Chapter 5 Counting

5.5 Generalized Permutations and Combinations

1. Introduction

- Introduction
  - Read Page 370

2. Permutations with Repetition

- Example 1 (page 371)
  - How many strings of length n can be formed from the English alphabet?
  - Solution: 26^n
- Theorem 1 (page 371)
  - The number of r-permutations of a set of n objects with repetition allowed is n^r.
  - Proof: 当允许重复时，在r-排列中对r-个位置中的每个位置有n种方式选择集合的元素，因为对每个选择，所有n个物体都是有效的。因此，由乘法规则，当允许重复时存在n^r 个r-排列。

3. Combinations with Repetition

- Example 2 (page 371)
  - How many ways are there to select four pieces of fruit from a bowl containing apples, oranges, and pears if the order in which the pieces are selected does not matter, only the type of fruit and not the individual pieces matters, and there are at least four pieces of each type of fruit in the bowl.

3. Combinations with Repetition

- Example 2 (page 371)
  - Solution: 15 ways

4 apples	4 oranges	4 pears
3 apples, 1 oranges	3 pears, 1 pear	3 oranges, 1 apple
3 oranges, 1 pear	3 pears, 1 apple	3 pears, 1 orange
2 apples, 2 oranges	2 apples, 2 pears	2 oranges, 2 pears

2 apples, 1 orange, 1 pear

2 oranges, 1 apple, 1 pear

2 pears, 1 apple, 1 orange

  - This solution is the number of 4-combinations with repetition allowed from a three-element set {apple, orange, pear

3. Combinations with Repetition

- Example 3 (page 372)
  - How many ways are there to select five bills from a cashbox containing \$1 bills, \$2 bills, \$5 bills, \$10 bills, \$20 bills, \$50 bills, and \$100 bills? Assume that the order in which the bills are chosen does not matter, that the bills of each denomination are indistinguishable (同种币值的纸币是不加区分的), and that there are at least five bills of each type

3. Combinations with Repetition

- Example 3 (page 372)
  - Solution: 要点如下（page 372~373）

$100	$50	$20	$10	$5	$2

用6 |　和　5 *来描述，　例如：

1. | | | * * | | | * * *表示2 $10s,3 $1s
2. * | * | * * | | * | | 表示 1 $100,2 $20s,1 $5
3. * | | | * * | | * | * 表示 1 $100,2 $10s,1 $2,1$1

- 问题转化为：在11个位置中选5个*号的位置（请仔细思考）

$$C(11, 5) = 11!/(5! \times 6!)$$

3. Combinations with Repetition

- Theorem 2
  - There are C(n+r-1, r) r-combinations from a set with n elements when repetition of elements are allowed.
  - Proof: 当允许重复时n个元素集合的每个r-组合可以用n-1条竖线和r颗星的表表示。这n-1条竖线是用来标记n个不同的单元。每当集合的第i个元素出现在组合中，第i个单元就包含一颗星。例如，4元素集合的一个6-组合用3条竖线和6颗星来表示。这里

    ＊＊｜＊｜　｜＊＊＊

    代表了恰好包含2个第一元素、1个第二元素、0个第三元素和3个第四元素。

3. Combinations with Repetition

- Theorem 2
  - 正如我们已经看到，包含n-1条竖线和r颗星的每一个不同的表对应于n元素集合的允许重复的一个r-组合。这种表的个数是C(n-1+r, r)，因为每个表对应于从包含r颗星和n-1条竖线的n-1+r个位置中取r个位置来放r颗星的一种选择。

3. Combinations with Repetition

- Example 4 (page 373)
  - Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen? Assume that only the type of cookie, and *not the individual cookies or the order in which they are chosen,* matters.
  - Solution:

four---------the number of elements

six------- 6-combination

C(4+6-1, 6) (可以直接套用公式)

Remark: Please use this theorem to consider example 2 (page 336): C(3+4-1, 6)

3. Combinations with Repetition

- Example 5 (page 373)
  - How many solutions does the equation

    $$x_1+x_2+x_3=11$$

    have, where x_1, x_2, and x_3 are nonnegative integers?
  - Solution

    ---- Selecting 11 items from a set with three elements.

    x1 items of type one

    x2 items of type two

    x3 items of type three

3. Combinations with Repetition

- Example 5 (page 373)
  - Solution (Cont.)

    ---- 11-combinations with repetition allowed from a set of three elements

    ---- answer: C(3+11-1, 11)

3. Combinations with Repetition

- Example 5 (page 373)
  - Remark: Variables can have constraints.

