Chapter 1
The Foundation: Logic and Proof

1.3 Predicates and Quantifiers

1. Introduction

- Question
  - The meaning of some statements cannot be expressed in propositional logic. How to extend?
    - Example:

1. "Every computer connected to the university network us functioning properly"
2. No rules of propositional logic allows us to conclude the truth of the statement
3. "MATH3 is functioning properly."

2. Predicates

- Consider the statement "x is greater than 3".
  - x---------subject of the statement
  - "is greater than"------predicate
- P(x) ------ "x is greater than 3"
  - P-----------"greater than 3"(predicate)
  - P(x) is also said to be the value of propositional function P at x.

2. Predicates

- P(x)--------"x is greater than 3"
  - P(4)---------true
  - P(2)--------false
- Example 3 (page 31)
  - Q(x,y)--------"x=y+3"
  - Q(1,2)--------false
  - Q(3,0)--------true
- In general,
  - A statement of the form P(x1,x2,…,xn) is the value of the propositional function P at the n-tuple (x1,x2,…,xn) , and P is called a predicate.

2. Universal Quantifier (全程量词)

- Universe of Discourse (or domain of discourse, or domain, 个体域)
  - a set containing all the values of a variable
- Definition 1 (page 34)
  - The universal quantification of P(x) is the proposition "P(x) is true for all values of x in the domain."
    - $\forall x\ P(x)$,
    - $\forall x\ P(x)$ is read as "for all x P(x)" or "for every x P(x)"

2. Universal Quantifier (全程量词)

- Example 8 (page 34)
  - P(x)-----”x+1>x”
    - domain------all real numbers
    - How about $\forall x\, P(x)$?
    - Answer: $\forall x\, P(x)$ ------true
- Example 9 (page 35)
  - Q(x)--------”x<2”
    - domain------all real numbers
    - How about $\forall x\, Q(x)$?
    - Answer: $\forall x\, Q(x)$ ------false

2. Universal Quantifier (全程量词)

- Further explanation
  - domain-----finite set {x1,x2,...,xn}
  - $\forall$x P(x) is the same as P(x1) $\wedge$ P(x2) $\wedge$ ...... $\wedge$ P(xn)
- Example 11 (page 35)
  - P(x)------” x2<10 ”
    - Domain---”the positive integers not exceeding 4” {1,2,3,4}
    - How about $\forall$x P(x) ?
    - Solution:
    1. $\forall$x P(x) is the same as P(1) $\wedge$ P(2) $\wedge$ P(3) $\wedge$ P(4)
    2. false

2. Universal Quantifier (全程量词)

- Example 13 (page 36)
  - $\forall x\ (x2 \geqslant x)$
    - domain ---- all integers
    - Solution:
    1. $x2 \geqslant x$ iff $x(x-1) \geqslant 0$ iff $x \geqslant 1$ or $x \leqslant 0$
    2. $\forall x\ (x2 \geqslant x)$--------true
    - How about domain ---- all real numbers
    1. $\forall x\ (x2 \geqslant x)$ -------- false

2. Universal Quantifier (全程量词)

- How to show $\forall x\, P(x)$ is false?
  - Try to find a counterexample (反例).
  - Example (补充):
    - P(x) ------- "x2>0",domain ---- all integers, How about $\forall x\, P(x)$?
      - Solution:
        - P(0) -------- false
        - $\forall x\, P(x)$ -------- false

3. Existential Quantifier (存在量词)

- Definition 2 (existential quantifier, page 36)
  - The existential Quantifier of P(x) is the proposition “There exists an element x in the domain such that P(x) is true”

    ---------denoted as $\exists x\ P(x)$

    - “There is an x such that P(x)”
    - “There is at least one x such that P(x)” or “For some x P(x)”

3. Existential Quantifier (存在量词)

- Example 14 (page 36)
  - P(x)---------”x>3”
  - domain-----all real numbers
  - Consider $\exists x\, P(x)$
    - Solution:

1. $\exists x\, P(x)$ is true
2. Why??? (x can be 3.5, 4, …., which makes is P(x) is true)

