II}, g(\theta^{II})\}$, letting p_1, p_2, p_3, p_4 denote the respective probabilities on the four types, an upper bound on the ratio of equilibrium disutility, to disutility from the best possible deviation, is[12]

[12] In the "disutility ratio" terms, the denominator supposes that whenever $\theta \in \{\theta^I, h(\theta^I)\}$, the DM chooses the myopically optimal action conditional on his information set; and that when $\theta \in \{\theta^{II}, g(\theta^{II})\}$, he chooses the corresponding myopically optimal action. In fact, at time 0, he knows only that $\theta \in \{\theta^I, h(\theta^I), \theta^{II}, g(\theta^{II})\}$, so he will typically do worse than if he knew the true pair: therefore, our expression gives an upper bound on the ratio of equilibrium to deviation disutilty. Section D.2 (see outline, specifically the paragraph referring to Proposition D6) proves that this upper bound is less than 1 in our original construction; since the continuous-time modification here leaves payoffs unchanged, this is then sufficient to establish that the DM cannot gain by deviating.

$$(p_1+p_2)\left[\begin{array}{c}\frac{\int_0^{t_1(\alpha_0)}e^{-r^{DM}t}dt}{\int_0^{\widehat{T}}e^{-r^{DM}t}dt}\left(\frac{E[\text{flow disutility from } v_1(\theta^I)|\{\theta^I,h(\theta^I)\}]}{E[\text{flow disutility from } \frac{p_1\theta^I+p_2h(\theta^I)}{p_1+p_2}|\{\theta^I,h(\theta^I)\}}\right)\\ +\frac{\int_{t_1(\alpha_0)}^{t_4}e^{-r^{DM}t}dt}{\int_0^{\widehat{T}}e^{-r^{DM}t}dt}\left(\frac{E[\text{flow disutility from } v_2(\theta^I)|\{\theta^I,h(\theta^I)\}]}{E[\text{flow disutility from } \frac{p_1\theta^I+p_2h(\theta^I)}{p_1+p_2}|\{\theta^I,h(\theta^I)\}}\right)\end{array}\right] \tag{9}$$
$$+\quad(p_3+p_4)\left[\begin{array}{c}\frac{\int_0^{t_2(\alpha_a)}e^{-r^{DM}t}dt}{\int_0^{\widehat{T}}e^{-r^{DM}t}dt}\left(\frac{E[\text{flow disutility from } u_1(\theta^{II})\mid\{\theta^{II},g(\theta^{II})\}}{E[\text{flow disutility from } \frac{p_3\theta^{II}+p_4g(\theta^{II})}{p_3+p_4}|\{\theta^{II},g(\theta^{II})\}}\right)\\ +\frac{\int_{t_2(\alpha_a)}^{t_3}e^{-r^{DM}t}dt}{\int_0^{\widehat{T}}e^{-r^{DM}t}dt}\left(\frac{E[\text{flow disutility from } u_2(\theta^{II})\mid\{\theta^{II},g(\theta^{II})\}}{E[\text{flow disutility from } \frac{p_3\theta^{II}+p_4g(\theta^{II})}{p_3+p_4}|\{\theta^{II},g(\theta^{II})\}}\right)\end{array}\right]$$

We have an equilibrium if (7) holds for every group I pair $\{\theta^I, h(\theta^I)\}$, the expression in (8) is weakly below 1 for every Group III pair $\{\theta^{III}, g(\theta^{III})\}$, and the expression in (9) is weakly below 1 for every information set of the form $\{\theta^I, h(\theta^I), \theta^{II}, g(\theta^{II})\}$. At $r^{DM} = r^E$, these reduce to precisely the inequalities proven to hold in Appendix D.2 (Propositions 5 and 6). (For example, consider (7) : at $r^{DM} = r^E$, using (6), the first "time ratio" term reduces to $\frac{T-\tau-2\alpha_0}{T-2\alpha_0}$. And since we have not made any modifications to the action functions or information sets, the constraint then simply says that the length of the v_2-recommendation phase in our original construction $(T-\tau-2\alpha_0)$, times the flow disutility from choosing v_2, must be smaller than the length of the remaining game $(T-2\alpha_0)$, times the flow disutility from choosing the myopically optimal action).

We complete the proof in the Appendix (Section E), showing that the constraints in (9), (8), and (7) become more relaxed as r^{DM} decreases. ■

As r^{DM} approaches zero and the horizon increases, we can push the result further, obtaining an equilibrium with attractive welfare properties for a large range of biases and priors:

Theorem 2 *If r^E is bounded above zero, preferences are quadratic (as in (1)), and $b < \frac{1}{16}$, then, for any prior F with a density that is everywhere bounded away from zero and infinity, there exists a horizon T^* and a discount rate $r^* > 0$ such that a fully revealing equilibrium exists whenever $r^{DM} < r^*$ and $\widehat{T} > T^*$. In this equilibrium, the decision-maker's average payoff goes to his full-information payoff as $\widehat{T} \to \infty$ and $r^{DM} \to 0$.*

Proof: Again use the timeline (6), together with all action and strategy specifications of our original construction; as above in Proposition 4, this leaves all utility expressions and analysis for the expert unchanged from our original construction; and as noted in the appendix, Proposition D2, our equilibrium works for the expert for all $b < \frac{1}{16}$. For the decision-maker, we just need to check the incentive constraints in (9), (8), and (7). We can conclude, from the fact that they held in our original construction, that the equilibrium flow disutilities (numerators in the second term of each expression) must be bounded. Our original construction also specifies "partner" functions which have positive and finite derivatives, which implies (see (5)) that for any prior which has a positive bounded density, the decision-maker's posteriors over all information sets assign a strictly positive probability to each type; this implies that the myopically optimal action at each information set is bounded away from the true state, and therefore the flow disutility to the decision-maker if he deviates to the myopically optimal action is bounded away from zero. We conclude that the second "flow disutility" ratios in (9), (8), and (7) are all finite. However, the first "time ratio" terms in these expressions go to zero as $r^{DM} \to 0$ and $\widehat{T} \to \infty$, noting that (6) and r^E bounded above zero imply that $t_1(\alpha_0), t_2(\alpha_a), t_3, t_4$ are all finite. Therefore, the ratio of the DM's equilibrium disutility, compared to his disutility from the best deviation, goes to zero as $r^{DM} \to 0$ and $\widehat{T} \to \infty$, implying that we have an equilibrium. Moreover, since the DM's equilibrium flow disutility is bounded up to time t_4 and zero thereafter, with t_4 finite, it follows that as $r^{DM} \to 0$, the DM's average expected payoff goes to his full-information payoff, zero. ■

5 Concluding Remarks

This paper shows that dynamic strategic communication differs fundamentally from its static counterpart.

The main novel ingredient of our model is that there are multiple rounds of communication, with a new action chosen after each round. The dynamic incentive considerations for the expert allow us to group together types that are far apart, forming "separable groups", which is the key to obtaining greater information revelation. Our dynamic setup also allows for future communication to be conditioned on past actions (trigger strategies); we show how information revelation can be facilitated through this channel.

The forces that we identify may be present in many dynamic environments with asymmetric information. Think, for example, of a dynamic contracting environment with limited commitment, or more generally, of a dynamic mechanism problem. In these models as well, past behavior sets the stage for future behavior. And, in contrast to the vast majority of the recent literature on dynamic mechanism design,[13] one needs to worry about both global and local incentive constraints, even with simple stage payoffs that satisfy the single-crossing property.

Lastly, given the important insights from cheap talk literature which have been widely applied in both economics and political science, we hope and expect that the novel aspects of strategic communication emphasized in our analysis will help shed light on many interesting dynamic problems.

[13] In recent years, motivated by the large number of important applications, there has been substantial work on dynamic mechanism design. See, for example, the survey of Bergemann and Said (2010), and the references therein, or Pavan, Segal and Toikka (2011).

A Proof of Proposition 2

When we restrict attention to monotonic partition equilibria, there will be some point in the game at which the last subdivision of an interval occurs, say period $\widehat{T} \leq T$. Assume (without loss of generality) that one interval is partitioned into two, inducing actions y_1 and y_2, and let $\widehat{\theta}$ be the expert type who is indifferent between y_1, y_2. Since no subdivision occurs after period $\widehat{T}$, it follows that type $\widehat{\theta}$'s indifference condition in period $\widehat{T}$ is

$$\left(1+\delta+..+\delta^{T-\widehat{T}-1}\right)u^E\left(y_1,\hat{\theta},b\right) \geq \left(1+\delta+..+\delta^{T-\widehat{T}-1}\right)u^E\left(y_2,\hat{\theta},b\right),$$

which reduces to the static indifference condition. But then, if this subdivision is possible, it cannot be the case that all static equilibria are equivalent babbling. This follows by Corollary 1 of Crawford and Sobel (1982)).

Observe that all the arguments in this proof go through even if we allow for trigger strategies. This is because at the point where the last subdivision occurs, it is impossible to incentivize the decision-maker to choose anything other than his myopic best response: he knows that no further information will be revealed, and so he knows that he cannot be rewarded in the future for choosing a suboptimal action now. So, the above argument applies.

B Monotonic partition equilibria with more partitions

Suppose that $\delta_E = \delta_{DM} = 1$, types are uniformly distributed on $[0,1]$ and preferences satisfy (1), with bias $b = \frac{1}{12}$. Using the standard arguments, one can establish that game has only two equilibria:[14] a babbling equilibrium, and an equilibrium with two partitions, $\left[0,\frac{1}{3}\right]\cup\left[\frac{1}{3},1\right]$, inducing actions $\frac{1}{6}$ and $\frac{4}{6}$. Now we show that when $T=2$, there exists a monotonic partition equilibrium where the state space is ultimately divided into three sub-intervals.

We look for an equilibrium with the following signaling rule:

$$\begin{aligned}
&\text{types in } [0,\theta_1] \text{ send message sequence } A = (m_{1(1)}, m_{2(1)}),\\
&\text{types in } [\theta_1,\theta_2] \text{ send message sequence } B = (m_{1(2)}, m_{2(2)}),\\
&\text{types in } [\theta_2,1] \text{ send message sequence } C = (m_{1(2)}, m_{2(3)}).
\end{aligned}$$

With this signaling rule, in the first period the interval $[0,1]$ is partitioned into $[0,\theta_1]$ and $[\theta_1,1]$. The indifference condition for type θ_2 in period 2 yields

$$\left(\frac{\theta_1+\theta_2}{2}-\theta_2-b\right)^2 = \left(\frac{1+\theta_2}{2}-\theta_2-b\right)^2 \Rightarrow \theta_2 = \frac{1}{3}+\frac{1}{2}\theta_1 \tag{12}$$

The second-period actions induced are $y_{2(1)} = \frac{\theta_1}{2}, y_{2(2)} = \frac{3}{4}\theta_1 + \frac{1}{6}$ and $y_{2(3)} = \frac{1}{4}\theta_1 + \frac{2}{3}$, and the first-period actions are $y_{1(1)} = \frac{\theta_1}{2}$ and $y_{1(2)} = \frac{1+\theta_1}{2}$.

[14] The largest number of subintervals that the type space can be divided into is the largest integer that satisfies

$$-2bp^2+2bp+1>0, \tag{10}$$

whose solution is

$$\left\langle -\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{2}{b}}\right\rangle, \tag{11}$$

and where $\langle x\rangle$ denotes the smallest integer greater than or equal to x.

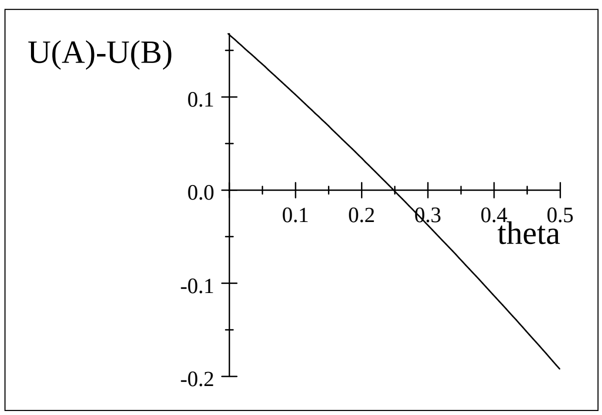
After any out-of- equilibrium message the decision maker assigns probability one to the state belonging in $[0, \theta_1]$ inducing $y^{out} = \frac{\theta_1}{2}$. With these out-of equilibrium beliefs it is immediate to see that no type has an incentive to send an out-of-equilibrium message.

At equilibrium, θ_1 must satisfy the following indifference condition:

$$\left(\underbrace{\frac{1+\theta_1}{2}}_{y_{1(2)}} - \theta_1 - \frac{1}{12}\right)^2 + \left(\underbrace{\frac{3}{4}\theta_1 + \frac{1}{6}}_{y_{2(2)}} - \theta_1 - \frac{1}{12}\right)^2 = 2\left(\underbrace{\frac{\theta_1}{2}}_{y_{1(1)}=y_{2(1)}} - \theta_1 - \frac{1}{12}\right)^2$$

which is solved by $\theta_1 = 0.2482$; together with (12) , we then obtain three final partitions, with cutoffs $\theta_1 = 0.2482$, $\theta_2 = 0.457\,43$; with this, the actions become $y_{1(1)} = y_{2(1)} = 0.124\,1$, $y_{1(2)} = 0.624\,1$, $y_{2(2)} = 0.352\,8$, and $y_{2(3)} = 0.728\,7$.

In constructing this strategy profile, we imposed only local incentive compatibility constraints, requiring that type θ_1 is indifferent in period 1 between inducing action sequence $\left(y_{1(1)}, y_{2(1)}\right)$ and $\left(y_{1(2)}, y_{2(2)}\right)$, and that type θ_2 is indifferent in period 2 between inducing actions $y_{2(2)}$ and $y_{2(3)}$. Now we want to verify that these conditions are sufficient for global incentive compatibility. In period 2 the game is isomorphic to the static one, where the fact that θ_2 is indifferent between $y_{2(2)}$ and $y_{2(3)}$ implies that all types above θ_2 prefer $y_{2(3)}$ and all types below θ_2 prefer $y_{2(2)}$. To verify that types below θ_1 prefer message sequence A and types above θ_1 prefer message sequence B, we plot the difference $U(A, \theta) - U(B, \theta)$ and show that it is positive for all $\theta < \theta_1$ and negative for $\theta > \theta_1$:



In our dynamic equilibrium, the expert's (ex ante) payoff is -0.0659 and the decision-maker's (ex ante) payoff is -0.052. If the most informative static equilibrium is played in both periods, payoffs are -0.069 to the expert, -0.055 to the decision-maker, both strictly worse than in our dynamic monotonic partition equilibrium.

C Pareto comparisons of dynamic cheap talk equilibria

The following example demonstrate that equilibria with more partitions can be Pareto inferior to the equilibria with fewer partitions

Take $\delta_E = \delta_{DM} = 1$ and $b = 0.08$, and consider the most informative static partition equilibrium where the number of partitions is $p = 3$.In this equilibrium the state space is divided into $[0, 0.013]$, $[0.013, 0.347]$ and $[0.347, 1]$. The corresponding optimal actions for the decision-maker are given by

$$y_1 = 0.0067 \quad y_2 = 0.18 \quad y_3 = 0.673,$$

from which we can calculate the ex-ante expected utility levels for the expert -0.032 and for the decision maker -0.0263. Then, at the equilibrium of the dynamic game where the most informative static equilibrium is played at $t = 1$ and babbling thereafter, the total expected utility is -0.065 for the expert, and -0.053 for the decision maker.

We now construct a dynamic equilibrium where the type space is subdivided into more subintervals, but both players' ex-ante expected payoffs are lower. We look for an equilibrium with the following signaling rule:

$$\begin{aligned}
&\text{types in } [0, \theta_1] \text{ send message sequence } (m_{1(1)}, m_{2(1)}) \\
&\text{types in } [\theta_1, \theta_2] \text{ send message sequence } (m_{1(2)}, m_{2(2)}) \\
&\text{types in } [\theta_2, \theta_3] \text{ send message sequence } (m_{1(2)}, m_{2(3)}) \\
&\text{types in } [\theta_3, 1] \text{ send message sequence } (m_{1(3)}, m_{2(4)}).
\end{aligned}$$

So types are partitioned into four intervals in stage 2, but in stage 1, the types in $[\theta_1, \theta_2]$ and $[\theta_2, \theta_3]$ pool together to send the same message $m_{1(2)}$. Since the signaling rule does not depend on the decision maker's action at stage 1, the decision maker will choose the following myopically optimal actions:

$$\begin{aligned}
y_{1(1)} &= y_{2(1)} = \frac{\theta_1}{2}, \\
y_{1(2)} &= \frac{\theta_1 + \theta_3}{2}, \; y_{2(2)} = \frac{\theta_1 + \theta_2}{2}, \; y_{2(3)} = \frac{\theta_2 + \theta_3}{2}, \\
y_{1(3)} &= y_{2(4)} = \frac{1 + \theta_3}{2}.
\end{aligned}$$

After any out-of-equilibrium message the decision maker assigns probability one to the state belonging in $[0, \theta_1]$ inducing $y^{out} = \frac{\theta_1}{2}$. With these out-of-equilibrium beliefs it is immediate to see that no type has an incentive to deviate.

