Chapter 2
Basic Structures: Sets, Functions, Sequences and Sum

2.3 Functions

1. Introduction

- What is function?
  - Definition 1 (page 133)
    - Let A and B be sets. A function from A to B is an assignment of exactly one element of B to each element of A.
    - We write $f(a)=b$ if b is the unique element of B assigned by the function f to the element a of A.
    - If f is a function from A to B, we write $f:A \to B$.
  - Example (page 133)
    - Assignment of Grades in a Discrete Mathematics Class

1. Introduction

- Domain (定义域), codomain (共域) and range (值域)
  - Definition 2 (page 134)
    - If f is a function from A to B, we say that A is the domain of f and B is the codomain of f
    - If f(a)=b, we say that b is the image (像) of a and a is a pre-image (原像) of b.
    - The range of f is the set of all images of elements of A. Also, if f is a function from A to B, we say f maps A to B.

1. Introduction

- Example 3 (page 135)
  - Let f: Z→Z assign the square of an integer to this integer, i.e., f(x)=x2.
  - What is the domain, codomain and range of function f ?
  - Solution: See book
- Definition 3 (page 135)
  - Let f_1 and f_2 be functions from A to R. Then f_1+f_2 and f_1 f_2 are also functions from A to R defined by
    1. $(f_1 + f_2)(x) = f_1 (x) + f_2(x)$
    2. $(f_1 f_2)(x) = f(x_1) f(x_2)$

1. Introduction

- Example 6 (page 135)
  - Let f_1 and f_2 be functions from R to R such that $f_1(x)=x^2$ and $f_1(x)=x-x^2$.
  - What are the functions f_1+f_2 and $f_1 f_2$?
  - Solution: See blackboard.

1. Introduction

- Definition 4 (page 136)
  - Let f be a function from the set A to the set B and let S be a subset of A. The image of S is the subset of B that consists of the images of the elements of S.
  - The notation: $f(S) = \{ f(s) \mid s \in S \}$
- Example 7 (page 136)
  - $A=\{a, b, c, d, e\}$
  - $B=\{1,2,3,4\}$
  - $f(a)=2$, $f(b)=1$, $f(c)=4$, $f(e)=1$
  - $S=\{b, c, d\}$
    - What is f(S)?
    - Solution: See book.

2. One-to-one and Onto functions

- One-to-one function (or injective, 单射)
  - Definition 5 (page 136)
    - A function f is said to be one-to-one, if and only if f(x)=f(y) implies that x=y for all x and y in the domain of f.
    - A function is said to be an injection if it is one-to-one.
  - Examples (pages 136-137)
    - Example 8
    - Determine whether the function f from {a,b,c,d} to {1,2,3,4,5} with f(a)=4, f(b)=5, f(c)=1, and f(d)=3 is one to one.
    - Solution: Function f is one-to-one.

2. One-to-one and Onto functions

- One-to-one function (or injective, 单射)
  - Examples (pages 136-137)
    - Example 9
    - Determine whether the function $f(x)=x^2$ from the set of integers to the set of integers is one-to-one.
    - Solution: No
    - However, if the domain is Z+, then f is one-to-one.

2. One-to-one and Onto functions

- One-to-one function (or injective, 单射)
  - Examples (pages 136-137)
    - Example 10
    - Determine whether the function $f(x)=x+1$ is one to one.
    - Solution: Yes.

2. One-to-one and Onto functions

- Some conditions that guarantee that a function is one to one (page 137)
  - Definition 6 (strictly increasing or descreasing function)
    - A function f whose domain and codomain are subset of the set of real numbers is called strictly increasing if $f(x)<f(y)$ whenever $x<y$ and x and y are in the domain of f.
    - strictly descreasing?

2. One-to-one and Onto functions

- Some conditions that guarantee that a function is one to one (page 137)
  - Definition 6 (strictly increasing or descreasing function)
    - If a function is either strictly increasing or strictly decreasing, it must be one to one.

2. One-to-one and Onto functions

- Onto (or surjective function, 满射)
  - Definition 7 (page 137)
    - A function f from A to B is called onto, or surjective, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a)=b$.
    - function f is called a surjection if it is onto.

2. One-to-one and Onto functions

- Examples (page 138)
  - Example 11
    - Let f be the function from {a,b,c,d} to {1,2,3} defined by f(a)=3, f(b)=2, f(c)=1, and f(d)=3.Is f an onto function?
    - Solution: Yes.
    - How about the answer if the codomain is {1,2,3,4}?

2. One-to-one and Onto functions

- Examples (page 138)
  - Example 12
    - Is the function f(x)=x2 from the set of integers to the set of integers onto?
    - Solution: No.
  - Example 13
    - Is the function f(x)=x+1 from the set of integers to the set of integers onto?
    - Solution: Yes.

2. One-to-one and Onto functions

- One-to-one correspondence (or bijection, 一一对应, 双射)
  - Definition 9 (page 138)
    - The function f is one-to-one correspondence or a bijection if it is both one-to-one and onto.
  - Example 14 (page 138)
    - Let f be the function from {a,b,c,d} to {1,2,3,4} with f(a)=4, f(b)=2, f(c)=1, and f(d)=3. Is f a bijective?
    - Solution: Yes.

2. One-to-one and Onto functions

- Identity function (恒等函数)
  - Let A be a set. The identity function on A
  - $i_A: A \to A$ where $i_A(x)=x$
  - where $x \in A$.

4. Inverse functions and compositions of functions

❑ Introduction

- Consider a one-to-one correspondence f from the set A to the set B.
- Since f is an onto function, every element of B is the image of some element in A.
- Because f is also one-to-one, every element of B is the image of a unique element of A.
- Therefore, we can define a new function from B to A that reverse the correspondence given by f.