3. Existential Quantifier (存在量词)

- Example 15 (page 36)
  - Q(x)-------”x=x+1”,domain---------all real numbers
  - Consider ∃xQ(x)
  - Solution:
    - For every real number x, Q(x) is false.
    - Therefore, ∃x Q(x) is false
- If the domain is a finite set, i.e., {x1, x2, ……, xn },then ∃x P(x) is the same as P(x1) ∨ P(x2) ∨…… ∨ P(xn)

3. Existential Quantifier (存在量词)

- Example 16 (page 37)
  - P(x)--------"x2>10", domain--------"all the positive integers not exceeding 4"
  - Consider $\exists x\ P(x)$
  - Solution:
    - domain={1, 2, 3, 4}
    - $\exists x\ P(x)$ is the same as
    - $P(1) \vee P(2) \vee P(3) \vee P(4)$
    - $\exists x\ P(x)$ is true because P(4) is true.

3. Existential Quantifier (存在量词)

- Summary (see Table 1 on page 34)

Statement	When True?	When False
$\forall x\ P(x)$	P(x) is true for all x	There is an x for which P(x) is false
$\exists x\ P(x)$	There is an x for which P(x) is true	P(x) is false for every x

4.Quantifiers with Restricted Domain

- Example 17 (page 38)
  - What do the statements below mean, where the domain in each case consists of the real numbers?
  - Solution: For the meanings, see page 38.
    - $\forall x<0\ (x2>0)$ --------- $\forall x\ (x<0 \rightarrow x2>0)$
    - $\forall y\neq 0\ (y3\neq 0)$ --------- $\forall y\ (y\neq 0 \rightarrow y3\neq 0)$
    - $\exists z>0\ (z2=2)$ --------- $\exists z\ (z>0\ /\backslash\ z2=2)$

5. Binding Variables (变量约束)

- Bound Variable and Free Variable (约束变量和自由变量)
  - Example 18 (page 38)
    - $\exists x\ (x+y=1)$
      1. x------bound variable
      2. y------free variable
    - $\underline{\exists x\ (\ P(x) \vee Q(x)\ )} \vee \underline{\forall x\ R(x)}$ the scope of bound variable

6. Logical Equivalence Involving Quantifiers

- Definition 3 (Page 39)
  - Statements involving predicates and quantifiers are logically equivalent if and only if they have the same true value no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions.
  - We use the notation $S \equiv T$ to indicate the two statements S and T involving predicates and quantifiers are logically equivalent.

6. Logical Equivalence Involving Quantifiers

- Example 19 (page 39)
  - Show that $\forall x\ (P(x) \land Q(x))$ and $\forall x\ P(x) \land \forall x\ Q(x)$ are logically the same?
  - Solution:
    - 我们要证明如下两点:
    1. 如果$\forall x\ (P(x) \land Q(x))$为真, 那么$\forall x\ P(x) \land \forall x\ Q(x)$为真
    2. 如果$\forall x\ P(x) \land \forall x\ Q(x)$为真, 那么$\forall x\ (P(x) \land Q(x))$为真,具体如下：
       - 首先证明第一点，假设$\forall x\ (P(x) \land Q(x))$为真，那么如果a是个体域中的一个元素，那么$P(a) \land Q(a)$为真，即$P(a)$为真且$Q(a)$为真。由于对个体域中的任何元素a， $P(a)$为真且$Q(a)$为真，所以我们可以得出$\forall x\ P(x)$和$\forall x\ Q(x)$都为真，即$\forall x\ P(x)$为$\forall x\ Q(x)$

6. Logical Equivalence Involving Quantifiers

- solution
  - 接下来证明第二点，假设$\forall x\, P(x) \land \forall x\, Q(x)$为真，那么$\forall x\, P(x)$为真且 $\forall x\, Q(x)$为真。所以若a是个体域中的一个元素，那么P(a)为真，且Q(a)为真。从而说明了对所有a, $P(a) \land Q(a)$为真，即$\forall x\, (P(x) \land Q(x))$为真。
  - 由上述可知：$\forall x\, (P(x) \land Q(x)) \equiv \forall x\, P(x) \land \forall x\, Q(x)$

5. Negating Quantified Expressions

- "Every student in the class has taken a course in calculus"
  - $\forall x\ P(x)$
  - Here, P(x)--------"x has taken a course in calculus"
- The negation is
  - "It is not the case that Every student in the class has taken a course in calculus"
  - or "There is a student in the class who has not taken a course in calculus"
    - $\exists x\ \neg P(x)$
    - $\neg\ \forall x\ P(x) \equiv \exists x\ \neg P(x)$