In equilibrium, type θ_1 is indifferent between action sequences $\left(y_{1(1)}, y_{2(1)}\right)$ and $\left(y_{1(2)}, y_{2(2)}\right)$, type θ_2 is indifferent between 2nd-period actions $y_{2(2)}$ and $y_{2(3)}$, and type θ_3 is indifferent between action sequences $\left(y_{1(2)}, y_{2(3)}\right)$ and $\left(y_{1(3)}, y_{2(4)}\right)$. Therefore, equilibrium cutoffs are the solution to the following system of equations:[15]

$$\begin{aligned}
2\left(\frac{\theta_1}{2} - \theta_1 - b\right)^2 - \left(\frac{\theta_1 + \theta_3}{2} - b - \theta_1\right)^2 - \left(\frac{\theta_1 + \theta_2}{2} - b - \theta_1\right)^2 &= 0, \\
\left(\frac{\theta_1 + \theta_2}{2} - b - \theta_2\right)^2 - \left(\frac{\theta_2 + \theta_3}{2} - b - \theta_2\right)^2 &= 0, \\
2\left(\frac{1 + \theta_3}{2} - b - \theta_3\right)^2 - \left(\frac{\theta_1 + \theta_3}{2} - b - \theta_3\right)^2 - \left(\frac{\theta_2 + \theta_3}{2} - b - \theta_3\right)^2 &= 0.
\end{aligned}$$

At $b = 0.08$, the only solution that gives numbers in $[0, 1]$ is $\theta_1 = 0.0056, \theta_2 = 0.015, \theta_3 = 0.345$, and the actions induced for $t = 1$ and for $t = 2$ are respectively given by $y_{1(1)} = y_{2(1)} = 0.00278$, $y_{1(2)} = 0.175$, $y_{2(2)} = 0.0105$, $y_{2(3)} = 0.18$ and $y_{1(3)} = y_{2(4)} = 0.673$. This implies the following total ex-ante expected utility for the expert -0.066, which is lower than $2(-0.033) = -0.0656$. The utility for the decision-maker is -0.053 which is lower than $2(-0.026) = 0.052$.

This example illustrates that although the interval is divided into more subintervals here, both players strictly worse off compared to the one where the most informative static equilibrium is played in the first period and babbling thereafter. The feature that less partitions lead to higher ex-ante welfare for both players also appears in example 1 of Blume, Board and Kawamura (2007).

[15] It is trivial to check exactly as we did in previous examples that these indifference conditions suffice for global incentive compatibility.

D Proof of Theorem 1

We will prove by construction that a fully revealing equilibrium exists. We first choose the endpoints $\theta_1, \theta_2, \theta_3$ described in the proof outline: for any bias $b < \frac{1}{61}$, define $a_\gamma < 0$ by

$$(a_\gamma - 2 + 2e^{-a_\gamma})e^2 - a_\gamma = \frac{1}{b} \tag{13}$$

and then set

$$\theta_3 = \frac{1}{b} + a_\gamma,\ \ \theta_2 = \theta_3 - 2,\ \ \theta_1 = \theta_2 - \theta_3 e^{-2} \tag{14}$$

It will be convenient to describe types parametrically, via functions $x : [-2, 0] \to [0, \theta_1]$, $g : [-2, 0] \to [\theta_2, \theta_3]$, $z : [a_\gamma, 0] \to [\theta_1, \theta_2]$, and $h : [a_\gamma, 0] \to [\theta_3, \frac{1}{b}]$. Then, let $u_1(a, \alpha_a)$, $u_2(a, \alpha_a)$ denote the first, second recommendations of types $x(a), g(a)$ (for all $a \in [-2, 0]$), and let $v_1(a, \alpha_0), v_2(a, \alpha_0)$ denote the first, second recommendations of types $(z(a), h(a))$ (for all $a \in [a_\gamma, 0]$). With this notation, Groups I,II, III described in the text are as follows:

$$\begin{aligned}
\text{Group } I &= \{z(a), h(a) \mid a \in [a_\gamma, 0]\} \\
\text{Group } II &= \{x(a), g(a) \mid a \in [-2, 0],\ \text{and } \exists\ a' \in [a_\gamma, 0] \text{ with } v_1(a', \alpha_0) = u_1(a, \alpha_a)\} \\
\text{Group } III &= \{x(a), g(a) |\ a \in [-2, 0],\ \text{and } x(a), g(a) \notin \text{Group } II\}
\end{aligned}$$

In our proposed equilibrium construction, each Group I pair $\{z(a), h(a)\}$ recommends $v_1(a, \alpha_0)$ for $2\alpha_0$ periods, then $v_2(a, \alpha_0)$ for $T - \tau - 2\alpha_0$ periods, then reveals the truth at time $T - \tau$; each Group II pair $\{x(a), g(a)\}$ recommends $u_1(a, \alpha_a)$ for $2\alpha_a$ periods, then $u_2(a, \alpha_a)$ for $2(1 - \alpha_a)$ periods, then separates and reveals the truth for the final $T - 2$ periods; and moreover, the recommendation $u_1(a, \alpha_a)$ coincides with the recommendation $v_1(a', \alpha_0)$ of some Group I pair $\{z(a'), h(a')\}$. Group III is identical to Group II, except that their recommendations *do not* coincide with those of any Group I pair.

We also specify the following off-path strategy for the expert: if the decision-maker ever deviates, by rejecting a recommendation that the expert made, then (i) if the expert himself has *not* previously deviated: send no further recommendations (equivalently, repeat the current recommendation in all subsequent periods). And (ii) if the expert *has* observably deviated in the past, behave as if the deviation did not occur. (For example, if he sends the initial recommendation $u_1(0, \alpha_0)$ prescribed for types $\{x(0), g(0)\}$, but then follows this with anything other than recommendation $u_2(0, \alpha_0)$ at time $2\alpha_0$, subsequently behave as if the deviation never occurred and he indeed sent $u_2(0, \alpha_0)$ at time $2\alpha_0$).

D.1 Optimality for the Expert

We prove that the expert wishes to follow the prescribed recommendation strategy via three propositions. Proposition D1 specifies strategies and beliefs for the decision-maker such that the expert has no incentive to send an out-of-equilibrium recommendation sequence, so we need only make sure that he does not wish to mimic any other type. Proposition D2 shows that in the prescribed revelation phase, the expert indeed finds it optimal to reveal the truth, provided that there have been no previous deviations. It remains only to show that the expert has no incentive to deviate prior to the prescribed revelation phase - by mimicking the initial recommendations of some other type - which we show in Proposition D3.

We specify the following strategy and beliefs for the decision-maker:

If there are no detectable deviations by the expert (i.e., he sends the equilibrium recommendation sequence for some type $\theta \in [0, \frac{1}{b}]$), then follow all recommendations, using Bayes' rule to assign beliefs at each information set. Following deviations: (i) If the expert observably deviates at time 0 (sending an off-path initial recommendation), subsequently adopt the strategy/beliefs that would follow if the expert had instead sent the recommendation $u_1(0, \alpha_0)$ prescribed for types $\{x(0), g(0)\}$; (ii) If the expert observably deviates on his 2nd recommendation (i.e., if an initial recommendation

$u_1(a, \alpha_a)$ (or $v_1(a, \alpha_0)$) is followed by something *other* than $u_2(a, \alpha_a)$ (or $v_2(a, \alpha_0)$), ignore it as an error, and subsequently adopt the strategy/beliefs that would follow had the deviation not occurred; (iii) If the expert deviates observably in the revelation phase, ignore it as an error, assigning probability 1 to the lowest type in the current information set, and accordingly choosing this as the myopically optimal action; (iv) And finally, if the decision-maker himself deviates, rejecting some recommendation by the expert, then he subsequently maintains the current (at time of deviation) beliefs, anticipating that the expert will subsequently repeat the current (at time of deviation) recommendation, and ignoring any other recommendations as errors.

D.1.1 Expert Optimality: Off-Path Behavior

Proposition D1: Under the above strategy and beliefs prescribed for the decision-maker, the expert has no incentive to choose an off-path recommendation sequence.
Proof of Proposition D1: Follows trivially from the specified strategy and beliefs for the decision-maker: (i) a deviation at time zero is equivalent to mimicking type $x(0)$ (who recommends $u_1(0, \alpha_0)$ at time $t = 0$); (ii) a deviation on the 2nd recommendation has no effect, since the decision-maker ignores it; (iii) a deviation in the revelation phase, if there have been no previous deviations, is equivalent to mimicking the strategy of the lowest type in the decision-maker's current (pre-revelation) information set; and (iv) if the decision-maker *has* previously deviated, then (by point (iv) of the above strategy-belief specification) he will chose whichever action was myopically optimal at the time of deviation, regardless of the expert's message; therefore, babbling is optimal for the expert, since his message has no effect on the decision-maker's action. ■

D.1.2 Expert Optimality: Truth Revelation Phase

Proposition D2: In the prescribed revelation phase, (i) if there have been no previous deviations by the decision-maker, then the expert finds it optimal to reveal the truth; (ii) if the decision-maker has ever deviated, then the expert finds it optimal to babble (e.g. by remaining silent).
Proof of Proposition D2: Part (ii) follows immediately from Proposition D1 (iv). For part (i): our specification of the expert strategy is such that at time $2\alpha_0$, the decision-maker's information set contains at most two types: either a pair $\{x(a), g(a)\}$ (in which case he plans to choose $g(a)$ if the expert recommends $g(a)$, $x(a)$ otherwise), or a pair $\{z(a), h(a)\}$ (in which case he plans to choose $h(a)$ if the expert recommends it, $z(a)$ otherwise). So, it suffices to show that each type would rather tell the truth than mimic his partner, which requires (in our rescaled state space) simply that all paired types be at least 2 units apart. By (15) we have

$$\min_{a \in [-2,0]} |g(a) - x(a)| = \theta_2 - \theta_1$$
$$\min_{a \in [a_\gamma,0]} |h(a) - z(a)| = \theta_3 - \theta_2$$

And by (14), $\theta_3 - \theta_2 = 2$, and $\theta_2 - \theta_1 = (a_\gamma - 2 + 2e^{-a_\gamma})$, which is greater than 2 whenever $a_\gamma < -.8951 \Leftrightarrow b < \frac{1}{15.67}$ (using (13)). This is in fact all that is needed for the construction to work for the expert, but we specify $b < \frac{1}{61}$ in (13) (see (64) for details) to make the construction work for the decision-maker. ■

D.1.3 Expert Optimality: Initial Recommendations

Propositions D1,D2 imply that once the expert has sent the initial recommendation (u_1 or v_1) prescribed for some type θ, it is optimal to follow also the continuation recommendations prescribed for that type. So, the only time when it could possibly be profitable to deviate is at time $t = 0$: we need to make sure that each type θ prefers to send the proposed equilibrium sequence of recommendations, rather than the sequence prescribed for any other type θ'.[16] We now choose parametrizations of functions x, g, z, h, along with action function u_1, u_2, v_1, v_2, which guarantee that the expert indeed finds it optimal to send the prescribed initial recommendation:

Proposition D3: Let the action functions and type parametrizations be as follows:

$$x(a) = \theta_3 + a - \theta_3 e^a, g(a) = \theta_3 + a,\ z(a) = \frac{1}{b} + a - 2e^{a-a_\gamma},\ h(a) = \frac{1}{b} + a \tag{15}$$

$$u_1(a, \alpha_a) = \theta_3 + K - \frac{T-2}{2}a - \sqrt{\frac{1-\alpha_a}{\alpha_a}}\sqrt{T-2}\sqrt{C_u + a\left(K - \frac{T}{4}a\right)} \tag{16}$$

$$u_2(a, \alpha_a) = \theta_3 + K - \frac{T-2}{2}a + \sqrt{\frac{\alpha_a}{1-\alpha_a}}\sqrt{T-2}\sqrt{C_u + a\left(K - \frac{T}{4}a\right)} \tag{17}$$

$$v_1(a, \alpha_0) = \theta_3 + \frac{2K-\tau(a-a_\gamma)}{T-\tau} - \frac{\sqrt{\frac{\tau(T-\tau-2\alpha_0)}{\alpha_0}}\sqrt{\frac{(T-\tau)(T-2)}{\tau}C_u + \left(\frac{T-\tau-2}{\tau}\right)K^2 + 2K(a-a_\gamma) - \frac{T}{2}(a-a_\gamma)^2}}{T-\tau} \tag{18}$$

$$v_2(a, \alpha_0) = \theta_3 + \frac{2K-\tau(a-a_\gamma)}{T-\tau} + \frac{\sqrt{\frac{4\tau\alpha_0}{T-\tau-2\alpha_0}}\sqrt{\frac{(T-\tau)(T-2)}{\tau}C_u + \left(\frac{T-\tau-2}{\tau}\right)K^2 + 2K(a-a_\gamma) - \frac{T}{2}(a-a_\gamma)^2}}{T-\tau} \tag{19}$$

for constants C_u, K, and for now taking T, α_0, α_a as given (T is the horizon, and α_a, α_0 relate to the duration of recommendations u_1, v_1 as described in the strategies above). Also set τ (length of the revelation phase for types in Group I) according to

$$\frac{\tau}{T-2} = \beta \equiv \frac{(\theta_2 - \theta_1)(\theta_2 - \theta_1 - 2)}{(\theta_4 - \theta_1)(\theta_4 - \theta_1 - 2)} \tag{20}$$

Then, for all types $\theta, \theta' \in [0, \frac{1}{b}]$, expert type θ prefers his equilibrium recommendation sequence to that sent by type θ', and in particular has no incentive to deviate at time $t = 0$.

Proof of Proposition D3:

Let $D_u(\theta'|\theta)$ denote the disutility to type θ from following the recommendation sequence prescribed for a type $\theta' \in [0, \theta_1] \cup [\theta_2, \theta_3]$, and let $D_v(\theta'|\theta)$ denote the disutility to type θ from following the strategy prescribed for a type $\theta' \in [\theta_1, \theta_2] \cup [\theta_3, \frac{1}{b}]$. The proof proceeds through two main Lemmas. Lemma D3.1 proves that the expert strategy is locally incentive compatible: for each interval $[\theta_i, \theta_{i+1}]$, $i \in \{0, 1, 2, 3\}$, no expert type $\theta \in [\theta_i, \theta_{i+1}]$ wishes to mimic any other type $\theta' \in [\theta_i, \theta_{i+1}]$ from the same interval. Lemma D3.2 proves that the expert strategy is also globally incentive compatible: no expert type wishes to mimic any type θ' from any other interval. The proofs will use calculations obtained below in Lemmas D3.3 and D3.4.

Lemma D3.1 (Local IC): For each interval $[\theta_i, \theta_{i+1}]$, with $i = 0, 1, 2, 3$, and any pair of types $\theta, \theta' \in [\theta_i, \theta_{i+1}]$, the disutility to type θ from mimicking type θ' is (weakly) *increasing* in $\left|\theta' - \theta\right|$, thus minimized when $\left|\theta' - \theta\right| = 0$. Therefore, for each $\theta \in [\theta_i, \theta_{i+1}]$, truth-telling is (weakly) better than mimicking any other type in the interval.

Proof of Lemma D3.1:

Differentiating disutility expressions (25), (26) , (27) , and (28) (obtained below in Lemma D3.3)

[16] This is what the text refers to as "providing incentives to join the right separable group". We need to make sure, for example, that type $\theta = 0$ prefers to induce the action sequence $(u_1(0), u_2(0), 0)$, rather than e.g. the sequence that type $\theta' \neq 0$ is supposed to send; by Propositions D1,D2, the choice to follow a different recommendation sequence can only be made at time $t = 0$.

gives

$$
\begin{aligned}
\frac{dD_u(x(a)|\theta)}{dx(a)} &= \frac{dD_u(g(a)|\theta)}{dx(a)} + \frac{2(T-2)\left(x(a)-\theta-1\right)x'(a) - 2(T-2)\left(g(a)-\theta-1\right)g'(a)}{x'(a)} \\
&= 0 + 2(T-2)\left(\theta_3 + a - \theta_3 e^a - \theta - 1 - \frac{\theta_3 + a - \theta - 1}{1-\theta_3 e^a}\right) \text{ (by (15))} \\
&= 2(T-2)\left(\frac{\theta_3 e^a}{\theta_3 e^a - 1}\right)\left(x(a)-\theta\right) \qquad (21) \\
\frac{dD_v(z(a)|\theta)}{dz(a)} &= \frac{dD_v(h(a)|\theta)/da}{z'(a)} + \frac{2\tau\left(z(a)-\theta-1\right)z'(a) - 2\tau\left(h(a)-\theta-1\right)h'(a)}{z'(a)} \\
&= 2\tau\left(\frac{2e^{a-a_\gamma}}{2e^{a-a_\gamma}-1}\right)\left(z(a)-\theta\right) \qquad (22) \\
\frac{dD_u(g(a)|\theta)}{dg(a)} &= 0 \qquad (23) \\
\frac{dD_v(h(a)|\theta)}{dh(a)} &= 0 \qquad (24)
\end{aligned}
$$

Consider first a type $\theta \in [0, \theta_1]$. By (21) , noting that $\frac{\theta_3 e^a}{\theta_3 e^a - 1} > 0$ (since $\theta_3 e^a \geq \theta_3 e^{-2} = \theta_2 - \theta_1 \geq 8$, by Proposition D2), we see that $\frac{dD_u(x(a)|\theta)}{dx(a)}$ has the same sign as $(x(a)-\theta)$. So if $x(a)-\theta > 0$, then $D_u(x(a)|\theta)$ is *increasing* in $x(a)$, thus increasing in $(x(a)-\theta)$; while if $x(a)-\theta < 0$, then $D_u(x(a)|\theta)$ is increasing in $(-x(a))$, thus increasing in $\theta - x(a)$. Combined, these establish that $D_u(x(a)|\theta)$ is strictly increasing in $|x(a)-\theta|$, as desired.

Next consider a type $\theta \in [\theta_1, \theta_2]$. By (22) , noting that $\left(\frac{2e^{a-a_\gamma}}{2e^{a-a_\gamma}-1}\right) > 0$ (since $a \in [a_\gamma, 0]$ implies $2e^{a-a_\gamma} \geq 2$), we see that $\frac{dD_v(z(a)|\theta)}{dz(a)}$ has the same sign as $z(a)-\theta$, and is thus positive (disutility increasing in $z(a)-\theta$) if $z(a) > \theta$, and negative (disutility increasing in $\theta - z(a)$) if $z(a) < \theta$. Combined, these establish that $D_u(z(a)|\theta)$ is strictly increasing in $|z(a)-\theta|$, as desired.