4. Inverse functions and compositions of functions

- Definition 9 (inverse function, page 140)
  - Let f be a one-to-one correspondence from the set A to the set B. The inverse function of f is the function that assigns to an element b belonging to B the unique element a such that f(a)=b.
  - The inverse function of f is denoted by f^{-1}.
  - Hence $f^{-1}(b)=a$ when $f(a)=b$
- Please note:
  - if f is not a one-to-one correspondence, we cannot define an inverse function of f.
  - (why??????????, please see page 140.)

4. Inverse functions and compositions of functions

- Examples (page 140)
  - Example 16
    - Let f be a function from {a,b,c} to {1,2,3} such that f(a)=2, f(b)=3, and f(c)=1. Is f invertible, and if it is, what is its inverse?
      - Solution: Yes.
    - Let f be the function from the set of integers to the set of integers such that f(x)=x+1. Is f invertible, and if it is, what is its inverse?
      - Solution: Yes.
      - $f^{-1}(y)=y^{-1}$

4. Inverse functions and compositions of functions

- Examples (page 140)
  - Example 18
    - Let f be the function from R to R with $f(x)=x^2$. Is f invertible?
    - Solution: Since f(1)=f(-1)=1, f is not one to one. Hence, f is not invertible.
    - 如果 f 限定如下：f is the function from the set of all nonnegative real numbers to the set of all nonnegative real numbers,那么：$f^{-1}(y)=sqrt(y)$

4. Inverse functions and compositions of functions

- Definition 10 (composition of two functions, 函数复合)
  - Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The composition of the function f and g, denoted by $f \circ g$, is defined by

$$(f \circ g)(a) = f(g(a))$$

- Please note:
  - For the direct explanation, please see page 141 (Figure 7).

4. Inverse functions and compositions of functions

- Examples
  - Let g be the function from the set {a,b,c} to itself such that g(a)=b, g(b)=c, and g(c)=a.
  - Let f be the function from the set {a,b,c} to the set {1,2,3} such that f(a)=3, f(b)=2, and f(c)=1.
    - What is the composition of f and g, and what is the composition of g and f?
    - Solution: See balckboard.

4. Inverse functions and compositions of functions

- Examples
  - Let f and g be the function from the set of integers to the set of integers defined by

    $f(x)=2x+3$ and $g(x)=3x+2$.

    - What is the composition of f and g?
    - What is the composition of g and f?
    - Solution: See blackboard.

4. Inverse functions and compositions of functions

- Remark:
  - Note that even if $f \circ g$ and $g \circ f$ are defined for functions f and g in Example 20, $f \circ g$ and $g \circ f$ are not equal.
  - In other words, the commutative law does not hold for the composition of functions

4. Inverse functions and compositions of functions

- The composition of a function and its inverse function (page 11)41
  - If f:A→B and f is a one-to-one correspoendence,

    then (i) f-1 ∘f = i_A and f ∘f -1 = i_B

    (ii) (f-1) -1 = f
  - Why?
  - Please explain (page 104).

5. The graph of functions

- Definition 11 (page 142)
  - Le f be the function from the set A to the set B. The graph of the function f is the set of ordered pairs

    $\{ (a,b) \mid a \in A \text{ and } f(a)=b \}$.
- Examples (page 142)
  - Example 22
  - Display the graph of the function $f(n)=2n+1$ from the set of integers to the set of integers.
  - Solution: Please see book.

5. The graph of functions

- Example 23
  - Display the graph of the function f(x)=x2 from the set of integers to the set of integers.
  - Solution: Please see book.

6. Some important functions

- Definition12 (floor function and ceiling function)
  - The floor function assigns to the real numbers x the largest integer that is less than or equal to x. The value of the floor function at x is denoted by $\lfloor \mathbf{x} \rfloor$.
  - The ceiling function assigns to the real number x the smallest integer that is greater than or equal to x. The value of the ceiling function at x is denoted by $\lceil \mathbf{x} \rceil$.

6. Some important functions

- Example 24 (page 143)
  - These are some values of the floor and ceiling functions.

$\lfloor 0.5 \rfloor$ =? $\lceil 0.5 \rceil$ =? $\lfloor -0.5 \rfloor$ =? $\lceil -0.5 \rceil$ =?

$\lfloor 3.1 \rfloor$ =? $\lceil 3.1 \rceil$ =? $\lfloor 7 \rfloor$ =? $\lceil 7 \rceil$ =?

  - Useful properties of the floor and ceiling function (page 143)

1. (1a) $\lfloor x \rfloor = n$ if and only if $n \leqslant x < n+1$
2. (1b) $\lceil x \rceil = n$ if and only if $n-1 < x \leqslant n+1$
3. (1c) $\lfloor x \rfloor = n$ if and only if $x-1 < n \leqslant x$
4. (1d) $\lceil x \rceil = n$ if and only if $x \leqslant n < x+1$
5. and ....

6. Some important functions

- Example 27 (page 145)
  - Prove that if x is a real number, then

    $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x+0.5 \rfloor$
  - Solution: See book.
- Example 28 (page 145)
  - Prove or disprove that $\lceil x+y \rceil = \lceil x \rceil + \lceil y \rceil$ for all real numbers x and y.
  - Solution: This statement is false. Counterexample: $x=0.5$ and $y=0.5$

6. Some important functions

- The factorial function
  - f: $N \rightarrow Z^+$
  - $f(n) = n!$
    - i.e., $f(n) = 1 \times 2 \times \ldots \times (n-1) \times n$ (and $f(0) = 0! = 1$)

Homework

- Page 146~149
  - 10, 12, 16, 18, 20, 26, 30, 31, 32, 38, 40, 66