# Chapter 1
# The Foundation: Logic and Proof

## 1.2 Propositional Equivalence

# 1.Introduction

- Example 1 (page 21)
  - p $\vee$ ¬p is always true. It is a tautology.
  - p $\wedge$ ¬p is always false. It is a contradiction.
- Definition 1 (see page 21)
  1. Tautology (永真公式)：A compound proposition that is always true, no matter what the truth values of the propositions that occur in it, is called a tautology.
  2. Contradiction (永假公式)：A compound proposition that is always false, no matter what the truth values of the propositions that occur in it, is called a contradiction.

# 1.Introduction(cont.)

- Definition 1 (see page 21)
  3. Contingency (中性公式)：A proposition that is neither a tautology nor a contradiction is called a contingency.

# 2. Logical Equivalence

- Example 2 (page 22)
  - Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.
    - Truth Table

| $p$ | $q$ | $p \vee q$ | $\neg(p \vee q)$ | $\neg p$ | $\neg q$ | $\neg p \wedge \neg q$ |
|---|---|---|---|---|---|---|
| T | T | T | **F** | F | F | **F** |
| T | F | T | **F** | F | T | **F** |
| F | T | T | **F** | T | F | **F** |
| F | F | F | **T** | T | T | **T** |

# 2. Logical Equivalence (cont.)

- Definition 2 (logical equivalence)
  - The propositions p and q are called logically equivalent if $p \leftrightarrow q$ is a **tautology**.
  - The notation $\mathbf{p \equiv q}$ denotes that p and q are logically equivalence.

# 3.More Examples

- Example 3 (page 23)
  - Show that $p \rightarrow q$ and $\neg p \vee q$ are logically equivalence.
    - **Solution:** We construct the truth table for these propositions in the table below. Since the truth values of $p \rightarrow q$ and $\neg p$ \/ $q$ agree, these propositions are logically equivalence.

| p | q | ¬p | ¬p ∨q | p→q |
|---|---|---|---|---|
| T | T | F | **T** | **T** |
| T | F | F | **F** | **F** |
| F | T | T | **T** | **T** |
| F | F | T | **T** | **T** |

# 3.More Examples

- Example 4 (page 23)
  - Show that the propositions $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent.
    - **Solution:** By constructing truth table.

| p | q | r | $q \wedge r$ | $p \vee (q \wedge r)$ | $p \vee q$ | $p \vee r$ | $(p \vee q) \wedge (p \vee r)$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T |
| T | T | F | F | T | T | T | T |
| T | F | T | F | T | T | T | T |
| T | F | F | F | T | T | T | T |
| F | T | T | T | T | T | T | T |
| F | T | F | F | F | T | F | F |
| F | F | T | F | F | F | T | F |
| F | F | F | F | F | F | F | F |

# 4. Some Important Equivalences

- Logical Equivalence ( Table a)
  - Identity laws(同一律)
    1. $p \wedge T \equiv p$
    2. $p \vee F \equiv p$
  - Domination laws(零律)
    1. $p \vee T \equiv T$
    2. $p \wedge F \equiv F$
  - Idempotent laws(幂等律)
    1. $p \vee p \equiv p$
    2. $p \wedge p \equiv p$

# 4. Some Important Equivalences

- Logical Equivalence ( Table a cont.)
  - Double negation law(双重否定律)
    1. $\neg(\neg p) \equiv p$
  - Commutative laws(交换律)
    1. $p \land q \equiv q \land p$
    2. $p \lor q \equiv q \lor p$
  - Associative laws(结合律)
    1. $(p \land q) \land r \equiv p \land (q \land r)$
    2. $(p \lor q) \lor r \equiv p \lor (q \lor r)$

# 4. Some Important Equivalences

- Logical Equivalence ( Table a cont.)
  - Distributive laws(分配律)
    1. $p \vee (q \wedge r) \equiv (p \wedge r) \vee (q \wedge r)$
    2. $p \wedge (q \vee r) \equiv (p \vee r) \wedge (q \vee r)$
  - De Morgan’s laws(德·摩根律)
    1. $\neg(p \wedge q) \equiv \neg p \vee \neg q$
    2. $\neg(p \vee q) \equiv \neg p \wedge \neg q$
  - Absorption laws(吸收律)
    1. $p \vee (p \wedge q) \equiv p$
    2. $p \wedge (p \vee q) \equiv p$

# 4. Some Important Equivalences

- Logical Equivalence ( Table a cont.)
  - Negative laws
    1. $p \wedge \neg p \equiv F$ (矛盾律)
    2. $p \vee \neg p \equiv T$（排中律）

# 4. Some Important Equivalences

- Logical Equivalence Involving Implication ( Table b)
  - 蕴含等值式
    1. $p \to q \equiv \neg p \lor q$
  - 假言易位
    1. $p \to q \equiv \neg q \lor \neg p$
  - Others
    1. $p \lor q \equiv \neg p \to q$
    2. $p \land q \equiv \neg(p \to \neg q)$
    3. $\neg(p \to q) \equiv p \land \neg q$

# 4. Some Important Equivalences (cont)

- Logical Equivalence Involving Implication ( Table b)
  - Others

4. $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$
5. $(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$
6. $(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$
7. $(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

# 4. Some Important Equivalences

- Logical Equivalence Involving Biconditionals ( Table c)
  - $p \leftrightarrow q \equiv (p \rightarrow q) \land (q \rightarrow p)$
  - $p \leftrightarrow q \equiv \neg q \leftrightarrow \neg p$
  - $p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q)$
  - $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$
- Questions:
  - How to verify these equivalences?
  - Answer: One way is by constructing the truth table.