By (23) and (24) , the disutility to type θ from mimicking a type $g(a) \in [\theta_2, \theta_3]$ or $h(a) \in [\theta_3, \frac{1}{b}]$ is independent of the particular type $g(a), h(a)$ chosen. Thus, $D_u(g(a)|\theta)$ is weakly increasing (in fact constant) in $|g(a)-\theta|$, and $D_u(h(a)|\theta)$ is weakly increasing (constant) in $|h(a)-\theta|$, completing the proof. ■

Lemma D3.2: For every interval $[\theta_i, \theta_{i+1}]$ $(i = 0, 1, 2, 3)$, and every $\theta \in [\theta_i, \theta_{i+1}]$, following the prescribed (truthful) recommendation sequence is better than mimicking any type θ' drawn from any *other* interval $[\theta_j, \theta_{j+1}]$ with $j \neq i$.

Proof of Lemma D3.2:

Consider first a type $\theta \in [0, \theta_1]$. By Lemma D3.1, truth-telling is better than mimicking any other type $\theta' \in [0, \theta_1]$, in particular type $\theta_1 = x(-2)$. By Lemma D3.4 (i) (below), type $\theta \in [0, \theta_1]$ prefers type $x(-2)$'s sequence to type $z(0)$'s sequence; and by Lemma D3.1, it is better to mimic type $z(0) = \theta_1$, than any other type $z(a) \in (\theta_1, \theta_2]$ (since $z(a) > \theta$ implies that $D_v(z(a)|\theta)$ is increasing in $z(a)-\theta$); together, these establish that mimicking a type $\theta' \in [\theta_1, \theta_2]$ is not optimal. By Lemma D3.4 (ii), type $\theta \leq \theta_2$ prefers $z(a_\gamma)$'s sequence (right endpoint of $[\theta_1, \theta_2]$) to $g(-2)$'s sequence (left endpoint of $[\theta_2, \theta_3]$); and by Lemma D3.1, disutility to type θ from mimicking type $g(a) \in [\theta_2, \theta_3]$ is independent of a; together, this implies that type θ also does not want to mimic any type $g(a) \in [\theta_2, \theta_3]$. And finally, by Lemma D3.4 (iii), type $\theta \leq \theta_2$ prefers the sequence prescribed for type $g(0)$ (right endpoint of $[\theta_2, \theta_3]$) to that prescribed for type $h(a_\gamma)$ (left endpoint of $[\theta_3, \frac{1}{b}]$), which (by Lemma D3.1) yields the same utility as mimicking any other type $h(a) \in [\theta_3, \frac{1}{b}]$, thus it is not optimal to mimic any type $h(a) \in [\theta_3, \frac{1}{b}]$. This establishes that type $\theta \in [0, \theta_1]$ does not wish to mimic any type θ' from any other interval.

Next consider type $\theta \in [\theta_1, \theta_2]$. By Lemma D3.1, truth-telling is better than mimicking any other type $z(a) \in [\theta_1, \theta_2]$, in particular type $z(0) = \theta_1$; by Lemma D3.4 (i), type $\theta \geq \theta_1$ prefers the sequence prescribed for type $z(0)$, to that prescribed for type $x(-2)$; and by Lemma D3.1, it is better to mimic $x(-2)$ (right endpoint of $[0,\theta_1]$) than any other type $x(a)\in[0,\theta_1]$, since $D_u(x(a)|\theta)$ is increasing in $|\theta-x(a)|$ and we have here $\theta>x(a)$; together, this implies that type θ does not wish to mimic any type $\theta'\in[0,\theta_1]$. The proof that he doesn't wish to mimic any type $g(a)\in[\theta_2,\theta_3]$ or $h(a)\in[\theta_3,\frac{1}{b}]$ is identical to the one given in the previous paragraph.

Now consider a type $\theta\in[\theta_2,\theta_3]$. As explained in the previous two paragraphs, following the truthful recommendation sequence yields the same utility as mimicking any other type $g(a)\in[\theta_2,\theta_3]$ or $h(a)\in[\theta_3,\theta_4]$, so we just need to make sure that it is not optimal to mimic types $\theta'\in[0,\theta_1]\cup[\theta_1,\theta_2]$. By Lemma D3.4 (ii), type $\theta\geq\theta_2$ prefers type $g(-2)$'s sequence (left endpoint of $[\theta_2,\theta_3]$) to type $z(a_\gamma)$'s sequence (right endpoint of $[\theta_1,\theta_2]$); by Lemma D3.1, such a type $\theta\geq\theta_2$ also prefers type $z(a_\gamma)$'s sequence to the one prescribed for any other (further-away) type $z(a)\in[\theta_1,\theta_2]$; combined, this establishes that mimicking a type $z(a)\in[\theta_1,\theta_2]$ is not optimal. By Lemma D3.4 (i), it is better to mimic type $z(0)$'s sequence than $x(-2)$'s sequence, which in turn is better (by Lemma D3.1) than any other type $x(a)$'s sequence. Thus, it is not optimal to mimic any type $x(a)\in[0,\theta_1]$, completing the proof for types $\theta\in[\theta_2,\theta_3]$.

The argument that types $\theta\in[\theta_3,\frac{1}{b}]$ don't wish to mimic types from other intervals is identical to the proof in the previous paragraph (for types $\theta\in[\theta_2,\theta_3]$).

This completes the proof of Lemma D3.2. ■

D.1.4 Expert Optimality: Preliminary Calculations

Lemma D3.3: Given the type parametrizations and action functions given in Proposition D3, disutility expressions $D_u\left(\theta'|\theta\right)$, $D_v(\theta'|\theta)$ are given by

$$\begin{aligned}
D_u(x(a)|\theta) &= D_u(g(a)|\theta)+(T-2)\left(x(a)-\theta-1\right)^2-(T-2)\left(g(a)-\theta-1\right)^2 && (25)\\
D_u(g(a)|\theta) &= T\left(\theta_3-\theta-1\right)^2+4K\left(\theta_3-\theta-1\right)+2K^2+2(T-2)C_u && (26)\\
D_v(z(a)|\theta) &= D_v(h(a)|\theta)-2\tau\left(h(a)-z(a)\right)\left(\frac{h(a)+z(a)}{2}-\theta-1\right) && (27)\\
D_v(h(a)|\theta) &= 2K^2+2(T-2)C_u+4\left(\theta_3-\theta-1\right)K+T\left(\theta-\theta_3+1\right)^2 && (28)
\end{aligned}$$

Proof of Lemma D3.3:

The disutility $D_u(g(a)|\theta)$ to expert type θ from following the strategy prescribed for type $g(a)\in[\theta_2,\theta_3]$, using (16), (17), is

$$\begin{aligned}
& 2\alpha_a\left(u_1(a)-\theta-1\right)^2+2(1-\alpha_a)\left(u_2(a)-\theta-1\right)^2+(T-2)\left(g(a)-\theta-1\right)^2 && (29)\\
= \; & 2(1-\alpha_a)\left(\theta_3+K-\frac{T-2}{2}a-\theta-1+\sqrt{\frac{\alpha_a}{1-\alpha_a}}\sqrt{T-2}\sqrt{C_u+a\left(K-\frac{T}{4}a\right)}\right)\\
& +2\alpha_a\left(\theta_3+K-\frac{T-2}{2}a-\theta-1-\sqrt{\frac{1-\alpha_a}{\alpha_a}}\sqrt{T-2}\sqrt{C_u+a\left(K-\frac{T}{4}a\right)}\right)^2\\
& +(T-2)\left(x(a)-\theta-1\right)^2
\end{aligned}$$

Expanding gives [17]

$$D_u(g(a)|\theta)=2\left(\theta_3-\theta-1+K-\frac{T-2}{2}a\right)^2+2(T-2)\left(C_u+a\left(K-\frac{T}{4}a\right)\right)+(T-2)\left(\theta_3+a-\theta-1\right)^2$$

If we now expand this expression, the coefficients on a^2, a reduce to zero (this is due to our choice

[17] Note that the coefficients on the square roots were chosen to make this independent of α_a, as mentioned in Appendix D3.1 (following (58)).

$g(a) = \theta_3 + a$), leaving

$$D_u(g(a)|\theta) = 2(\theta_3 - \theta - 1 + K)^2 + 2(T-2)C_u + (T-2)(\theta_3 - \theta - 1)^2$$

which rearranges to expression (26).

The disutility to type θ from following the strategy prescribed for type $x(a) \in [0, \theta_1]$, $D_u(x(a)|\theta)$, is given by (29), just replacing $g(a)$ with $x(a)$: this gives the desired expression (25).

The disutility to type θ from following the strategy prescribed for type $h(a) \in [\theta_3, \theta_4]$ is

$$D_v(h(a)|\theta) = 2\alpha_0 (v_1(a) - \theta - 1)^2 + (T - \tau - 2\alpha_0)(v_2(a) - \theta - 1)^2 + \tau (h(a) - \theta - 1)^2$$

Again, the coefficients on the square root terms in v_1, v_2 were chosen to make both disutility and average action independent of α_0 : substituting (18), (19) into the above expression and expanding, we get

$$\begin{aligned} D_v(h(a)|\theta) &= (T-\tau)\left(\theta_3 + \frac{2K + \tau a_\gamma}{T-\tau} - \frac{\tau}{T-\tau}a - \theta - 1\right)^2 + \tau (h(a) - \theta - 1)^2 \\ &\quad + 2\tau \left(\frac{(T-2)}{\tau} C_u + \frac{\left(\frac{T-\tau-2}{\tau}\right) K^2 + 2K(a - a_\gamma) - \frac{T}{2}(a - a_\gamma)^2}{T-\tau} \right). \end{aligned}$$

Substituting in $h(a) = \theta_4 + a$, using $\theta_3 = \theta_4 + a_\gamma$, and expanding, we find (this is due to our choice $h'(a) = 1$) that the coefficients on both a^2, a reduce to zero, so that our expression simplifies further to (28). Finally, using the fact that the strategies for types $h(a), z(a)$ differ only in the revelation phase, so

$$D_v(h(a)|\theta) - D_v(z(a)|\theta) = \tau (h(a) - \theta - 1)^2 - \tau (z(a) - \theta - 1)^2$$

we obtain (27). This completes the proof. ■

Lemma D3.4: (utility at the endpoints)

Under the expressions given in Proposition D3, we have that (i) endpoint $\theta_1 = x(-2) = z(0)$: type θ (weakly) prefers type $x(-2)$'s recommendation sequence to $z(0)$'s sequence iff $\theta \in [0, \theta_1]$; (ii) endpoint $\theta_2 = z(a_\gamma) = g(-2)$: type θ prefers $z(a_\gamma)$'s sequence to $g(-2)$'s sequence iff $\theta \in [0, \theta_2]$; and (iii) endpoint $\theta_3 = g(0) = h(a_\gamma)$: all types are indifferent between the sequences sent by types $g(0), h(a_\gamma)$.[18]

Proof of Lemma D3.4:

At $\theta_1 = x(-2) = z(0)$, we have (using the expressions in Lemma D3.3 and simplifying) that $D_v(z(0)|\theta) - D_u(x(-2)|\theta)$ equals

$$(T-2)(\theta_2 - \theta_1)(\theta_1 + \theta_2 - 2\theta - 2) - \tau(\theta_4 - \theta_1)(\theta_4 + \theta_1 - 2\theta - 2)$$

Using $\tau(\theta_4 - \theta_1) = (T-2)\frac{(\theta_2 - \theta_1)(\theta_2 - \theta_1 - 2)}{(\theta_4 - \theta_1 - 2)}$ (by (20)), this simplifies to

$$D_v(z(0)|\theta) - D_u(x(-2)|\theta) = 2(T-2)(\theta_2 - \theta_1)(\theta_4 - \theta_2)\left(\frac{\theta_1 - \theta}{\theta_4 - \theta_1 - 2}\right) \tag{30}$$

This is negative, meaning that type θ prefers $z(0)$'s strategy to $x(-2)$'s strategy, iff $\theta > \theta_1$, thus establishing part (i).

[18] For example, consider part (i). In our construction, type θ_1 is both the right endpoint $x(-2)$ of the interval $[0, \theta_1]$, and the left endpoint $z(0)$ of the interval $[\theta_1, \theta_2]$: part (i) says that type θ_1 is indifferent between the two sequences prescribed for his type, and that everyone below θ_1 prefers the strategy of type $x(-2)$, everyone above θ_1 prefers the strategy of type $z(0)$.

At $\theta_2 = g(-2) = z(a_\gamma)$, we have (by (27) and (26)

$$\begin{aligned} D_u(g(-2)|\theta) - D_v(z(a_\gamma)|\theta) &= \tau(\theta_3 - \theta_2)(\theta_3 + \theta_2 - 2\theta - 2) \\ &= 4\tau(\theta_2 - \theta) \ \text{(using } \theta_3 - \theta_2 = 2) \end{aligned} \tag{31}$$

This is negative, meaning that type θ prefers $g(-2)$'s strategy to $z(a_\gamma)$'s strategy, iff $\theta > \theta_2$, proving part (ii).

At θ_3, we have (by (26) and (28)),

$$D_u(g(0)|\theta) - D_v(h(a_\gamma)|\theta) = 0 \tag{32}$$

so that all types are indifferent between the strategies prescribed for type $g(0) = \theta_3$, $h(a_\gamma) = \theta_3$, as desired to complete the proof. ■

D.2 Optimality for the decision-maker

Let the expert strategy be as specified in the previous subsection, using the action functions and parametrizations from Proposition D3, with $\tau = \beta(T-2)$ as in (20). Recall that we had the following free parameters: constants K, C_u, the horizon T, a number $\alpha_0 \in [0,1]$, and numbers $\alpha_a \in [0,1]\ \forall a \in [-2,0]$. We wish to show that the specified strategies constitute a fully revealing PBE: since we established expert optimality in the previous section, and since the beliefs and off-path strategies specified for the decision-maker (see Proposition D1) trivially satisfy all PBE requirements, all that remains is to prove that the decision-maker's on-path strategy is optimal.

Recall the timeline presented in Figure 1 (Section 4). It is immediately clear that during the revelation phase, when the expert's recommendation is equal (with probability 1) to the true state, the decision-maker indeed finds it optimal to follow the recommendation. In between time $2\alpha_0$ (when Group I separates from Group II by switching to v_2) and the revelation phase, no new information is revealed, but any failure by the decision-maker to follow the expert's recommendations will result in the expert subsequently babbling, rather than revealing the truth. So, the best possible deviation is to choose the myopically optimal action in all subsequent periods, and the strongest incentive to do so occurs at the *earliest* time that new information is revealed (when the "reward phase", revelation of the truth, is furthest away). So to prove decision-maker optimality, we need only show that he does not want to deviate to the myopically optimal action either at time $t = 0$, or at time $t = 2\alpha_0$ if he learns that he is in fact facing a Group I pair. We summarize this as:

Observation D4: If the decision-maker cannot gain by deviating at time $t \in \{0, 2\alpha_0\}$, then the prescribed strategy is optimal.

D.2.1 Optimality for the decision-maker: Outline and Parameter Choices

Given T, α_0 (and with $\tau = \beta(T-2)$ as specified by (20), we set the constants C_u, K according to

$$C_u = \frac{1-\alpha_0}{\alpha_0}\frac{K^2}{T-2} \tag{33}$$

$$K = \frac{\alpha_0 \tau a_\gamma \left(1 + \sqrt{\frac{(T-2\alpha_0)(T-\tau)}{2\tau\alpha_0}}\right)}{(T-\tau-2\alpha_0)} \tag{34}$$

And choose a horizon T satisfying

$$\begin{cases} \text{if } \beta a_\gamma^2 < 8: & T - \tau = 4 \Leftrightarrow T = T_1 \\ \text{if } \beta a_\gamma^2 > 8: & 6 \le T \le \min\{T_1, T_2\} \end{cases} \tag{35}$$

where $T_1 = \frac{2(2-\beta)}{1-\beta}$, and $T_2 = \frac{2\widehat{\phi}^2}{1-\widehat{\phi}^2(1-\beta)}$, with $\widehat{\phi}$ solving $\ln\left(\frac{\widehat{\phi}^2}{1+\widehat{\phi}}\right) = 1 \Leftrightarrow \widehat{\phi} \cong 3.4959$. Note that with β as specified in (20), $\beta a_\gamma^2 < 8 \Leftrightarrow a_\gamma \gtrapprox -3.18$. In the range $\beta a_\gamma^2 > 8$, we also have $\min\{T_1, T_2\} = T_2$ whenever $\beta > .95825$, and that $T_1 > 11.5$, $T_2 > 24$. All proofs use α_0 near 1 when $\beta a_\gamma^2 < 8$, and α_0 near 0 when $\beta a_\gamma^2 > 8$. The parameter α_a (relating to the time $2\alpha_a$ at which Group II and III pairs $\{x(a), g(a)\}$ switch from u_1 to u_2) may depend on the specific pair $\{x(a), g(a)\}$, but is chosen in Lemma D7.1 to satisfy $\alpha_0 \leq \alpha_a \leq 1$ $\forall a$.

The constants C_u, K from (33), (34) are chosen in Lemma D7.1, in order to facilitate the overlap of the functions u_1, v_1. The need for this overlap is as follows: the decision-maker's gain to following the expert's advice is large at information sets containing only a Group II or III pair, but would be negative at time $t = 0$, for *all* priors, if his information set contained only a Group I pair.[19] So, for the equilibrium to work, we need to make sure that each Group I pair's initial message coincides with that of a Group II pair, and then ensure (via the prior and construction details) that the weight the decision-maker places on the Group II pair is high enough to make him want to follow the recommendation. Specifically, Lemma D7.1 proves that under (33), (34), if we set $T \leq T_1$, with equality if $\beta a_\gamma^2 < 8$, then for every $a' \in [a_\gamma, 0]$, there exists $a \in [-2, 0]$ and $\alpha_a \in [\alpha_0, 1]$ such $v_1(a', \alpha_0) = u_1(a, \alpha_a)$.

Proposition D6 shows that for a range of priors, the decision-maker's gain to deviating at time $t = 0$ (or later) is strictly negative at any information set containing only a group II or III pair $\{x(a), g(a)\}$. This then implies also that at time $t = 0$, if he gets a message $v_1(a, \alpha_0)$ which could have been sent by either a Group II pair (in which case he wants to follow the advice) or a Group I pair (in which case he might want to reject the advice), he will find it optimal to follow the recommendation as long as his posterior beliefs assign a high enough weight to the Group II pair, so we conclude that there exist beliefs for which the decision-maker has no incentive to deviate at time $t = 0$. It is also Proposition D6 that places an upper bound on the biases b for which the equilibrium works. The proof specifically uses posteriors near 50-50, but in fact the range of posteriors that work is quite large.