    For example, $x1 \geqslant 1$, $x2 \geqslant 2$, and $x3 \geqslant 3$.
    - Solution:
    1. Firstly, choose one item from type one, two of type two, and three of type three.
    2. Then, select five additional items
    - answer: C(3+5-1, 5)
- Example 6 (page 374)
  - Solution: Please read it by yourself.

4. Permutations with Indistinguishable Objects

- 具有不可区别物体的集合的排列
- Example 7 (page 375)
  - How many different strings can be made by reordering the letters of the word

SUCCESS

  - Solution:
    - Success-------3 Ss, 2 Cs, 1 U, 1 E

4. Permutations with Indistinguishable Objects

- Example 7 (page 375)
  - Solution:
    - 3 Ss can be placed among 7 positions in C(7, 3) ways.
    - 2 Cs can be placed in C(4,2) ways.
    - U can be placed in C(2,1) ways.
    - E can be placed in C(1,1) way.
    - Answer

      -------- $C(7,3) \times C(4,2) \times C(2,1) \times C(1,1)$

4. Permutations with Indistinguishable Objects

- Theorem 3 (page 375)
  - the number of different permutations of n objects, where they are

    n_1 indistinguishable objects of type 1,

    n_2 indistinguishable objects of type 2,

    ......, and

    n_k indistinguishable objects of type k, is

    $$n! \,/\, (n_1! \times n_2! \ldots\ldots n_k!)$$
  - Proof: similar to example 2.

5. Distributed Objects into Boxes

- 把物体放入盒子
- Example 8 (page 378)
  - How many ways are there to distribute hands of 5 cards of four player from the standard of deck of 52 cards.
  - Solution:

    $C(52,5) \times C(47,5) \times C(42,5) \times C(37,5)$

5. Distributed Objects into Boxes

- Theorem 4 (page 377)
  - The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objetcs are placed into box i, i=1,2, ..., k, equals

$$n! / (n_1! \times n_2! \ldots\ldots n_k!)$$

5. Distributed Objects into Boxes

- Indistinguishable Objects and Distinguishable Boxes
  - Question and Method
    - Counting the number of ways of placing n indistinguishable objects into k distinguishable boxes turns out to be the same as counting the number of n-combinations for a set with k elements when repetition is allowed

      ------ C(k+n-1, n)

5. Distributed Objects into Boxes

- Example 9 (page 377)
  - How many ways are there to place 10 indistinguishable balls into eight distinguishable bins?
  - Solution:

----- C(8+10-1,10) = C(17,10)

= 17!/(10! 7!)

5. Distributed Objects into Boxes

- Distinguishable Objects and indistinguishable Boxes
  - Question and Method
    - Counting the ways to place n distinguishable objects into k indistinguishable boxes is more difficult than counting the ways to place objects, distinguishable or indistinguishable objects, into distinguishable boxes.

      -----No simple closed formula for this counting problem

5. Distributed Objects into Boxes

- Example 10 (Page 377~378)
  - How many ways are there to put four different employees into three indistinguishable offices, when each office can contain any number of employees?
  - Solution:

    4 different employees----A, B, C, D Ways:

    - { {A, B, C, D} }
    - { {A, B, C}, {D} }
    - { {A, B, D}, {C} }
    - { {A, C, D}, {B} }
    - { {B, C, D}, {A} }

5. Distributed Objects into Boxes

- Example 10 (Page 377~378)
  - How many ways are there to put four different employees into three indistinguishable offices, when each office can contain any number of employees?
  - Solution:

4 different employees----A, B, C, D Ways:

1. { {A, B, C, D} }
2. { {A, B, C}, {D} }
   { {A, B, D}, {C} }
   { {A, C, D}, {B} }
   { {B, C, D}, {A} }

5. Distributed Objects into Boxes

- Example 10 (Page 377~378)
  - Solution:

    4 different employees----A, B, C, D Ways:

3. { {A, B}, {C, D} }
   { {A, C}, {B, D} }
   { {A, D}, {B, C} }
4. { {A, B}, {C}, {D} }
   { {A, C}, {B}, {D} }
   { {A, D}, {B}, {C} }
   { {B, C}, {A}, {D} }
   { {B, D}, {A}, {C} }
   { {C, D}, {A}, {B} }

Homework

- Page 379~382
  - 14, 16, 30, 32, 38