# 5. Extension of De Morgan's Law

- Extension of De Morgan's Law
  - $\neg(p \land q) \equiv \neg p \lor \neg q$ can be extended to

    $\neg(p_1 \land p_2 \land \ldots \land p_n) \equiv \neg p_1 \lor \neg p_2 \lor \ldots \lor \neg p_n$
  - $\neg(p \lor q) \equiv \neg p \land \neg q$ can be extended to

    $\neg(p_1 \lor p_2 \lor \ldots \lor p_n) \equiv \neg p_1 \land \neg p_2 \land \ldots \land \neg p_n$

# 6. Constructing New Logical Equivalence

- Example 6: Show that $\neg(p \to q)$ and $p \wedge \neg q$ are logically equivalent. (page 26)
  - Proof

$\neg(p \to q)$

$\equiv \neg(\neg p \vee q)$ by Example 3

$\equiv \neg(\neg p) \wedge \neg q$ by the second De Morgan law

$\equiv p \wedge \neg q$ by the double negation law

# 6. Constructing New Logical Equivalence

- Example 6: Show that $\neg(p \rightarrow q)$ and $p \wedge \neg q$ are logically equivalent. (page 26)
  - 中文表达

$\neg(p \rightarrow q)$

$\equiv \neg(\neg p \vee q)$ 蕴含等值试

$\equiv \neg(\neg p) \wedge \neg q$ 德摩根律

$\equiv p \wedge \neg q$ 双重否定律

# 6. Constructing New Logical Equivalence

- Example 7: Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.
  - **Proof**

$\neg(p \vee (\neg p \wedge q))$

$\equiv \neg p \wedge \neg(\neg p \wedge q))$ by the second De Morgan law

$\equiv \neg p \wedge (\neg(\neg p) \vee \neg q)$ by the first De Morgan law

$\equiv \neg p \wedge (p \vee \neg q)$ by the double negation law

$\equiv (\neg p \wedge p) \wedge (\neg p \wedge \neg q)$ by the second distributed law

$\equiv F \vee (\neg p \wedge \neg q)$ because $\neg p \wedge p \equiv F$

$\equiv (\neg p \wedge \neg q) \wedge F$ by the communicative law

$\equiv \neg p \wedge \neg q$ by the identify law for F

# 6. Constructing New Logical Equivalence

- Example 7: Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.
  - 中文表达

| | |
|---|---|
| $\neg(p \vee (\neg p \wedge q))$ | |
| $\equiv \neg p \wedge \neg(\neg p \wedge q))$ | 德摩根律 |
| $\equiv \neg p \wedge (\neg(\neg p) \vee \neg q)$ | 德摩根律 |
| $\equiv \neg p \wedge (p \vee \neg q)$ | 双重否定律 |
| $\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$ | 分配律 |
| $\equiv F \vee (\neg p \wedge \neg q)$ | 矛盾律 |
| $\equiv (\neg p \wedge \neg q) \vee F$ | 交换律 |
| $\equiv \neg p \wedge \neg q$ | 同一律 |

# 6. Constructing New Logical Equivalence

- Example 8: Show that $(p \land q) \to (p \lor q)$ is a tautology.
  - **Solution:**

$(p \land q) \to (p \lor q)$

| | |
|---|---|
| $\equiv \neg(p \land q) \lor (p \lor q)$ | by Example 3 |
| $\equiv (\neg p \lor \neg q) \lor (p \lor q)$ | by the first De Morgan law |
| $\equiv (\neg p \lor p) \lor (\neg q \lor q)$ | by the associative and communicative law for disjunction |
| $\equiv T \lor T$ | by example 1 and the communicative law for disjunction |
| $\equiv T$ | by domination law |

# 6. Constructing New Logical Equivalence

- Example 8: Show that $(p \land q) \to (p \lor q)$ is a tautology.
  - 中文表达：

$(p \land q) \to (p \lor q)$

$\equiv \neg(p \land q) \lor (p \lor q)$　　蕴含等值式

$\equiv (\neg p \lor \neg q) \lor (p \lor q)$　　德摩根律

$\equiv (\neg p \lor p) \lor (\neg q \lor q)$　　交换律、结合律

$\equiv T \lor T$　　排中律

$\equiv T$　　零律

# Homework

- Page 28~30
  - 4, 10, 14, 28, 30, 60