Proposition D5 shows that if the expert sends a message $v_2(a, \alpha_0)$ at time $t = 2\alpha_0$, thus revealing to the decision-maker that he is facing a Group I pair $\{z(a), h(a)\}$, then there exist posteriors on the two types for which the decision-maker will find it optimal to choose the action $v_2(a, \alpha_0)$. Specifically, we prove that if $\beta a_\gamma^2 < 8$ the incentive constraints are slack in the limit as $\alpha_0 \to 1$, and if $\beta a_\gamma^2 > 8$ the incentive constraints are slack in the limit as $\alpha_0 \to 0$. It is also this result which places the additional upper bound $T \leq T_2$ (from (35)) on the horizon.

Proposition D7 completes the proof, by proving that there exist priors which generate the posteriors used Propositions D5, D6.

Before proceeding with the proof, we briefly comment on the timeline. First, note that Theorem 1 places only a lower bound on the horizon T^*, whereas the constraint (35) in fact also places an *upper* bound on the horizon (as $b \to 0 \Leftrightarrow a_\gamma \to -\infty$, the constraint $T \leq T_2$ from Proposition D5 binds, with $T_2 \to 24.443$). However, the construction may trivially be extended for larger horizons in two ways: (i) add a babbling phase at the beginning; or (ii) scale everything up. All that matters is the ratios – the length of each initial recommendation compared to the length of the revelation phase – so e.g. the analysis for a T-period equilibrium (where Groups II, III reveal the truth at time 2, Group I at time $T - \tau$) is identical to the analysis for a $T\lambda$-period equilibrium (with Groups II, III revealing the truth at time 2λ, Group I at time $(T - \tau)\lambda$, for λ any positive number).[20] It should also be noted that the times at which the expert instructs the decision-maker to change his action – namely, times $2\alpha_0, 2\alpha_a, T - \tau$ – are not necessarily integers in our construction. In a continuous time setting, where action changes can be made frequently, this clearly poses no problem. In a discrete time setting, it is possible to scale everything up to obtain integers.[21]

[19] It may in fact be shown (see Section D.3.2) that this is necessarily true of any fully revealing construction: the expert's local + global IC constraints imply a sufficiently large distortion in some interval of types' (and their partners') recommendations that, if the DM were certain that he was facing one of these pairs, he would rather forego learning the exact truth than follow their advice.

[20] This follows immediately from the derivations in Appendix D3.

[21] This is not completely trivial, due to the fact that the coefficient $\frac{\tau}{T-2}$ specified in (20) may not be rational, but

D.2.2 Optimality for the decision-maker: Deviations at time $t = 2\alpha_0$

Proposition D5: Suppose that the decision-maker receives recommendation $v_2(a)$ at time $t = 2\alpha_0$, and assigns probabilities $q_a, 1-q_a$ to the two types $z(a), h(a)$ in his information set. Under $(33), (34),$ and (35), there exists a range of values for $\frac{q_a}{1-q_a}$ such that (i) if $\beta a_\gamma^2 < 8$, the decision-maker's gain to deviating at time $2\alpha_0$ is strictly negative for α_0 near 1, and (ii) if $\beta a_\gamma^2 > 8$, the gain to deviating is strictly negative for α_0 near zero.

Proof of Proposition D5:

If the decision-maker follows recommendation $v_2(a)$ (expecting to choose this action until time $T-\tau$, then learn the truth), his expected disutility is

$$(T-\tau-2\alpha_0)\left(q_a\left(v_2(a)-z(a)\right)^2+(1-q_a)\left(v_2(a)-h(a)\right)^2\right)+\tau(0)$$

The best possible deviation is to instead choose myopically optimal action $q_a z(a)+(1-q_a)h(a)$ in all remaining $T-2\alpha_0$ periods, for disutility

$$\begin{aligned}&(T-2\alpha_a)\left(q_a\left(q_a z(a)+(1-q_a)h(a)-z(a)\right)^2+(1-q_a)\left(q_a z(a)+(1-q_a)h(a)-h(a)\right)^2\right)\\ =\;&(T-2\alpha)q_a(1-q_a)\left(h(a)-z(a)\right)^2\end{aligned}$$

So to prove the Proposition, we need to show that there are q_a's s.t. the following inequality holds:

$$\begin{aligned}0 \;\geq\;& \left(q_a\left(v_2(a)-z(a)\right)^2+(1-q_a)\left(v_2(a)-h(a)\right)^2\right)-\frac{(T-2\alpha_0)}{(T-\tau-2\alpha_0)}q_a(1-q_a)\left(h(a)-z(a)\right)^2\\ =\;& q_a\left(2\left(\frac{v_2(a)-h(a)}{h(a)-z(a)}\right)+1\right)+\left(\frac{v_2(a)-h(a)}{h(a)-z(a)}\right)^2-\frac{T-2\alpha_0}{T-\tau-2\alpha_0}q_a(1-q_a)\end{aligned}\tag{36}$$

This is easiest to satisfy (RHS is minimized) at

$$q_a = q_a^* \equiv \frac{1}{2}-\frac{\left(\frac{v_2(a)-h(a)}{h(a)-z(a)}\right)+\frac{1}{2}}{\frac{T-2\alpha_0}{T-\tau-2\alpha_0}}\tag{37}$$

where (36) becomes

$$\frac{\phi^2-1}{4\phi^2}\left(2\left(\frac{v_2(a)-h(a)}{h(a)-z(a)}\right)+\phi+1\right)\left(2\left(\frac{v_2(a)-h(a)}{h(a)-z(a)}\right)+1-\phi\right)\;\leq\;0\tag{38}$$

$$\text{with } \phi^2 \;=\; \frac{T-2\alpha_0}{T-\tau-2\alpha_0}\tag{39}$$

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this can be easily modified via minor changes to the construction. Specifically, given bias b, the easiest way to make β rational is to change the h-function (specifying how Group I types are paired), as follows: first, choose a rational number $\widehat{\beta}$ which is *smaller* than β. Solve the pair of equations $\widehat{\beta} = \frac{\left(\lambda a_\beta-2+2e^{-a_\beta}\right)\left(\lambda a_\beta-4+2e^{-a_\beta}\right)}{(\lambda-1)a_\beta\left(2e^{-a_\beta}+\lambda a_\beta\right)+2e^{-a_\beta}\left((\lambda-1)a_\beta+2e^{-a_\beta}-2\right)}$, $\frac{1}{b} = \left(\lambda a_\beta-2+2e^{-a_\beta}\right)e^2-\lambda a_\beta$ for λ and a_β ($\widehat{\beta}<\beta$ will give $\lambda>1$, as required by the expert's S.O.C.'s). In all analysis, replace a_γ with a_β, and change the h-function from $h(a)=\frac{1}{b}+a$ to $h(a)=\frac{1}{b}+\lambda a$; this requires changing the endpoints specified in (14) to $\theta_3=\frac{1}{b}+\lambda a_\beta$, with (as before) $\theta_2=\theta_3-2$, $\theta_1=\theta_2-\theta_3 e^{-2}$. To satisfy the expert's modified local IC constraints, simply plug the new h-function into the recommendation functions v_1, v_2 specified by (59). To satisfy the expert's modified global IC constraints, replace β with $\widehat{\beta}$, and change the constant C_v (from (62)) in the v-functions (specified in (59)) by replacing a_γ with a_β, and also subtracting the term $\frac{(\lambda-1)}{2}a_\beta\left(a_\beta+2\right)$; no change to the constant k_v. For $\widehat{\beta}$ sufficiently close to β, the changes to the analysis for the DM are negligible.

Thus, a sufficient condition for the decision-maker's IC constraint (36) to hold is

$$-\phi - 1 \leq 2\left(\frac{v_2(a) - h(a)}{h(a) - z(a)}\right) \leq \phi - 1 \tag{40}$$

which implies both that (38) holds (so the decision-maker does not want to deviate), and that q_a^* (from (37)) is indeed a probability, since we have

$$\begin{aligned} 2\left(\frac{v_2(a)-h(a)}{h(a)-z(a)}\right)+1+\phi^2 &> 2\left(\frac{v_2(a)-h(a)}{h(a)-z(a)}\right)+1+\phi \geq 0 \Rightarrow q_a^* < 1 \\ 2\left(\frac{v_2(a)-h(a)}{h(a)-z(a)}\right)+1-\phi^2 &< 2\left(\frac{v_2(a)-h(a)}{h(a)-z(a)}\right)+1-\phi \leq 0 \Rightarrow q_a^* = 0 \end{aligned}$$

Substituting $C_u = \frac{1-\alpha_0}{\alpha_0}\frac{K^2}{T-2}$ (from (33)) into (19) , we obtain that $v_2(a, \alpha_0) - h(a)$ equals

$$\theta_3 + \frac{2K-\tau(a-a_\gamma)}{T-\tau} + \frac{\sqrt{\frac{4\tau\alpha_0}{T-\tau-2\alpha_0}}\sqrt{\frac{(T-\tau)(T-2)}{\tau}\frac{1-\alpha_0}{\alpha_0}\frac{K^2}{T-2}+\left(\frac{T-\tau-2}{\tau}\right)K^2+2K(a-a_\gamma)-\frac{T}{2}(a-a_\gamma)^2}}{T-\tau} - h(a)$$

Simplifying, and using $h(a) - z(a) = 2^{a-a_\gamma}$ and $\theta_3 - \theta_4 = a_\gamma$ (from (15) and (14)), this implies that

$$\begin{aligned} 2\frac{v_2(a,\alpha_0)-h(a)}{h(a)-z(a)} &= \frac{2K - T(a-a_\gamma) + \sqrt{\frac{2\alpha_0}{T-\tau-2\alpha_0}}\sqrt{2\frac{T-\tau-2\alpha_0}{\alpha_0}K^2 + 2\tau\left(2K\left(a-a_\gamma\right) - \frac{T}{2}\left(a-a_\gamma\right)^2\right)}}{(T-\tau)e^{a-a_\gamma}} \\ &= \frac{k - ty + \sqrt{k^2 + \left(\phi^2 - t\right)\left(2ky - ty^2\right)}}{e^y} \end{aligned}$$

where $k \equiv \frac{2K}{T-\tau}$, $t \equiv \frac{T}{T-\tau}$, and $y \equiv a - a_\gamma$ (with ϕ defined in (39)). So by (40), we wish to show that for all $y \in [0, -a_\gamma]$,

$$\frac{k - ty + \sqrt{k^2 + \left(\phi^2 - t\right)\left(2ky - ty^2\right)}}{e^y} \in [-\phi - 1, \phi - 1] \tag{41}$$

By construction, the value of K specified in (34) sets the square root portion of v_1, v_2 equal to zero at $a = 0 \Leftrightarrow y = -a_\gamma$ (see Lemma D7.1), so we have

$$k = a_\gamma\left(\phi^2 - t + \phi\sqrt{\phi^2 - t}\right)$$

We first prove (ii). Suppose $\beta a_\gamma^2 > 8$, in which case Lemma D7.1 requires $\frac{T-\tau}{2} \leq 2$, and $\alpha_0 \leq \overline{\alpha_0}$ (for some $\overline{\alpha}_0 \in (0,1)$). Consider what happens as $\alpha_0 \to 0$ (in which case we trivially satisfy $\alpha_0 \leq \overline{\alpha_0}$): then $\phi^2 - t = \frac{T-2\alpha_0}{T-\tau-2\alpha_0} - \frac{T}{T-\tau} \to 0$, implying also that $k \to 0$, so we have

$$\begin{aligned} \min_{y\in[0,-a_\gamma]}\left(k - ty + \sqrt{k^2 + \left(\phi^2 - t\right)\left(2ky - ty^2\right)} + (1+\phi)e^y\right) &\to \min_{y\in[0,-a_\gamma]}\left(-\phi^2 y + (\phi+1)e^y\right) \\ \max_{y\in[0,-a_\gamma]}\left(k - ty + \sqrt{k^2 + \left(\phi^2 - t\right)\left(2ky - ty^2\right)} + (1-\phi)e^y\right) &\to \max_{a\in[0,-a_\gamma]}\left(-\phi^2 y + (1-\phi)e^y\right) \end{aligned}$$

In the second line, the limit is clearly decreasing in y (noting that $\phi \to \sqrt{\frac{T}{T-\tau}} > 1$), thus maximized at $y = 0$: here, the value is $1 - \phi$, the desired upper bound in (41). In the first line, note that if we set ϕ s.t.

$$(\phi+1)e > \phi^2 \Leftrightarrow 1 > \ln \frac{\phi^2}{\phi+1} \Leftrightarrow \phi < 3.4959$$

then the limiting expression $(-\phi^2 y + (1+\phi)e^y)$, which is minimized at $y = \ln \frac{\phi^2}{\phi+1} < 1$, has a minimum value of

$$-\phi^2 \ln \frac{\phi^2}{\phi+1} + \phi^2 = \phi^2 \left(1 - \ln \frac{\phi^2}{\phi+1}\right)$$

which is positive, thus trivially satisfying the desired lower bound in (41).

So to complete the proof of (ii), we just need to show that for α_0 near zero, we can find a horizon T which satisfies both the requirement $\Delta \leq 2$ (from Proposition D6 and Lemma D7.1), as well as

$$\sqrt{\frac{T}{T-\tau}} = \lim_{\alpha \to 0} \phi < 3.4959$$

Using $\tau = \beta(T-2)$, which implies $T = \frac{2(\Delta-\beta)}{1-\beta}$, the above inequality binds if $\beta > \frac{(3.4959)^2-1}{(3.4959)^2-\frac{1}{2}} = 0.957\,34$, and otherwise is implied by $\Delta \leq 2 \Leftrightarrow T \leq \frac{4-2\beta}{1-\beta}$; this gives the horizon constraint (35) specified in the Proposition.

Finally we prove (i). Suppose $\beta a_\gamma^2 < 8$, in which case Lemma D7.1 uses $\frac{T-\tau}{2} \equiv \Delta = 2$ and requires $\alpha_0 \geq \underline{\alpha_0}$, for some $\underline{\alpha_0} \in [0,1]$. So, set $\Delta = 2$, and consider what happens as $\alpha_0 \to 1$: then

$$t \to \frac{T}{4} = \frac{2-\beta}{2(1-\beta)},\ \phi^2 \to \frac{T-2}{T-\tau-2} = \frac{1}{1-\beta},\ k \to \frac{\beta a_\gamma}{2(1-\beta)}\left(1 + \sqrt{\frac{2}{\beta}}\right)$$

Define $\xi(y) \equiv k - ty + \sqrt{k^2 + (\phi^2 - t)(2ky - ty^2)}$; we want to prove that

$$-(1+\phi) \leq \frac{\xi(y)}{e^y} \leq \phi - 1 \text{ for all } y \in [0, -a_\gamma] \tag{42}$$

To this end, note that

$$\begin{aligned} \xi'(y) &= -t + \frac{(\phi^2 - t)(k - ty)}{\sqrt{k^2 + (\phi^2 - t)(2ky - ty^2)}} < 0 \text{ (by } \phi^2 - t > 0, y > 0, \text{ and } k < 0) \\ \xi''(y) &= \frac{-(\phi^2 - t)k^2\phi^2}{\left(k^2 + (\phi^2 - t)(2ky - ty^2)\right)^{\frac{3}{2}}} < 0 \end{aligned}$$

This implies that $\xi(y)$ reaches a maximum (over $y \in [0, -a_\gamma]$) at $y = 0$, and lies above the straight line connecting the points $(0, \xi(0))$ and $(-a_\gamma, \xi(-a_\gamma))$: since we have $\xi(-a_\gamma) = k + ta_\gamma$ (the square root term is zero here by construction), and $\xi(0) = k + \sqrt{k^2} = 0$, this line $\widetilde{\xi}$ is given by

$$\widetilde{\xi}(y) - \widetilde{\xi}(0) = \frac{\widetilde{\xi}(-a_\gamma) - \widetilde{\xi}(0)}{-a_\gamma}(y - 0) \Rightarrow \widetilde{\xi}(y) = \left(\frac{-k}{a_\gamma} - t\right) y$$

Substituting in $k = \frac{\beta a_\gamma}{2(1-\beta)}\left(1 + \sqrt{\frac{2}{\beta}}\right)$, this becomes $\widetilde{\xi}(y) = -\left(\frac{1+\sqrt{\frac{\beta}{2}}}{1-\beta}\right) y$. Then, the upper bound in (42) follows from the fact that

$$\max_{y\in[0,-a_\gamma]} \frac{\xi(y)}{e^y} \leq \frac{\max_{y\in[0,-a_\gamma]} \xi(0)}{\min_{y\in[0,-a_\gamma]} e^y} = 0 \leq \phi - 1$$

while the lower bound in (42) follows from

$$\min_{y\in[0,-a_\gamma]} \frac{(y)}{e^y} \geq \min_{y\in[0,-a_\gamma]} \frac{\widetilde{\xi}(y)}{e^y} = \min_{y\in[0,-a_\gamma]} -\left(\frac{1+\sqrt{\frac{\beta}{2}}}{1-\beta}\right)\frac{y}{e^y} = \frac{-\left(1+\sqrt{\frac{\beta}{2}}\right)}{(1-\beta)e}$$

So, using $\phi = \frac{1}{\sqrt{1-\beta}}$, to show that $\min_{y\in[0,-a_\gamma]} \frac{\xi(y)}{e^y} + (1+\phi) \geq 0$, it suffices to show that

$$\frac{-\left(1+\sqrt{\frac{\beta}{2}}\right)}{(1-\beta)e} + 1 + \sqrt{\frac{1}{1-\beta}} \geq 0 \Leftrightarrow e \geq \frac{\left(1+\sqrt{\frac{\beta}{2}}\right)}{1-\beta+\sqrt{1-\beta}}$$