5. Negating Quantified Expressions

- "There is a student in the class who has taken a course in calculus"
  - $\exists x\ Q(x)$
  - Here, Q(x)--------"x has taken a course in calculus"
- The negation is
  - "It is not the case that there is a student in the class who has taken a course in calculus"
  - or "Every student in the class has not taken a course in calculus"
    - $\forall\ x\ \neg Q(x)$
    - $\neg\ \exists x\ Q(x) \equiv \forall\ x\ \neg Q(x)$

5. Negating Quantified Expressions

- Example 21 (page 41)
  - What is the negations of the statements $\forall x\ (x2>x)$ and $\exists x\ (x2=2)$
  - Solution:
    - $\neg\forall x\ (x2>x) \equiv \exists x\ \neg(x2>x)$

      $\equiv \exists x\ (x2 \leqslant x)$ ....(1)
    - $\neg\exists x\ (x2=2) \equiv \forall x \neg(x2=2)$

      $\equiv \forall x\ (x2 \neq 2)$ ....(2)
  - The truth values of these statements depends on the universe of discourse.

1. For (1), use [0.5, 3] and [2, 5] to check
2. For (2), use [0,1] and [1,2] to check

5. Negating Quantified Expressions

- Example 22 (page 41)
  - Show that $\neg\forall x\,(P(x) \rightarrow Q(x))$ and $\exists x\,(P(x) \wedge \neg Q(x))$ are logically equivalent.
  - Proof:
    - 由量词否定等值式，可得$\neg\forall x\,(P(x) \rightarrow Q(x))$与 $\exists x\,\neg(P(x) \rightarrow Q(x))$ 逻辑等价，
    - 由书上§1.2的表格7中的第五个等值式可知， 对任何x，$\neg(P(x) \rightarrow Q(x))$ 与$P(x) \wedge \neg Q(x)$逻辑等价，由于在逻辑等价中，我们可以用一个逻辑表达式来替换另一个与之等价的表达式，
    - 由此可得：$\neg\forall x\,(P(x) \rightarrow Q(x))$与$\exists x\,\neg(P(x) \wedge \neg Q(x))$ 逻辑等价

6. Translating from English into logical Expression

- Example 23 (page 42)
  - Express the statement <u>“Every student in this calss has studied calculus”</u> in predicates and quantifiers.
  - Solution 1:
    - C(x)-----”x has studied calculus”,
    - domian: -----all the students in the class
    - $\forall$ x C(x)
  - solution 2:
    - domain-------all people
    - “For every person x, if person x is in this class then x has studied calculus.”

1. S(x)-------person x is in this class
2. C(x)------person x has studied calculus
3. $\forall$ x (S(x)→C(x)) (correct)
4. $\forall$ x (S(x) /\ C(x)) (wrong, why????)

6. Translating from English into logical Expression

- Example 24 (page 42)
  - Express the statements below in predicates and quantifiers.
    - “Some students in this class has visited Mexico” ----------------------(1)
    - “Every student in this class has visited Canada or Mexico” --------------------(2)

6. Translating from English into logical Expression

- Example 24 (page 42)
  - Solution for (1)
    - Way 1:
      - M(x)------x has visited Mexico.
      - domian------all the students in this class
      - $\exists x\ M(x)$
    - Way 2:
      - domain-------all people
      - S(x)------”x is a student in this class”
      - $\exists x\ (\ S(x) \wedge M(x)\ )$--------correct
      - $\exists x\ (\ S(x) \rightarrow M(x)\ )$---------wrong

6. Translating from English into logical Expression

- Example 24 (page 42)
  - Solution for (2)
    - Way 1:
      - C(x)------”x has visited Canada.”
      - M(x)------”x has visited Mexico”
      - domain-------”all the students in this class”
      - $\forall x\ (\ C(x) \lor M(x)\ )$
    - Way 2:
      - domain-------”all people”
      - S(x)-------”x is a student in this class.”
      - C(x)------”x has visited Canada.”
      - M(x)------”x has visited Mexico”
      - $\forall x\ (\ S(x) \rightarrow (C(x) \lor M(x))\ )$

Homework

- Page 46~50
  - 12, 14, 16, 18, 20, 30, 36