Chapter 5 Counting

5.4 Binomial Coefficients

1. The Binomial Theorem

- Example 1 (page 363)
  - $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
- Theorem 1
  - The Binomial Theorem (page 363)

    $(x+y)^n$

    $= \sum_{j=0}^{n} C(n,j)\, x^{n-j} y^j$

    $= C(n,0)x^n + C(n,1)x^{n-1}y + C(n,2)x^{n-2}y^2 + \ldots\ldots\ldots$

    $+ C(n,n-1)xy^{n-1} + C(n,n)y^n$

1. The Binomial Theorem

- Theorem 1
  - Proof
    - 当乘积被展开时其中的项都是下述形式：$x^{n-j}y^j$, $j=0,1,2, ..., n$。为计数形如$x^{n-j}y^j$的项数，观察到必须从n个和中选n-j个x（从而乘积中其他的j个项都是y）才能得到这种项。因此， $x^{n-j}y^j$的系数是

$$C(n,n-j)=C(n,j)$$

定理得证

1. The Binomial Theorem

- Theorem 1
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定理得证

1. The Binomial Theorem

- Example 2
  - What is the expansion of $(x+y)^4$?
  - Solution

$$
\begin{aligned}
&(x+y)^4 \\
&= \sum_{j=0}^{n} C(4,j)\ x^{4-j} y^j \\
&= C(4,0)x^4 + C(4,1)x^3y + C(4,2)x^2y^2 \\
&\qquad + C(4,3)xy^3 + C(4,4)y^4 \\
&= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4
\end{aligned}
$$

1. The Binomial Theorem

- Example 3
  - What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$?
  - Solution

$$C(25,13)$$

$$= 25! / (13! \times 12!)$$

1. The Binomial Theorem

- Example 4
  - What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?
  - Solution

$$(2x+(-3y))^{25}$$

$$= \sum_{j=0}^{25} C(25,j)\ (2x)^{25-j}\ (-3y)^{j}$$

- The coefficient of $x^{12}y^{13}$ is

$$C(25,j)\ 2^{12}\ (-3)^{13}$$

$$= -\ 25!/(13!\times 12!)\ 2^{12}3^{13}$$

1. The Binomial Theorem

- Corollary 1
  - Let n be a nonnegative integer. The

    $\sum_{k=0}^{n} C(n,k) = 2^n$
  - Proof:

    2^n

    $= (1+1)^n$

    $= \sum_{k=0}^{n} C(n,k)\ 1^k\ 1^{n-k}$

    $= \sum_{k=0}^{n} C(n,k)$

1. The Binomial Theorem

- Corollary 2
  - Let n be a positive integer. The

    $\sum_{k=0}^{n} (-1)^k C(n,k) = 0$
  - Proof:

    0

    $= 0^n$

    $= ((-1) + 1)^n$

    $= \sum_{k=0}^{n} (-1)^k 1^{n-k}$

    $= \sum_{k=0}^{n} (-1)^k C(n,k)$

1. The Binomial Theorem

- Remark
  - Corollary 2 implies that

$$C(n,0) + C(n,2) + C(n,4) + \ldots\ldots$$
$$= C(n,1) + C(n,3) + C(n,5) + \ldots\ldots$$

1. The Binomial Theorem

- Corollary 3
  - Let n be a nonnegative integer. Then

    $\sum_{k=0}^{n} C(n,k) = 3^n$
  - Proof:

    $(1+2)^n$

    $= \sum_{k=0}^{n} C(n,k)\ 1^{n-k}\ 2^k$

    $= \sum_{k=0}^{n} C(n,k)\ 2^k$

2. Pascal's Identity and Triangle

- Theorem 2 (Pascal's Identity)
  - Let n and k be positive integers with $n \geqslant k$. Then $C(n+1,k) = C(n,k-1) + C(n,k)$
  - Proof:
    - 假设T是一个含有n+1个元素的集合，令a是T中的一个元素，且S=T-{a}。注意到T的包含k个元素的子集有C(n+1,k)个。然而T的含k个元素的子集或者包含a和S中的k-1个元素，或者不包含a但包含S中的k个元素。由于S的k-1元素子集有C(n,k-1)个，故T含a在内的k元素子集有C(n,k-1)个。又由于S的k元素子集有C(n,k)个，故T的不含a的k元子集有C(n,k)个。从而可得：

      $$C(n+1,k) = C(n,k-1) + C(n,k)$$

2. Pascal's Identity and Triangle

- Remark
  - Pascal's Identity, together with the initial conditions $C(n,0)=C(n,n)=1$ for all integers n, can be used to recursively define the binomial cofficients

2. Pascal's Identity and Triangle

- Pascal's Triangle

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1

………………………..

3. Some Other Identities of the Binomial Coefficients

- Theorem 3 (Vandermondes's Identity)
  - Let m, n, and r be nonnegative integers with r not exceeding either m or n. Then

    $$C(m+n, r) = \sum_{k=0}^{r} C(m, r-k)\, C(n,k)$$

  - Proof

    假定在第一集合中有m个项，第二集合中有n个项。从这两个集合的并集中取r个元素的方式数是C(m+n,r)。从并集中取r个元素的另一种方式是先从第一个集合中取k个元素，接着从第二个集合中取r-k个元素，其中k是满着0≤k≤r的整数。用乘积法则，这可以用C(m,k) C(n,r-k)种方式完成。所以，从这个并集中选取 r个元素的总方式数等于

    $$C(m+n, r) = \sum_{k=0}^{r} C(m, r-k)\, C(n,k)$$

    这就证明了范德蒙德恒等式

3. Some Other Identities of the Binomial Coefficients

- Corollary 4
  - If n is a nonnegative integer. Then

    $$C(2n, n) = \sum_{k=0}^{n} C(n,k)^2$$

  - Proof

    我们使用范德蒙德恒等式且m=r=n：

    $$C(2n, n)$$

    $$= \sum_{k=0}^{n} C(n, n-k)\, C(n,k)$$

    $$= \sum_{k=0}^{n} C(n, k)^2$$

3. Some Other Identities of the Binomial Coefficients

- Theorem 4
  - Let n and r be nonnegative integers with r⩽n. Then

    $C(n+1, r+1) = \sum_{j=r}^{n} C(j,r)$
  - Proof:
    - See Next Slide

3. Some Other Identities of the Binomial Coefficients

- Theorem 4
  - Proof:

由§5.3中的定理 10可知，等式的左边，C(n+1, r+1)计算了长度为n+1的二进制位串中含r+1个1 的位串的个数。

现考虑等式右边，考虑二进制位串中最后一个1的位置，显然可知，最后一个1的位置肯定在位置r+1, r+2, ... ,或n+1。更进一步讲，若最后1的位置是k，那么二进制位串中前k-1个位置必须有r个位置上是1。所以， §4.3中的定理 10，相应的二进制位串有C(k-1,r)个。由于k的取值范围是：r+1≤k≤n+1，所以我们有

$$\sum_{k=r+1}^{n+1} C(k-1, r) = \sum_{j=r}^{n} C(j, r)$$

个含有r+1个1 的长度为n的二进制位串。

Homework

- Page 369~370
  - 8, 10, 20, 22, 24, 28