Noting that the RHS is clearly increasing in β, it may be easily verified numerically that the above expression holds for all $\beta \leq 0.82035$, which is implied by the range under consideration ($\beta a_\gamma^2 < 8 \Leftrightarrow a_\gamma \gtrapprox -3.18$ implies $\beta < 0.8$). This completes the proof for $a_\gamma \geq -3.2 \Leftrightarrow \beta a_\gamma^2 < 8$, showing that if we set $T = \frac{4-2\beta}{1-\beta}$, then all decision-maker IC constraints are slack in the limit as $\alpha_0 \to 1$ (and so, by continuity, hold for all α_0 sufficiently high, which satisfy the bounds in Lemma D7.1). This completes the proof. ■

D.2.3 Optimality for the decision-maker: Deviations at time $t = 0$

Proposition D6: For all $b < \frac{1}{61} \Leftrightarrow a_\gamma \leq -1.7726$, and for all $a \in [-2, 0]$, there exists a range of values p_a such that if the decision-maker assigns probabilities $p_a, 1-p_a$ to types $x(a), g(a)$, then his gain to deviating at information set $\{x(a), g(a)\}$ is strictly negative at any time $t \geq 0$.
Proof of Proposition D6: As explained in Observation D4, it suffices to prove that the gain to deviating is negative at time $t = 0$. Substituting $x(a) = \theta_3 + a - \theta_3 e^a$ and $g(a) = \theta_3 + a$ into (63), we obtain that the decision-maker's gain to deviating at information set $\{x(a), g(a)\}$, if he assigns probability p_a to type $x(a)$, is

$$\begin{aligned} & 2p_a\left(K - \frac{T}{2}a + \theta_3 e^a\right)^2 + 2(1-p_a)\left(K - \frac{T}{2}a\right)^2 + 2(T-2)a\left(K - \frac{T}{4}a\right) \\ & -Tp_a(1-p_a)\left(\theta_3 e^a\right)^2 + 2(T-2)C_u \\ = \quad & 2K^2 + 4K\left(p_a\theta_3 e^a - a\right) + T\left(p_a\theta_3 e^a - a\right)^2 - (T-2)p_a(\theta_3 e^a)^2 + 2(T-2)C_u \end{aligned} \tag{43}$$

If we make $p_a = p$ constant, the derivative of this expression w.r.t. a is

$$\begin{aligned} & 2(p\theta_3 e^a - 1)\left[2K - Ta + Tp\theta_3 e^a - \frac{(T-2)p\left(\theta_3 e^a\right)^2}{(p\theta_3 e^a - 1)}\right] \\ \leq \quad & 2(p\theta_3 e^a - 1)\left[2K - Ta - ((1-p)T - 2)\theta_3 e^a\right] \end{aligned} \tag{44}$$

So, it suffices to choose a value p s.t. (44) is negative $\forall a$ (implying that the decision-maker's gain to deviating is largest–making the IC constraint most difficult to satisfy–at $a = -2$), and such that this maximal gain to deviating at $a = -2$ is negative. In the proof below, we will set $p \leq \frac{1}{2}$, in which case (35) implies that $(1-p)T - 2 > 0$, from which it immediately follows that the square bracketed term in (44) is decreasing in a, thus negative everywhere if it is negative at $a = -2$. Using this and rearranging, (44) requires

$$\theta_3 e^{-2} \geq \frac{2K + 2T}{(1-p)T - 2} \tag{45}$$

For the maximal gain to deviating to be negative, evaluate (43) at $a = -2$ and solve for $\theta_3 e^{-2}$, to get

$$\theta_3 e^{-2} \geq \frac{(2K + 2T) + \sqrt{\frac{2(T-2)}{p}\left((1-p)K^2 + 4K + 2T + ((1-p)T - 2)\, C_u\right)}}{(1-p)T - 2} \tag{46}$$

This trivially implies the inequality (45). So, it is sufficient to choose a value $p \leq \frac{1}{2}$ which satisfies (46), and then set $p_a = p \ \forall a \in [-2, 0]$.

If $a_\gamma > -1.7786 \Leftrightarrow \theta_3 e^{-2} < 8$, this is impossible (see Section D.3.2: this is what yields the upper bound $b \cong \frac{1}{61}$ on the biases for which our construction works).

If $a_\gamma \in [-2, -1.7786]$, set $p = \frac{1}{4}$; by (35) and (34), recalling that in this range we use $\alpha_0 \to 1$, we have

$$T = \frac{4 - 2\beta}{1 - \beta}, \ K \to \frac{\beta a_\gamma}{1 - \beta}\left(1 + \sqrt{\frac{2}{\beta}}\right), C_u \to 0$$

Substituting these values into (46), along with $\theta_3 e^{-2} = a_\gamma - 2 + 2e^{-a_\gamma}$, and simplifying, (46) becomes

$$0 \geq a_\gamma - 2 + 2e^{-a_\gamma} - 8\left(\frac{1 + \sqrt{\frac{2}{\beta}}}{1 + \frac{2}{\beta}}\right)\left(\left(\frac{a_\gamma}{2} - 1 + \sqrt{\frac{2}{\beta}}\right) + \sqrt{\frac{3}{4}\left(\frac{a_\gamma^2}{1-\beta}\right) + \frac{4}{\beta}\frac{\left(a_\gamma - 1 + \sqrt{\frac{2}{\beta}}\right)}{\left(1 + \sqrt{\frac{2}{\beta}}\right)}}\right) \tag{47}$$

Using $\beta = \frac{(a_\gamma - 2 + 2e^{-a_\gamma})(a_\gamma - 4 + 2e^{-a_\gamma})}{2e^{-a_\gamma}(2e^{-a_\gamma} - 2)}$ from (20), this holds for $a_\gamma \in [-2, -1.7726]$, as shown in the graph below.

If $a_\gamma \leq -2$, set $p = \frac{1}{2}$, so that (46) becomes

$$\theta_3 e^{-2} \geq 4\frac{(K + T) + \sqrt{\left(\frac{T-2}{2}\right)\left((K+4)^2 + 4(T-4) + (T-4)\, C_u\right)}}{T - 4}$$

Substituting in (33) and (34), this becomes

$$\frac{\theta_3 e^{-2}}{4} \geq \frac{\frac{\alpha_0 \tau a_\gamma\left(1 + \sqrt{\frac{(T-2\alpha_0)(T-\tau)}{2\tau\alpha_0}}\right)}{(T-\tau-2\alpha_0)} + T + \sqrt{\frac{(T-4+2\alpha_0)}{2\alpha_0}\left(\frac{\alpha_0 \tau a_\gamma\left(1 + \sqrt{\frac{(T-2\alpha_0)(T-\tau)}{2\tau\alpha_0}}\right)}{(T-\tau-2\alpha_0)}\right)^2 + (T-2)\left(\frac{4\alpha_0 \tau a_\gamma\left(1 + \sqrt{\frac{(T-2\alpha_0)(T-\tau)}{2\tau\alpha_0}}\right)}{(T-\tau-2\alpha_0)} + 2T\right)}}{T-4} \tag{48}$$

If $\beta a_\gamma^2 > 8 \Leftrightarrow a_\gamma \lessapprox -3.18$, our proof uses $\alpha_0 \to 0$: in this case, taking limits in (48) yields

$$\frac{\theta_3 e^{-2}}{4} \geq \frac{T + \sqrt{T(T-2)\left(\frac{(T-4)}{4}\frac{\beta a_\gamma^2}{(T-\tau)} + 2\right)}}{T - 4}$$

Since $\beta < 1$ and $T - \tau - 2 = (1 - \beta)(T - 2) > 0$ (by (20)), an upper bound on the RHS is attained at $\beta = 1$, $T - \tau = 2$; with this, using $\theta_3 e^{-2} = a_\gamma - 2 + 2e^{-a_\gamma}$, and rearranging, it is sufficient to prove that

$$\left(\frac{T-4}{4}\left(a_\gamma - 2 + 2e^{-a_\gamma}\right) - T\right)^2 \geq T(T-2)\left(\frac{(T-4)}{8}a_\gamma^2 + 2\right)$$

Expanding and dividing through by $T\,(T-4)$, this becomes

$$\left(\frac{T-4}{4T}\right)\left(\frac{a_\gamma - 2 + 2e^{-a_\gamma}}{2}\right)^2 - \left(\frac{a_\gamma - 2 + 2e^{-a_\gamma}}{2}\right) - \frac{(T-2)}{8}a_\gamma^2 - 1 \geq 0$$

The derivative of the LHS is increasing in T, so that the expression becomes easier to satisfy as T increases, whenever

$$\left(\frac{a_\gamma - 2 + 2e^{-a_\gamma}}{2T}\right)^2 - \frac{a_\gamma^2}{8} \geq 0 \Leftrightarrow T \leq \frac{a_\gamma - 2 + 2e^{-a_\gamma}}{\frac{-a_\gamma}{\sqrt{2}}}$$

Using (20), this is implied by the restriction $T \leq \frac{4-2\beta}{1-\beta}$ from (35) (whenever $a_\gamma \lesssim -1.3691$), and so our expression is most difficult to satisfy when T is small; choosing $T \geq 6$ is sufficient for the range $a_\gamma \lesssim -3.18$ under consideration, yielding the lower bound on T given in (35).

Finally, for $a_\gamma \in [-3.18, -2]$, we proved Proposition D5 using $\alpha_0 \to 1$ (which implies $C_u \to 0$, $K \to \frac{\beta a_\gamma}{1-\beta}\left(1+\sqrt{\frac{2}{\beta}}\right)$) and $T = \frac{4-2\beta}{1-\beta}$. Substituting into (48), we need

$$\left(\frac{2}{\beta}\right)\left(\beta a_\gamma\left(1+\sqrt{\frac{2}{\beta}}\right) + 4 - 2\beta + \sqrt{\left((1-\beta)\left(\frac{\beta a_\gamma}{1-\beta}\left(1+\sqrt{\frac{2}{\beta}}\right)+4\right)^2 + 8\beta\right)}\right) - a_\gamma + 2 - 2e^{-a_\gamma} \leq 0 \qquad (49)$$

Substituting in $\beta = \frac{(a_\gamma - 2 + 2e^{-a_\gamma})(a_\gamma - 4 + 2e^{-a_\gamma})}{2e^{-a_\gamma}(2e^{-a_\gamma}-2)}$ (from (20)), this holds as desired for $a_\gamma \in [-3.18, -2]$, as shown in the Fig D below.

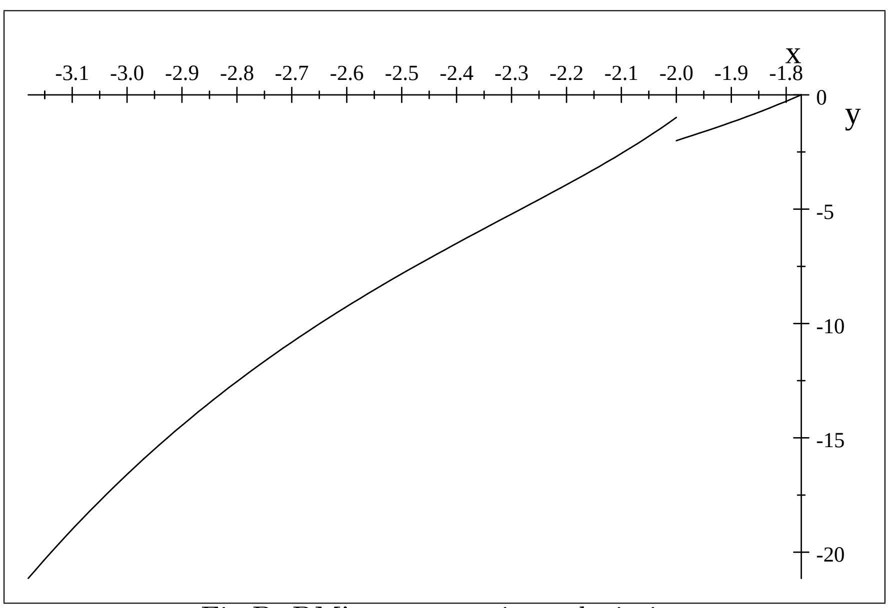

Fig D: DM's ex ante gain to deviating

(This shows (49) for $a_\gamma \in [-3.18, -2]$, and (47) for $a_\gamma \in [-1.7726]$, both strictly negative as desired). This completes the proof of Proposition D6.[22] ■

[22] It should be noted that, for brevity of exposition, we made p equal a constant $p_a \in \{\frac{1}{4}, \frac{1}{2}\}$, which works

D.2.4 Optimality for the decision-maker: Completing the Proof

As explained at the beginning of this section, it remains only to prove that there are many priors μ generating posteriors which satisfy the conditions in Propositions D5, D6, which we prove here:
Proposition D7: For any continuous functions $p : [-2, 0] \to [0, 1]$, $q : [a_\gamma, 0] \to [0, 1]$, and $r : [a_\gamma, 0] \to [0, 1]$, there exists a density f over the state space such that, in our construction, a Bayesian decision-maker will hold the following beliefs: (i) $\Pr(x(a)|\{x(a), g(a)\}) = p(a)$; (ii) $\Pr(z(a)|\{z(a), h(a)\}) = q(a)$; (iii) $\Pr(\{z(a), h(a)\} \mid \{z(a), h(a), x(\widehat{a}), g(\widehat{a})\} = r(a)$.
Proof of Proposition D7: Bayesian beliefs satisfy

$$\begin{aligned}
\frac{\Pr(x(a)|\{x(a), g(a)\})}{\Pr(g(a)|\{x(a), g(a)\})} &= \frac{f(x(a))}{f(g(a))}(\theta_3 e^a - 1) \\
\frac{\Pr(z(a)|\{z(a), h(a)\})}{\Pr(h(a)|\{z(a), h(a)\})} &= \frac{f(z(a))}{f(h(a))}\left(2e^{a-a_\gamma} - 1\right) \\
\frac{\Pr(\{z(a), h(a)\})}{\Pr(\{x(\widehat{a}), g(\widehat{a})\})} &= \frac{f(z(a))}{f(\widehat{x}(a))}\frac{p(\widehat{a})}{q(a)}\frac{2e^{a-a_\gamma} - 1}{\theta_3 e^{\widehat{a}(a)} - 1} \cdot \frac{1}{|\widehat{a}'(a)|}
\end{aligned}$$

where $\widehat{a}(a) = u_1^{-1}(v_1(a'))$ (as explained in Section D.3). We want the first expression to equal $\frac{p(a)}{1-p(a)}$, the second expression to equal $\frac{q(a)}{1-q(a)}$, and the third expression to equal $\frac{r(a)}{1-r(a)}$. It is straightforward to construct such a density f : for example, for each $a \in [-2, 0]$, set

$$f(x(a)) = \frac{1}{M}$$

with M a constant to be determined (this assigns probabilities to $[0, \theta_1]$). Then, assign probabilities to types $g(a) \in [\theta_2, \theta_3]$ using this and the first expression above, i.e. set

$$f(g(a)) = f(x(a))(\theta_3 e^a - 1)\left(\frac{1 - p(a)}{p(a)}\right) = \frac{(\theta_3 e^a - 1)}{M}\frac{1 - p(a)}{p(a)}$$

Next, for each $a \in [a_\gamma, 0]$, set $f(z(a))$ to satisfy the 3rd bound above, i.e.

$$f(z(a)) = \frac{\frac{q(a)}{p(\widehat{a})}\frac{\theta_3 e^{\widehat{a}(a)}-1}{2e^{a-a_\gamma}-1} \cdot |\widehat{a}'(a)|}{M}\frac{r(a)}{1 - r(a)}$$

(This assigns a prior for types $z(a) \in [\theta_1, \theta_2]$). And similarly, now use this and the second expression above to assign beliefs to types $h(a) \in [\theta_3, \theta_4]$. Finally, choose M so that the total measure of the type space integrates to 1 (This is possible since (16), (18) imply that u_1, v_1 and their derivatives w.r.t. a are finite and non-zero except perhaps at a single point a, from which it follows that $|\widehat{a}'(a)|$ is bounded (again, except perhaps at a single point a), which implies that all of the desired probabilities $\mu(x(a)), \mu(g(a)), \mu(z(a))$ specified above are bounded - in particular finite numbers divided by a number M - so that integrating over the state space will yield a finite number divided by M. Choose M so that this number equals 1.) ■

throughout the range. But for each specific value of a_γ, the set of beliefs that work is in fact *much* wider than a small interval around p_a.

D.2.5 Optimality for the decision-maker: Preliminary Calculations

Lemma D7.1: Define

$$[\underline{\alpha_0}, \overline{\alpha_0}] \equiv \left[\frac{(T-\tau) + \beta a_\gamma \sqrt{\frac{T-\tau}{2\beta}}}{2 + (-\beta a_\gamma)\left(\frac{2(T-2)}{T} + \frac{T-4}{T}\sqrt{\frac{T-\tau}{2\beta}}\right)}, 1 - (T-\tau-2)\left(\sqrt{\frac{1}{2\beta(T-\tau)}} - \frac{1}{-\beta a_\gamma}\right) \right]$$

Then (i) $T-\tau \leq 4$ and $\beta a_\gamma^2 > 8$ implies that $\underline{\alpha}_0 < 0 < \overline{\alpha_0}$, (ii) $T-\tau = 4$ and $\beta a_\gamma^2 < 8$ implies $\underline{\alpha}_0 < 1 < \overline{\alpha}_0$, (iii) $\alpha_0 \in [\underline{\alpha_0}, \overline{\alpha_0}]$ implies that u_t, v_t are both real-valued, and (iv) $\alpha_0 \in [\underline{\alpha_0}, \overline{\alpha_0}]$ implies that for every $a \in [a_\gamma, 0]$, there exists a pair $(\widehat{a}(a), \alpha_a)$, with $\alpha_a \geq \alpha_0$ and $\widehat{a}(a) \in [-2, 0]$, such that

$$u_1(\widehat{a}(a), \alpha_a) = v_1(a, \alpha_0)$$

Proof of Lemma D7.1: We first prove (i). If $T - \tau \leq 4 < \frac{\beta a_\gamma^2}{2}$, then

$$(T-\tau) < \sqrt{\frac{\beta a_\gamma^2 (T-\tau)}{2}} = -\beta a_\gamma \sqrt{\frac{T-\tau}{2\beta}}$$

Thus the numerator in the expression for $\underline{\alpha}_0$ is negative, while the denominator is clearly positive, so we obtain $\underline{\alpha}_0 < 0$. To complete the proof of (i), we need to show that $\overline{\alpha}_0 > 0$: we have

$$\begin{aligned}
T-\tau \in [1,4] \Rightarrow (T-\tau-2)^2 &\leq (T-\tau) \\
&< 2\beta(T-\tau) \text{ for } \beta > \frac{1}{2} \text{ (implied by } \beta a_\gamma^2 > 8) \\
\Rightarrow T-\tau-2 &< \frac{1}{\sqrt{\frac{1}{2\beta(T-\tau)}}} < \frac{1}{\left(\sqrt{\frac{1}{2\beta(T-\tau)}} - \frac{1}{-\beta a_\gamma}\right)}
\end{aligned}$$

and so $\overline{\alpha}_0 = 1 - (T-\tau-2)\left(\sqrt{\frac{1}{2\beta(T-\tau)}} - \frac{1}{-\beta a_\gamma}\right) > 0$, as desired.

For part (ii), set $T - \tau = 4$: then

$$\beta a_\gamma^2 < 8 \Rightarrow \frac{\beta a_\gamma^2}{2} < T-\tau \Rightarrow \sqrt{\frac{1}{2\beta(T-\tau)}} < \sqrt{\frac{1}{\beta^2 a_\gamma^2}} \Rightarrow \overline{\alpha_0} > 1$$

so we just need to prove that

$$\underline{\alpha}_0 \equiv \frac{(T-\tau) + \beta a_\gamma \sqrt{\frac{T-\tau}{2\beta}}}{2 + (-\beta a_\gamma)\left(\frac{2(T-2)}{T} + \frac{T-4}{T}\sqrt{\frac{T-\tau}{2\beta}}\right)} < 1$$

Rearranging gives

$$\frac{T}{2} < \frac{(T-2)}{T-\tau-2}(-\beta a_\gamma)\left(1 + \sqrt{\frac{T-\tau}{2\beta}}\right) = \left(\frac{-\beta a_\gamma}{1-\beta}\right)\left(1 + \sqrt{\frac{T-\tau}{2\beta}}\right)$$

Setting $\Delta \equiv \frac{T-\tau}{2} = \frac{(1-\beta)T + 2\beta}{2} \Leftrightarrow \frac{T}{2} = \frac{\Delta - \beta}{1-\beta}$, this rearranges to the inequality $a_\gamma < 1 - \sqrt{\frac{\Delta}{\beta}}$, which is implied by $a_\gamma \leq -1.7726 \Rightarrow \beta \geq .4172$ and $\Delta = 2$, establishing (ii).

Now for (iii) and (iv): with C_u as specified by (33), the equations in (16), (17) become

$$u_1(a,\alpha_a)-\theta_3 = K-\frac{T-2}{2}a-\sqrt{\frac{1-\alpha_a}{\alpha_a}\frac{1-\alpha_0}{\alpha_0}K^2+\frac{1-\alpha_a}{\alpha_a}(T-2)a\left(K-\frac{T}{4}a\right)} \quad (50)$$

$$v_1(a,\alpha_0)-\theta_3 = \frac{2K-\tau(a-a_\gamma)-\sqrt{\frac{T-\tau-2\alpha_0}{2\alpha_0}}\sqrt{2\frac{T-\tau-2\alpha_0}{\alpha_0}K^2+2\tau\left(2K\left(a-a_\gamma\right)-\frac{T}{2}\left(a-a_\gamma\right)^2\right)}}{T-\tau} \quad (51)$$

We first prove (iii). To show that $v_1(\cdot)$ is real-valued, note that the term in square roots is decreasing in a (derivative w.r.t. a has the same sign as $2K-T(a-a_\gamma)$, which is negative by $K<0$ and $a\geq a_\gamma$), so we just need to show that minimum value, which is at $a=0$, is non-negative: that is, we need

$$0\leq\frac{T-\tau-2\alpha_0}{\tau\alpha_0}K^2-2Ka_\gamma-\frac{T}{2}a_\gamma^2$$

The value of K specified in (34) is precisely the negative root of this equation, so the above inequality holds by construction. For $u_1(\cdot)$ to be real-valued, we need

$$0\leq\frac{1-\alpha_0}{\alpha_0}K^2+(T-2)\min_{a\in[-2,0]}a\left(K-\frac{T}{4}a\right)$$

The 2nd term is either minimized at $a=0$ (in which case the expression clearly holds), or at $a=-2$, in which case it holds (for $K<0$) iff

$$K\leq\frac{\alpha_0(T-2)}{1-\alpha_0}\left(1-\sqrt{\frac{T-2\alpha_0}{\alpha_0(T-2)}}\right)$$

For this to be satisfied by the K-value in our construction, we need

$$\begin{aligned}&\frac{\alpha_0\tau(-a_\gamma)\left(1+\sqrt{\frac{(T-2\alpha_0)(T-\tau)}{2\tau\alpha_0}}\right)}{(T-\tau-2\alpha_0)}\geq\frac{\alpha_0(T-2)}{1-\alpha_0}\left(\sqrt{\frac{T-2\alpha_0}{\alpha_0(T-2)}}-1\right)\\ \Leftrightarrow\ &\frac{(-\beta a_\gamma)(1-\alpha_0)}{(T-\tau-2\alpha_0)}\geq\frac{T(1-\alpha_0)}{\left(\sqrt{\alpha_0(T-2)(T-2\alpha_0)}\left(1+\sqrt{\frac{T-\tau}{2\beta}}\right)+(T-2\alpha_0)\sqrt{\frac{T-\tau}{2\beta}}+\alpha_0(T-2)\right)}\end{aligned}$$

(2nd line obtained from 1st by (i) replacing τ with $\beta(T-2)$ in the LHS numerator; (ii) multiplying both sides by $\frac{1-\alpha_0}{\alpha_0(T-2)\left(1+\sqrt{\frac{(T-2\alpha_0)(T-\tau)}{2\tau\alpha_0}}\right)}$; and finally (iii) on the resulting RHS, multiplying both numerator and denominator by $\sqrt{\alpha_0(T-2)}\left(\sqrt{T-2\alpha_0}+\sqrt{\alpha_0(T-2)}\right)$). Note that $\sqrt{\alpha_0(T-2)(T-2\alpha_0)}\geq\alpha_0(T-2)$, so the RHS is at most

$$\frac{T(1-\alpha_0)}{\left(\alpha_0(T-2)\left(1+\sqrt{\frac{T-\tau}{2\beta}}\right)+(T-2\alpha_0)\sqrt{\frac{T-\tau}{2\beta}}+\alpha_0(T-2)\right)}=\frac{T(1-\alpha_0)}{\left((T-2\alpha_0+\alpha_0(T-2))\sqrt{\frac{T-\tau}{2\beta}}+2\alpha_0(T-2)\right)}$$

Therefore, to prove the desired inequality, it is sufficient to show that the following holds:

$$\begin{aligned}
\frac{(-\beta a_\gamma)}{(T-\tau-2\alpha_0)} &\geq \frac{T}{\left((T-2\alpha_0+\alpha_0(T-2))\sqrt{\frac{T-\tau}{2\beta}}+2\alpha_0(T-2)\right)} \\
&\Leftrightarrow \quad \alpha_0 \geq \frac{(T-\tau)+\beta a_\gamma\sqrt{\frac{T-\tau}{2\beta}}}{2+(-\beta a_\gamma)\left(\frac{2(T-2)}{T}+\frac{T-4}{T}\sqrt{\frac{T-\tau}{2\beta}}\right)}
\end{aligned}$$

The RHS is precisely the value $\underline{\alpha_0}$ defined in the Lemma, so $\alpha_0 \geq \underline{\alpha_0}$ is sufficient to guarantee that $u_t(\cdot)$ is real-valued. This completes the proof of (iii).

Finally, to prove (iv): by (50) and (51) we have $u_1(0,\alpha_0) = K - \frac{1-\alpha_0}{\alpha_0}\sqrt{K^2} = K\left(1+\frac{1-\alpha_0}{\alpha_0}\right) = \frac{K}{\alpha_0}$, and $u_1(-2,1) = K+T-2$. So, if we set $\alpha(0)=\alpha_0$ and $\alpha(-2)=1$, then $\left[\frac{K}{\alpha_0}, K+T-2\right] \subseteq [u_1^{\min}, u_1^{\max}]$. On the other hand,

$$\begin{aligned}
\frac{T-\tau}{\tau}\frac{dv_1(a,\alpha_0)}{da} &= -1+\frac{\left(\frac{T}{2}(a-a_\gamma)-K\right)}{\sqrt{K^2+\frac{\tau\alpha_0}{T-\tau-2\alpha_0}\left(2K\left(a-a_\gamma\right)-\frac{T}{2}\left(a-a_\gamma\right)^2\right)}} \\
\frac{T-\tau}{2}\frac{d^2v_1(a,\alpha_0)}{da^2} &= \frac{\frac{T}{2}\sqrt{K^2+\frac{\alpha_0}{T-\tau-2\alpha_0}\tau\left(2K\left(a-a_\gamma\right)-\frac{T}{2}\left(a-a_\gamma\right)^2\right)}^2+\frac{\tau\alpha_0}{T-\tau-2\alpha_0}\left(\frac{T}{2}(a-a_\gamma)-K\right)^2}{\sqrt{K^2+\frac{\tau\alpha_0}{T-\tau-2\alpha_0}\left(2K\left(a-a_\gamma\right)-\frac{T}{2}\left(a-a_\gamma\right)^2\right)}^3}
\end{aligned}$$

Clearly the second expression is positive, so we have that

$$a \geq a_\gamma \Rightarrow \frac{T-\tau}{\tau}\frac{dv_1(a,\alpha_0)}{da} \geq \frac{T-\tau}{\tau}\frac{dv_1(a,\alpha_0)}{da}\bigg|_{a=a_\gamma} = -1+\frac{(-K)}{\sqrt{K^2}} = 0$$

That is, $v_1(\cdot)$ is strictly increasing on $[a_\gamma, 0]$, with

$$\begin{aligned}
v_1^{\min}-\theta_3 &= v_1(a_\gamma,\alpha_0)-\theta_3 = \frac{2K-\sqrt{\frac{T-\tau-2\alpha_0}{2\alpha_0}}\sqrt{2\frac{T-\tau-2\alpha_0}{\alpha_0}K^2}}{T-\tau} = \frac{2K\left(1+\frac{T-\tau-2\alpha_0}{2\alpha_0}\right)}{T-\tau} = \frac{K}{\alpha_0} \\
v_1^{\max}-\theta_3 &= v_1(0,\alpha_0)-\theta_3 = \frac{2K+\tau a_\gamma-\sqrt{\frac{T-\tau-2\alpha_0}{2\alpha_0}}\sqrt{2\tau\left(\frac{T-\tau-2\alpha_0}{\tau\alpha_0}K^2-2Ka_\gamma-\frac{T}{2}a_\gamma^2\right)}}{T-\tau} \\
&= \frac{2K+\tau a_\gamma}{T-\tau}
\end{aligned}$$

So we immediately have $v_1^{\min} = u_1(0,\alpha_0)$. To show that $v_1^{\max} \leq u_1(-2,1)$, we need to show that

$$K+T-2 \geq \frac{2K+\tau a_\gamma}{T-\tau} \Leftrightarrow \frac{(T-\tau-2)}{(T-2)}K > \frac{\tau a_\gamma}{T-2}-(T-\tau)$$

Substituting in our value $K = \frac{a_\gamma\left(\tau\alpha_0+\sqrt{\frac{(T-2\alpha_0)(\tau\alpha_0)(T-\tau)}{2}}\right)}{(T-\tau-2\alpha_0)}$, using $\tau=\beta(T-2)$, and dividing by a_γ

(which flips the inequality), this becomes

$$\frac{a_\gamma\left(\tau\alpha_0+\sqrt{\frac{(T-2\alpha_0)(\tau\alpha_0)(T-\tau)}{2}}\right)}{(T-\tau-2\alpha_0)} > \frac{\beta a_\gamma-(T-\tau)}{1-\beta} \Leftrightarrow \frac{(T-2)\left(\beta\alpha_0+\sqrt{\frac{(T-2\alpha_0)(\beta\alpha_0)(T-\tau)}{2(T-2)}}\right)}{(T-\tau-2\alpha_0)} < \frac{\beta+\frac{(T-\tau)}{(-a_\gamma)}}{1-\beta}$$

For this, it is sufficient to prove the following (which replaces the 2nd numerator term on the LHS with its maximum value (at $\alpha_0 = 1$)):

$$\frac{(T-2)\left(\beta\alpha_0+\sqrt{\frac{\beta(T-\tau)}{2}}\right)}{(T-\tau-2\alpha_0)} < \frac{\beta+\frac{(T-\tau)}{-a_\gamma}}{1-\beta}$$

Multiplying both sides by $(1-\beta)(T-\tau-2\alpha_0)$, this rearranges as

$$(T-\tau-2)\sqrt{\frac{\beta\,(T-\tau)}{2}}-(T-\tau-2\alpha_0)\,\frac{(T-\tau)}{-a_\gamma} < \beta\,(T-\tau)\,(1-\alpha_0)$$

Replacing the LHS with the upper bound at $\alpha_0 = 1$, and dividing through by $\beta(T-\tau)$, we obtain the following sufficient condition:

$$\alpha_0 \leq 1-(T-\tau-2)\left(\sqrt{\frac{1}{2\beta(T-\tau)}}-\frac{1}{-\beta a_\gamma}\right)$$

The RHS is precisely the value $\overline{\alpha_0}$ defined in the Lemma: that is, $\alpha_0 \leq \overline{\alpha_0}$ implies that $[v_1^{\min}, v_1^{\max}] \subseteq [u_1(0,\alpha_0), u_1(-2,1)]$, which, by continuity, implies that for every $a \in [a_\gamma, 0] \Leftrightarrow v_1(a) \in [v_1^{\min}, v_1^{\max}]$, there exists $\widehat{a}(a) \in [-2,0]$ and $\alpha_a \geq \alpha_0$ s.t. $v_1(a,\alpha_0) = u_1(\widehat{a}(a),\alpha_a)$. This proves (iv). ■

D.3 Derivations

In this section, we explain how the functions and parameters in our fully revealing construction were chosen.

D.3.1 For the Expert:

Suppose we wanted an equilibrium in which each type $\theta \in [0,\theta_1]$ pools with a type $g(\theta) \in [\theta_2,\theta_3]$, to recommend an action $u_1(\theta)$ in period 1, $u_2(\theta)$ in period 2, and then reveal the truth for the final $T-2$ periods. The disutilities to types $\theta, g(\theta)$ from following (respectively) the strategies prescribed for types $\theta', g(\theta')$ are then

$$\begin{aligned} D_u(\theta'|\theta) &= \left(u_1(\theta')-\theta-1\right)^2+\left(u_2(\theta')-\theta-1\right)^2+(T-2)\left(\theta'-\theta-1\right)^2 \qquad (52)\\ D_u(g(\theta')|g(\theta)) &= \left(u_1(\theta')-g(\theta)-1\right)^2+\left(u_2(\theta')-g(\theta)-1\right)^2+(T-2)\left(g(\theta')-g(\theta)-1\right)^2 \end{aligned}$$

In order for this to be an equilibrium, it must be that $D_u(\theta'|\theta)$ reaches a minimum over $[0,\theta_1]$ at $\theta' = \theta$ (so that type θ earns a lower disutility by telling the truth than by mimicking any other type θ' in the interval $[0,\theta_1]$), and that $D_u(g(\theta')|g(\theta))$ reaches a minimum at $g(\theta') = g(\theta)$. We can do this by simply choosing functions that satisfy the corresponding first- and second-order conditions:

beginning with the F.O.C.'s, we need

$$0 = \frac{1}{2}\left.\frac{dD_u(\theta'|\theta)}{d\theta'}\right|_{\theta'=\theta} = u_1'(\theta)\left(u_1(\theta)-\theta-1\right)) + u_2'(\theta)\left(u_2(\theta)-\theta-1\right) - (T-2) \quad (53)$$

$$0 = \frac{1}{2}\left.\frac{dD_u(g(\theta')|g(\theta))}{dg(\theta')}\right|_{\theta'=\theta} = \frac{u_1'(\theta)}{g'(\theta)}\left(u_1(\theta)-g(\theta)-1\right) + \frac{u_2'(\theta)}{g'(\theta)}\left(u_2(\theta)-g(\theta)-1\right) - (T-2) \quad (54)$$

Subtracting the 2nd expression from the 1st, we get

$$\left(u_1'(\theta)+u_2'(\theta)\right)\left(g(\theta)-\theta\right) = (T-2)(1-g'(\theta))$$

If we define $a(\theta) \equiv \ln\frac{g(\theta)-\theta}{g(\theta_1)-\theta_1}$, so that $a'(\theta) = \frac{g'(\theta)-1}{g(\theta)-\theta}$, this becomes

$$u_1'(\theta)+u_2'(\theta) = -(T-2)a'(\theta) \Rightarrow u_1(\theta)+u_2(\theta) = k_u-(T-2)a(\theta),\ k_u \text{ a constant} \quad (55)$$

Now: the disutility from telling the truth is

$$\begin{aligned} D_u(\theta|\theta) &\equiv D_u(\theta) = (u_1(\theta)-\theta-1)^2 + (u_2(\theta)-\theta-1)^2 + (T-2) \quad (56)\\ &\Rightarrow \frac{D_u'(\theta)}{2} = \left(u_1'(\theta)-1\right)\left(u_1(\theta)-\theta-1\right) + \left(u_2'(\theta)-1\right)\left(u_2(\theta)-\theta-1\right) \end{aligned}$$

Substituting (53) into this expression, we get

$$\begin{aligned} \frac{D_u'(\theta)}{2} &= T-2-(u_1(\theta)+u_2(\theta))+2(\theta+1)\\ &= T+2\theta-k_u+(T-2)a(\theta) \text{ (by (55))} \end{aligned}$$

Integrating w.r.t. θ, we get

$$D_u(\theta) = D_u(0) + 2\theta\left(T+\theta-k_u\right) - 2(T-2)\int_0^\theta (a(\theta'))d\theta'$$

Setting $u_1(0) \equiv u_0$ and using expression (55) to obtain $u_2(0) = k_u-(T-2)a(0)-u_1(0) = k_u-u_0$, this becomes

$$\begin{aligned} D_u(\theta) &= \underbrace{(u_0-1)^2 + (k_u-u_0-1)^2 + (T-2)}_{D_u(0)} + 2\theta(T+\theta-k_u) + 2(T-2)\int_0^\theta a(\theta')d\theta'\\ &= 2\left(\frac{k_u}{2}-u_0\right)^2 + 2\left(\frac{k_u}{2}-\theta-1\right)^2 + 2\theta\left(T-2\right) + 2(T-2)\int_0^\theta a(\theta')d\theta' + (T-2) \end{aligned}$$

It will be convenient to change variables: rather than describing g as a function from $[0,\theta_1] \rightarrow [\theta_2,\theta_3]$, and using $a(\theta) \equiv \ln\frac{g(\theta)-\theta}{g(0)-0}$, we "flip" variables, describing each type $\theta \in [0,\theta_1]$ as a parametric function $x(a)$ of the variable a, and each type in $[\theta_2,\theta_3]$ as a parametric function $g(a)$ of the variable a, where a takes on all values between 0 and $a_1 = \ln\frac{g(\theta_1)-\theta_1}{g(0)-0}$, and $g(a), x(a)$ hold the relationship $(g(a)-x(a)) = (g(0)-x(0))\,e^a$. With this, rewriting $\int_0^\theta a(\theta')d\theta'$ as $\int_0^a sx'(s)ds$, and noting that $\theta = \int_0^a x'(s)ds$, our above disutility expression for type $\theta = x(a)$ becomes

$$D_u(x(a)) = 2\left(\frac{k_u}{2} - x(a) - 1\right)^2 + 2(T-2)\int_0^a (s+1)x'(s)ds + (T-2) + 2(T-2)C_u \tag{57}$$

where $C_u \equiv \frac{\left(\frac{k_u}{2}-u_0\right)^2}{T-2}$ may be any non-negative constant. Setting this equal to type $x(a)'s$ truth-telling disutility (evaluate (56) at $\theta = x(a)$), using $u_2(a) = k - (T-2)a - u_1(a)$ (from (55)), and solving for $u_1(a), u_2(a)$, we obtain

$$\begin{aligned} u_1(a) &= \frac{k_u}{2} - \frac{T-2}{2}a - \sqrt{T-2}\sqrt{C_u + \frac{k_u}{2}a - a(x(a)+1) - \frac{T-2}{4}a^2 + \int_0^a (s+1)x'(s)ds} \quad (58) \\ &= \frac{k_u}{2} - \frac{T-2}{2}a - \sqrt{T-2}\sqrt{C_u + \frac{k_u}{2}a - a(g(a)+1) - \frac{T-2}{4}a^2 + \int_0^a (s+1)g'(s)ds} \end{aligned}$$

with $u_2(a) = k_u - (T-2)a - u_1(a)$. Evaluating this at $x(a) = \theta_3 + a - \theta_3 e^a$ and $\frac{k_u}{2} = K + \theta_3$ gives precisely our expression $u_1(a, \alpha_a)$ in (16) evaluated at $\alpha_a = \frac{1}{2}$; the expressions in (16), (17) were "rescaled" (via the coefficients on the square roots) such that both disutility and average actions are independent of α_a.

Now, for our S.O.C.'s: differentiating (52) w.r.t. θ' gives

$$\begin{aligned} \frac{1}{2}\frac{dD_u(\theta'|\theta)}{d\theta'} &= \underbrace{u_1'(\theta')\left(u_1(\theta') - \theta' - 1\right) + u_2'(\theta')\left(u_2(\theta') - \theta' - 1\right) - (T-2)}_{=0 \text{ by } (53)} \\ &\quad + (\theta' - \theta)\underbrace{\left(u_1'(\theta') + u_2'(\theta') + T - 2\right)}_{=\frac{d}{d\theta'}(u_1(\theta') + u_2(\theta') + (T-2)\theta')} \end{aligned}$$

This implies that a sufficient condition for truth-telling to indeed yield a *minimum* on disutility is that the average action induced by each type θ, $u_1(\theta) + u_2(\theta) + (T-2)\theta$, be increasing: in this case, $\frac{dD_u(\theta'|\theta)}{d\theta'}$ is positive for any $\theta' > \theta$ (as type θ contemplates mimicking types θ' further above him, disutility increases, making him worse off), and negative for $\theta' < \theta$ (as he moves further below the truth, disutility increases, also making him worse off), but zero at $\theta' = \theta$: thus, telling the truth is better than mimicking any other type in the interval.

To sum up, this has shown that given arbitrary interval endpoints $\theta_1, \theta_2, \theta_3$, functions $x : [a_1, 0] \to [0, \theta_1]$ and $g : [a_1, 0] \to [\theta_2, \theta_3]$, and with $a_1 \equiv \ln \frac{g(a_1) - x(a_1)}{g(0) - x(0)} = \ln \frac{\theta_2 - \theta_1}{\theta_3 - 0}$ and $g(a) - x(a) = \theta_3 e^a$, if we want an equilibrium in which types $x(a), g(a)$ recommend $u_1(a)$ for one period, then $u_2(a)$ for one period, then separate and reveal the truth, then truth-telling satisfies the F.O.C. for disutility minimization iff u_1, u_2 are as specified by (58) and $u_2(a) = k_u - (T-2)a - u_1(a)$. If we additionally impose the requirement that average action be increasing in type, then we satisfy also the S.O.C.'s: this requires that each of $x'(a), g'(a)$ is either negative or ≥ 1. Analogously, for arbitrary functions $z : [a_\gamma, 0] \to [\theta_1, \theta_2]$, $h : [a_\gamma, 0] \to [\theta_3, \theta_4]$, with $a_\gamma = \ln \frac{h(a_\gamma) - z(a_\gamma)}{h(0) - z(0)} = \frac{\theta_3 - \theta_2}{\theta_4 - \theta_1}$ and $h(a) - z(a) = (\theta_4 - \theta_1)e^a$, if we want an equilibrium in which types $z(a), h(a)$ recommend $v_1(a)$ for $\frac{T-\tau}{2}$ periods, then $v_2(a)$ for $\frac{T-\tau}{2}$ periods, then separate and reveal the truth, the F.O.C.'s for truth-telling to minimize disutility yield the following equations:

$$v_1(a) = \frac{k_v}{2} - \frac{\tau}{T-\tau}a - \sqrt{\frac{2\tau}{T-\tau}}\sqrt{C_v + \frac{k_v}{2}a - a(h(a)+1) - \frac{\tau}{2(T-\tau)}a^2 + \int_0^a (s+1)h'(s)ds} \tag{59}$$

with $v_2(a) = k_v - \frac{2\tau}{T-\tau}a - v_1(a)$, k_v and C_v constants. And the S.O.C.'s, guaranteeing that truth-telling indeed yields a disutility *minimum* over the interval, reduce to the requirement that each of $z'(a), h'(a)$ is either negative or ≥ 1. The proof that no expert type wishes to deviate after the initial recommendation follows almost trivially from the prescribed strategies.

It remains to show that no expert type wishes to mimic the initial recommendation of any type from any *other* interval. This reduces to the additional requirements that at each endpoint $\theta_i \in \{\theta_1, \theta_2, \theta_3\}$, the average action is non-decreasing at θ_i (if discontinuous), and type θ_i is indifferent between the two sequences that he can induce. Our construction chooses the specific parametrizations $g(a) = \theta_3 + a$ and $h(a) = \theta_4 + a$, with $x(a) = g(a) - \theta_3 e^a$, $z(a) = h(a) - (\theta_4 - \theta_1)e^a$. Then we have $x'(a) = 1 - \theta_3 e^a \leq 1 - \theta_3 e^{-2}$, $z'(a) = 1 - 2e^{a-a_\gamma} \leq -1$, and $g'(a) = h'(a) = 1$, which clearly satisfy the S.O.C.'s (provided that $\theta_3 e^{-2} \geq 2$; we in fact will restrict to $\theta_3 e^{-2} \geq 8$). With this, the expressions in (58), (59) become (with $K \equiv \frac{k_u}{2} - \theta_3$)

$$u_1(a) = K + \theta_3 - \frac{T-2}{2}a - \sqrt{T-2}\sqrt{C_u + Ka - \frac{T}{4}a^2} \tag{60}$$

$$v_1(a) = \frac{k_v}{2} - \frac{\tau}{T-\tau}a - \sqrt{\frac{2\tau}{T-\tau}}\sqrt{C_v + \left(\frac{k_v}{2} - \theta_4\right)a - \frac{T}{2(T-\tau)}a^2} \tag{61}$$

We chose $a_1 = -2$ and $g'(a) = 1$ because this is in fact the *only* way that the indifference constraint at θ_2 can hold simultaneously with both the indifference constraint at θ_3, and the increasing-average-action requirement at θ_2. We chose $h'(a) = 1$ just for simplicity. With it, the remaining increasing-average-action constraints do not bind, and the indifference conditions reduce to the following requirements on the relationships between k_v, C_v, τ (parameters from the v_t-functions) and k_u, C_u, T (parameters from the u_t-functions):

$$\begin{aligned} &\tfrac{k_v}{2} - \theta_3 = \tfrac{2K + \tau a_\gamma}{T-\tau} \\ &C_v = \tfrac{(T-2)C_u}{\tau} + \tfrac{\frac{T-\tau-2}{\tau}K^2 - 2Ka_\gamma - \frac{T}{2}a_\gamma^2}{(T-\tau)} \\ &\tfrac{\tau}{T-2} = \beta = \tfrac{(\theta_2-\theta_1)(\theta_2-\theta_1-2)}{(\theta_4-\theta_1)(\theta_4-\theta_1-2)} \end{aligned} \tag{62}$$

With this, the expressions in (60), (61) simplify exactly to the expressions for $u_1(a, \alpha_a)$ in (16) and $v_1(a, \alpha_0)$ in (18), at $\alpha_a = \frac{1}{2}$, $\alpha_0 = \frac{T-\tau}{4}$; in Proposition D3, we then rescaled (16), (18) for other values of α_a, α_0 in such a way that incentives were not affected.

D.3.2 For the decision-maker:

Suppose the decision-maker receives the recommendation $u_1(a, \alpha_a)$ in period 1. If he assigns probabilities $(p_a, 1-p_a)$ to types $x(a), g(a)$, then his disutility from following all recommendations is

$$\begin{aligned} &p_a\left(2\alpha_a\left(u_1(a,\alpha_a) - x(a)\right)^2 + 2(1-\alpha_a)\left(u_2(a,\alpha_a) - x(a)\right)^2\right) \\ &+(1-p_a)\left(2\alpha_a\left(u_1(a,\alpha_a) - g(a)\right)^2 + 2(1-\alpha_a)\left(u_2(a,\alpha_a) - g(a)\right)^2\right) \end{aligned}$$

Substituting in the expression for $u_1(a, \alpha_a)$,$u_2(a, \alpha_a)$ from (16), (17), and expanding, this becomes

$$2p_a\left(K + \theta_3 - \frac{T-2}{2}a - x(a)\right)^2 + 2(1-p_a)\left(K + \theta_3 - \frac{T-2}{2}a - g(a)\right)^2 + 2(T-2)C_u + 2(T-2)a\left(K - \frac{T}{4}a\right)$$

The best possible deviation is to choose the myopically optimal action $p_a x(a) + (1 - p_a) g(a)$ in all T periods, resulting in disutility

$$\begin{aligned} & Tp_a \left(p_a x(a) + (1 - p_a) g(a) - x(a)\right)^2 + T(1 - p_a)\left(p_a x(a) + 1 - p_a) g(a) - g(a)\right)^2 \\ = \ & Tp_a (1 - p_a)\left(g(a) - x(a)\right)^2 \end{aligned}$$

Therefore, incentive compatibility of our strategies for the decision-maker refers that the following expression (the gain to deviating at $\{x(a), g(a)\}$) be weakly negative for all $a \in [-2, 0]$:

$$\begin{aligned} & 2p_a \left(K + \theta_3 - \frac{T-2}{2} a - x(a)\right)^2 + 2(1 - p_a)\left(K + \theta_3 - \frac{T-2}{2} a - g(a)\right)^2 \\ & + 2(T-2)C_u + 2(T-2) a \left(K - \frac{T}{4} a\right) - Tp_a (1 - p_a)\left(g(a) - x(a)\right)^2 \end{aligned} \tag{63}$$

Substituting in $g(a) = \theta_3 + a, x(a) = \theta_3 + a - \theta_3 e^a$ and solving for K, we obtain that the decision-maker's gain to deviating at information set $\{x(a), g(a)\}$ is negative if and only if

$$K \in \left[a - p_a \theta_3 e^a - \sqrt{T-2}\Delta(a),\ a - p_a \theta_3 e^a + \sqrt{T-2}\Delta(a)\right] \tag{64}$$

$$\text{where } \Delta(a) \equiv \sqrt{\frac{1}{2} p_a (1 - p_a)\left(\theta_3 e^a\right)^2 + p_a \left(\theta_3 a e^a\right) - \frac{a^2}{2} - C_u}$$

For there to exist a value of K which satisfies this expression, we need $\Delta(a)$ to be real-valued, i.e. the term in square roots must be positive; at $a = -2$, this holds iff

$$p_a \in \left[\frac{\theta_3 e^{-2} - 4}{2(\theta_3 e^{-2})} - \frac{1}{2}\sqrt{\frac{\theta_3 e^{-2} - 8}{\theta_3 e^{-2}}}, \frac{\theta_3 e^{-2} - 4}{2(\theta_3 e^{-2})} + \frac{1}{2}\sqrt{\frac{\theta_3 e^{-2} - 8}{\theta_3 e^{-2}}}\right]$$

which in turn is *possible* (for some belief system) to satisfy only if $\theta_3 \geq 8e^2$. For our construction, with $g' = h' = 1$, this corresponds to

$$a_\gamma = \ln \frac{\theta_3 - \theta_2}{\theta_4 - \theta_1} = \ln \frac{2}{-a_\gamma + 2 + \theta_3 e^{-2}} \leq -1.7726 \Leftrightarrow b \leq \frac{1}{60.885}$$

That is, it is possible to satisfy the decision-maker IC constraints in a straightforward manner only if the bias satisfies $b \leq \frac{1}{60.885}$, which is why our construction specifies $b < \frac{1}{61}$.

Similarly, if the decision-maker receives the recommendation $v_1(a)$ in period 1 and assigns probabilities $q_a, 1 - q_a$ to types $z(a), h(a)$, his maximum gain to deviating is

$$\begin{aligned} & q_a \left(2\alpha_0 \left(v_1(a, \alpha_0) - z(a)\right)^2 + (T - \tau - 2\alpha_0)\left(v_2(a, \alpha_0) - z(a)\right)^2\right) \\ & + (1 - q_a)\left(2\alpha_0 \left(v_1(a, \alpha_0) - h(a)\right)^2 + (T - \tau - 2\alpha_0)\left(v_2(a, \alpha_0) - h(a)\right)^2\right) - Tq_a (1 - q_a)\left(h(a) - z(a)\right)^2 \end{aligned}$$

Recalling that the expressions in (18), (19) were scaled to make the above expression independent of α_0, we can without loss of generality set $\alpha_0 = 1$, in which case v_1 is given by (61), and $v_2 = k_v - \frac{2\tau}{T-\tau} a - v_1$; substituting into the above expression for the decision-maker's gain to deviating, we obtain

$$\begin{aligned}&(T-\tau)\left(q_a\left(\frac{k_v}{2}-\frac{\tau}{T-\tau}a-z\right)^2+(1-q_a)\left(\frac{k_v}{2}-\frac{\tau}{T-\tau}a-h\right)^2\right)+2\tau C_v\\&+2\tau a\left(\frac{k_v}{2}-\theta_4-\frac{T}{2(T-\tau)}a\right)-Tq_a(1-q_a)(h-z)^2\end{aligned}\tag{65}$$

Setting $h(a)=\theta_4+a$, $z(a)=\theta_4+a-(\theta_4-\theta_1)e^a=\theta_4+a-2e^{a-a_\gamma}$ and solving for $\frac{k_v}{2}$, we obtain that the decision-maker's gain to deviating at $\{z(a),h(a)\}$ is negative if and only if

$$\begin{aligned}\frac{k_v}{2} &\in \left[\theta_4+a-q_a\left(2e^{a-a_\gamma}\right)-\sqrt{\frac{2\tau}{T-\tau}}\widetilde{\Delta}(a),\ \theta_4+a-q_a\left(2e^{a-a_\gamma}\right)+\sqrt{\frac{2\tau}{T-\tau}}\widetilde{\Delta}(a)\right]\\ \text{where } \widetilde{\Delta}(a) &\equiv \sqrt{\frac{1}{2}q_a(1-q_a)(2e^{a-a_\gamma})^2+q_a\left(2ae^{a-a_\gamma}\right)-\frac{a^2}{2}-C_v}\end{aligned}$$

This constraint by itself is problematic. To understand the difficulty, note that at $a=a_\gamma$, there exists a value of k_v satisfying the above expression only if $\widetilde{\Delta}(a_\gamma)$ is real-valued, requiring

$$2q_a(1-q_a+a_\gamma)-\frac{a_\gamma^2}{2}-C_v\geq 0$$

We showed in the previous paragraph that the IC constraints at information sets of the form $\{x(a),g(a)\}$ can only hold if $a_\gamma\leq-1.7726$, in this case, the first term in the above inequality is negative (since we need $q_a\geq 0$ and $1+a_\gamma<0$), the second term is clearly negative, and the third must be negative (i.e. we need $C_v\geq 0$) in order for the functions $v_1,v_2$to be real-valued at $a=0$. Therefore, if the decision-maker finds it optimal to follow all recommendations sent by pairs $\{x(a),g(a)\}$, then he necessarily will have an incentive to deviate if his information set contains only types $\{z(a_\gamma),h(a_\gamma)\}=\{\theta_2,\theta_3\}$. To solve this problem, we will "bunch" pairs - scaling our action functions such that whenever the decision-maker would have an incentive to deviate after a recommendation v_1 sent by a pair $\{z(a),h(a)\}$, he believes that the recommendation is also sent (for the same length of time) by a pair $\{x(a'),g(a')\}$, and such that the expected benefit to following the recommendation (likelihood that it was sent by the pair $(x(a'),g(a'))$, times the gain in this case) exceeds the cost (which is the likelihood that it was sent by pair $(z(a),h(a))$, times the cost in this case).

D.3.3 Decision-maker Beliefs

Our incentive constraints for the DM were specified in terms of arbitrary probabilities p_a,q_a, which in turn depend both on his prior F, and on the precise details of our construction. As explained in Section 4, we assume that the DM is Bayesian. For our construction, (5) becomes:

- after a message (or message sequence) sent by types $\{x(a),g(a)\}$, $a\in[-2,0]$:

$$\frac{p_a}{1-p_a}\equiv\frac{\Pr(x(a))}{\Pr(g(a))}=\frac{f\left(x(a)\right)}{f\left(g(a)\right)}\cdot\left|\frac{x'(a)}{g'(a)}\right|=\frac{f\left(x(a)\right)}{f\left(g(a)\right)}(\theta_3e^a-1)$$

- after a message (or message sequence) sent by types $\{z(a),h(a)\}$, $a\in[a_\gamma,0]$:

$$\frac{q_a}{1-q_a}\equiv\frac{\Pr(z(a)}{\Pr(h(a)}=\frac{f\left(z(a)\right)}{f\left(h(a)\right)}\cdot\left|\frac{z'(a)}{h'(a)}\right|=\frac{f\left(z(a)\right)}{f\left(h(a)\right)}(2e^{a-a_\gamma}-1)$$

- and, after a message sent by types $\{z(a), h(a), x(\widehat{a}(a)), g(\widehat{a}(a))\}$, with $u_1(\widehat{a}(a)\alpha_{\widehat{a}}) = v_1(a, \alpha_0)$:

$$\frac{\Pr(x(\widehat{a}(a)))}{\Pr(z(a))} = \frac{f(x(\widehat{a}))}{f(z(a))} \cdot \left|\frac{x'(\widehat{a}))}{z'(a)}\right| \cdot \left|\widehat{a}'(a)\right|$$

so, denoting $\widehat{I}(\widehat{a}) = \{x(\widehat{a}), g(\widehat{a})\}$ and $I(a) = \{z(a), h(a)\}$, the decision-maker's beliefs at the pooled (4-type) information set $\widehat{I}(\widehat{a}) \cup I(a)$ satisfy

$$\begin{aligned}\frac{P^*}{1-P^*} &\equiv \frac{\Pr(\widehat{I}(\widehat{a}))}{\Pr(I(a))} = \frac{\Pr(x(\widehat{a})) \cdot \left(1 + \frac{\Pr(g(\widehat{a}))}{\Pr(x(\widehat{a}))}\right)}{\Pr(z(a)) \cdot \left(1 + \frac{\Pr(h(a))}{\Pr(z(a))}\right)} \\ &= \frac{\mu(x(\widehat{a}))}{\mu(z(a))} \cdot \left(\frac{\theta_3 e^{\widehat{a}(a)} - 1}{2e^{a-a_\gamma} - 1}\right) \left|\widehat{a}'(a)\right| \cdot \left(\frac{q_a}{p_a}\right)\end{aligned}$$

(with p_a, q_a as defined in the first two bullet points; the final term in the first line numerator is $1 + \frac{1-p_a}{p_a} = \frac{1}{p_a}$, and similarly final denominator term is $1 + \frac{1-q_a}{q_a} = \frac{1}{q_a}$).

E Proof of Proposition 4

To prove that the decision-maker's incentive constraints are relaxed as he becomes more patient, thus completing the proof of Proposition 4, it suffices to prove that the "time ratio" terms in $(7), (8)$, and $(9)$are increasing in r^{DM} (so that a decrease in r^{DM} below r^E causes a decrease in the expressions, thus making deviations even less attractive). For future reference, recall that our parameter outline in Section D.2.1 specified $T - \tau \leq 4$, $2\alpha_0 \leq 2\alpha_a$, and $T \geq 6$: by (6), this implies

$$t_1(\alpha_0) \leq t_2(\alpha_a) \leq t_3 = \frac{1}{r^E} \ln\left(\frac{1}{1-2\phi r^E}\right),\ t_4 \leq \frac{1}{r^E} \ln\left(\frac{1}{1-4\phi r^E}\right),\ \text{and } \widehat{T} \geq \frac{1}{r^E} \ln\left(\frac{1}{1-6\phi r^E}\right) \tag{66}$$

Now: to prove that $\left(\frac{\int_0^{t_1(\alpha_0)} e^{-r^{DM}} dt}{\int_0^{\widehat{T}} e^{-r^{DM}} dt}\right)$ and $\left(\frac{\int_0^{t_2(\alpha_a)} e^{-r^{DM}} dt}{\int_0^{\widehat{T}} e^{-r^{DM}} dt}\right)$ are increasing in r, we will show that $\frac{d}{dr}\left(\frac{\int_0^t e^{-rt} dt}{\int_0^T e^{-rt} dt}\right) > 0$ for any $t < T$ (so, by (66), this in particular holds at $t \in \{t_1(\alpha_0), t_2(\alpha_a)\}$ and $T = \widehat{T}$). We have,

$$\frac{d}{dr}\left(\frac{1-e^{-rt}}{1-e^{-rT}}\right) = \frac{\left(1-e^{-rT}\right) te^{-rt} - Te^{-rT}\left(1-e^{-rt}\right)}{\left(1-e^{-rT}\right)^2} = (T-t)e^{-r(T+t)} \frac{1 - \left(\frac{Te^{rt} - te^{rT}}{T-t}\right)}{\left(1-e^{-rT}\right)^2}.$$

This is positive whenever $\left(\frac{Te^{rt} - te^{rT}}{T-t}\right) < 1$; and since $\frac{d}{dr}\left(\frac{Te^{rt} - te^{rT}}{T-t}\right) = \frac{Tt}{(T-t)}\left(e^{rt} - e^{rT}\right) < 0$ for $t < T$, it follows that the term $\left(\frac{Te^{rt} - te^{rT}}{T-t}\right)$ is decreasing in r, hence maximized at $r = 0$, where it exactly equals 1; for any $r > 0$, the term $\left(\frac{Te^{rt} - te^{rT}}{T-t}\right)$ is strictly below 1, implying that $\frac{d}{dr}\left(\frac{1-e^{-rt}}{1-e^{-rT}}\right) > 0$, as desired.

Next, we prove that $\left(\frac{\int_{t_1(\alpha_0)}^{t_4} e^{-r^{DM}} dt}{\int_{t_1(\alpha_0)}^{\widehat{T}} e^{-r^{DM}} dt}\right)$ is increasing in $e^{-r^{DM}}$. We have,

$$\frac{d}{dr}\left(\frac{e^{-rt_1} - e^{-rt_4}}{e^{-rt_1} - e^{-r\widehat{T}}}\right) = \frac{(T-t_4)e^{-r(T+t_4)} + (t_4-t_1)e^{-r(t_4+t_1)} - (T-t_1)e^{-r(T+t_1)}}{\left(e^{-rt_1} - e^{-r\widehat{T}}\right)^2}$$

We want to show that this expression is positive $\forall r > 0$. Suppose, by contradiction, that it is negative: then,

$$\frac{(T-t_4)}{(T-t_1)}e^{-r(T+t_4)} + \frac{(t_4-t_1)}{(T-t_1)}e^{-r(t_4+t_1)} - e^{-r(T+t_1)} < 0 \tag{67}$$

The derivative of the LHS of this expression w.r.t r, divided by $(T+t_1)$, is

$$-\left(\frac{T-t_4}{T-t_1}\right)\left(\frac{T+t_4}{T+t_1}\right)e^{-r(T+t_4)} - \left(\frac{t_4-t_1}{T-t_1}\right)\left(\frac{t_4+t_1}{T+t_1}\right)e^{-r(t_4+t_1)} + e^{-r(T+t)}$$

Substituting in $e^{-r(T+t_1)} > \frac{(T-t_4)}{(T-t_1)}e^{-r(T+t_4)} + \frac{(t_4-t_1)}{(T-t_1)}e^{-r(t_4+t_1)}$ from (67), and factoring, we obtain that this is greater than

$$\left(\frac{T-t_4}{T-t_1}\right)\frac{(t_4-t_1)}{(T-t_1)}\left(e^{-r(t_4+t_1)} - e^{-r(t_4+T)}\right)$$

which is strictly positive $\forall r > 0$ by $t_1 < t_4 < T$. That is, the LHS of (67) is increasing in r whenever it is negative; therefore, if it is negative at some $\widehat{r} > 0$, it is also negative for all $r < \widehat{r}$. So, in particular, (67) can only hold for some $\widehat{r} > 0$ if it also holds strictly at $r = 0$; but since this is in fact *not* the case – the LHS of (67) is exactly zero at $r = 0$ – we conclude that (67) cannot hold for any $r > 0$. Therefore, as desired, we have that $\frac{d}{dr}\left(\frac{e^{-rt_1}-e^{-rt_4}}{e^{-rt_1}-e^{-r\widehat{T}}}\right) > 0 \; \forall r > 0$.

And finally, to prove that $\frac{\int_{t_1(\alpha_0)}^{t_4} e^{-r^{DM}} dt}{\int_0^{\widehat{T}} e^{-r^{DM}} dt}$ and $\frac{\int_{t_2(\alpha_a)}^{t_3} e^{-r^{DM}} dt}{\int_0^{\widehat{T}} e^{-r^{DM}} dt}$ are increasing in r^{DM}, we will show that the condition $\frac{d}{dr}\left(\frac{\int_t^{t+\Delta} e^{-r\tau} d\tau}{\int_0^{\widehat{T}} e^{-r\tau} d\tau}\right) > 0$ becomes more difficult to satisfy as $r, t, t+\Delta$ increase, and easier to satisfy as $\widehat{T}$ increases: so, it is sufficient to prove that the inequality holds if we replace $r, t, t+\Delta$ with upper bounds, and $\widehat{T}$ with a lower bound. We specifically need $\frac{d}{dr}\left(\frac{\int_t^{t+\Delta} e^{-r\tau} d\tau}{\int_0^{\widehat{T}} e^{-r\tau} d\tau}\right) > 0$ to hold for $(t, t+\Delta) \in \{(t_1(\alpha_0), t_4), (t_2(\alpha_a), t_3)\}$, $T = \widehat{T}$, and $r \leq r^E$, which gives the upper bounds (by (66)) $r = r^E$, $t = \frac{1}{r^E}\ln\left(\frac{1}{1-2\phi r^E}\right)$, and $t + \Delta = \frac{1}{r^E}\ln\left(\frac{1}{1-4\phi r^E}\right)$, and the lower bound $\widehat{T} = \frac{1}{r^E}\ln\left(\frac{1}{1-T\phi r^E}\right) \geq \frac{1}{r^E}\ln\left(\frac{1}{1-6\phi r^E}\right)$.

To this end, we differentiate $\frac{\int_t^{t+\Delta} e^{-r\tau} d\tau}{\int_0^{\widehat{T}} e^{-r\tau} d\tau} = \left(\frac{e^{-rt}-e^{-r(t+\Delta)}}{1-e^{-r\widehat{T}}}\right)$, obtaining

$$\begin{aligned} \frac{d}{dr}\left(\frac{e^{-rt}-e^{-r(t+\Delta)}}{1-e^{-r\widehat{T}}}\right) &= (1-e^{-r\widehat{T}})\left((t+\Delta)e^{-r(t+\Delta)} - te^{-rt}\right) - \widehat{T}e^{-rT}\left(e^{-rt} - e^{-r(t+\Delta)}\right) \\ &> 0 \text{ whenever } \frac{\widehat{T}}{(e^{r\widehat{T}}-1)} < \frac{(t+\Delta)\,e^{rt} - te^{r(t+\Delta)}}{e^{r(t+\Delta)} - e^{rt}} \end{aligned} \tag{68}$$

We first show that this becomes harder to satisfy as r increases. We have,

$$\begin{aligned}\frac{d}{dr}\left(\frac{t+\Delta-te^{r\Delta}}{e^{r\Delta}-1}-\frac{\widehat{T}}{(e^{rT}-1)}\right) &= -\Delta^2\frac{e^{r\Delta}}{(e^{r\Delta}-1)^2}+\widehat{T}^2\frac{e^{\widehat{T}r}}{\left(e^{\widehat{T}r}-1\right)^2} \\ &< 0 \text{ whenever } e^{r\widehat{T}}-\frac{\widehat{T}}{\Delta}e^{\left(\frac{T+\Delta}{2}\right)r}+\frac{\widehat{T}}{\Delta}e^{\left(\frac{\widehat{T}-\Delta}{2}\right)r}-1>0 \quad (69)\end{aligned}$$

The derivative of the LHS of (69) w.r.t. r, setting $\lambda \equiv \frac{\widehat{T}}{\Delta}$, is

$$\frac{1}{2}\Delta\lambda e^{r\Delta\left(\frac{\lambda-1}{2}\right)}\left(2e^{r\Delta\left(\frac{\lambda+1}{2}\right)}+(\lambda-1)-(\lambda+1)\,e^{r\Delta}\right).$$

This is positive for any $r>0$, since the bracketed term equals zero at $r=0$, and is increasing in r (the derivative is $\Delta(\lambda+1)\,e^{r\Delta}\left(e^{r\Delta\left(\frac{\lambda-1}{2}\right)}-1\right)$, which is positive by $\lambda=\frac{\widehat{T}}{\Delta}>1$). That is, the LHS of the expression in (69) is increasing in r; and since it is exactly equal to zero at $r=0$, we conclude that (69) holds: that is, our desired inequality (68) becomes harder to satisfy as r increases. Also, $\frac{t+\Delta-te^{r\Delta}}{e^{r\Delta}-1}$ is decreasing in both t and Δ (since $\frac{d}{dt}\left(\frac{t+\Delta-te^{r\Delta}}{e^{r\Delta}-1}\right)=-1$, and $\frac{d}{d\Delta}\left(\frac{t+\Delta-te^{r\Delta}}{e^{r\Delta}-1}\right)=\frac{\left((1-r\Delta)e^{r\Delta}-1\right)}{(e^{r\Delta}-1)^2}$, which is negative: the denominator is positive, and the numerator is decreasing in r (derivative w.r.t. r is $-r\Delta^2e^{r\Delta}$) with a maximum value, at $r=0$, of zero).

So, it is sufficient to prove that (68) holds at $r=r^E$, $t=\frac{1}{r^E}\ln\left(\frac{1}{1-2\phi r^E}\right)$, $t+\Delta=\frac{1}{r^E}\ln\left(\frac{1}{1-4\phi r^E}\right)$, $\widehat{T}=\frac{1}{r^E}\ln\left(\frac{1}{1-\phi T r^E}\right)$ (where T is the horizon from the original construction; recall that Section D.2.1 specifies $T\geq 6$) : here, (68) becomes

$$\begin{aligned}&\frac{\frac{1}{r^E}\ln\left(\frac{1}{1-T\phi r^E}\right)}{e^{\ln\left(\frac{1}{1-T\phi r^E}\right)}-1}<\frac{\frac{1}{r^E}\ln\left(\frac{1}{1-4\phi r^E}\right)e^{\ln\left(\frac{1}{1-2\phi r^E}\right)}-\frac{1}{r^E}\ln\left(\frac{1}{1-2\phi r^E}\right)e^{\ln\left(\frac{1}{1-4\phi r^E}\right)}}{e^{\ln\left(\frac{1}{1-4\phi r^E}\right)}-e^{\ln\left(\frac{1}{1-2\phi r^E}\right)}} \quad (70)\\ \Leftrightarrow\ &\frac{2}{T}\left(1-T\phi r^E\right)\ln\left(1-T\phi r^E\right)-\left(1-4\phi r^E\right)\ln\left(1-4\phi r^E\right)+\left(1-2\phi r^E\right)\ln\left(1-2\phi r^E\right)>0\end{aligned}$$

Note that the LHS of (70) exactly equals zero at $\phi r^E=0$; so, to show that (70) holds $\forall r^E>0$, we just need to show that the LHS expression is increasing in ϕr^E. To this end, note that

$$\begin{aligned}&\frac{d^2}{d(\phi r^E)^2}\left(\frac{2}{T}\left(1-T\phi r^E\right)\ln\left(1-T\phi r^E\right)-\left(1-4\phi r^E\right)\ln\left(1-4\phi r^E\right)+\left(1-2\phi r^E\right)\ln\left(1-2\phi r^E\right)\right)\\ =\ &\frac{2\left(T-6+8\phi r^E\right)}{(1-T\phi r^E)\left(4\phi r^E-1\right)\left(2\phi r^E-1\right)}\end{aligned}$$

This is positive (by $T\geq 6$ and $2\phi r^E<4\phi r^E<T\phi r^E<1$), and therefore the first derivative w.r.t. r of the LHS of (70) reaches a minimum at $r=0$, where it equals zero. We conclude that the LHS of (70) is increasing in ϕr^E, therefore strictly positive for any $\phi r^E\in(0,\frac{1}{T})$, as desired.

This completes the proof that all incentive constraints for the decision-maker are relaxed as he becomes more patient; therefore, our modified timeline yields a fully revealing equilibrium (for some priors) for any $r^{DM}\leq r^E$. ■

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