FOURTH ORDER EQUATIONS IN CONFORMAL GEOMETRY

by

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Abstract. — In this article we review some recent work on fourth order equations in conformal geometry of three and four dimensions. We discuss some an existence result for a Yamabe-type equation in dimension three. We examine a generalization of the Cohn-Vossen inequality to dimension four. Finally, we review an application of the fourth order equation to a fully nonlinear equation in dimension four that involves the Ricci tensor.

Résumé **(Équations d'ordre quatre en Géométrie Conforme).** —

1. Introduction

In this article we discuss some new developments in the fourth order equations in conformal geometry of three and four dimensions. We refer the reader to [**CY2**] for a survey of some earlier work in this area.

On a Riemannian manifold (M^n, g) of dimension n, the Laplace Beltrami operator is the natural geometric operator. Under conformal change of metric $g_w = e^{2w} g$, when the dimension is two, Δ_{g_w} is related to Δ_g by the simple formula:

$$\Delta_{g_w}(\varphi) = e^{-2\omega} \Delta_g(\varphi) \quad \text{for all} \quad \varphi \in C^\infty(M^2) \tag{1}$$

In dimension greater than two, similar transformation property continues to hold for a modification of the Laplacian operator called the conformal Laplacian operator $L \equiv -\frac{4(n-1)}{n-2}\Delta + R$ where R is the scalar curvature of the metric. We have

$$L_{g_w}(\varphi) = e^{-\frac{n+2}{2}\omega} L_g \left(e^{\frac{n-2}{2}\omega} \varphi \right) \tag{2}$$

for all $\varphi \in C^\infty(M)$.

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In general, we call a metrically defined operator A conformally covariant of bidegree (a, b), if under the conformal change of metric $g_\omega = e^{2\omega} g$, the pair of corresponding operators A_ω and A are related by

$$A_\omega(\varphi) = e^{-b\omega} A(e^{a\omega}\varphi) \quad \text{for all} \quad \varphi \in C^\infty(M^n) \tag{3}$$

A particularly interesting such operator is a fourth order operator on 4-manifolds discovered by Paneitz [**Pa**] in 1983:

$$P\varphi \equiv \Delta^2\varphi + \delta\left(\frac{2}{3}RI - 2\mathrm{Ric}\right)d\varphi \tag{4}$$

where δ denotes the divergence, d the deRham differential and Ric the Ricci tensor of the metric. The *Paneitz* operator P is conformal covariant of bidegree $(0, 4)$ on 4-manifolds, i.e.

$$P_{g_w}(\varphi) = e^{-4w} P_g(\varphi) \quad \text{for all} \quad \varphi \in C^\infty(M^4) \tag{5}$$

A fourth order curvature invariant $Q = \frac{1}{12}\{-\Delta R + R^2 - 3|Rc|^2\}$ is associated to the Paneitz operator:

$$Pw + 2Q = 2Q_w e^{4w}$$

In dimension four, the Paneitz equation has close connection with the Chern-Gauss-Bonnet formula. For a compact oriented 4-manifold,

$$\chi(M) = \frac{1}{4\pi^2}\int_M (\frac{|W|^2}{8} + Q)dV \tag{6}$$

where $\chi(M)$ denotes the Euler characteristic of the manifold M, and $|W|^2$= norm squared of the Weyl tensor. Since $|W|^2 dV$ is a pointwise invariant under conformal change of metric, QdV is the term which measures the conformal change in formula (6).

For a 4-manifold with boundary, [**CQ**] defines a third order boundary operator P_3 which is conformally covariant of bidegree (1,3):

$$P_3 = -\frac{1}{2}\frac{\partial}{\partial n}\Delta - \tilde{\Delta}\frac{\partial}{\partial n} - \frac{2}{3}H\tilde{\Delta} + L_{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta + (\frac{1}{3}R - R_{\alpha N\alpha N})\frac{\partial}{\partial n} + \frac{1}{3}\tilde{\nabla}H\cdot\tilde{\nabla} \tag{7}$$

where ∂n is the unit interior normal, $\tilde{\Delta}$ is the boundary Laplacian, H is the mean curvature, $L_{\alpha\beta}$ the second fundamental form, and $\tilde{\nabla}$ the boundary gradient. The boundary P_3 operator defines the third order curvature invariant T through the equation:

$$-P_3 w + T_w e^{3w} = T \quad \text{on} \quad \partial M \tag{8}$$

where

$$T = \frac{1}{12}\frac{\partial}{\partial n}R + \frac{1}{6}RH - R_{\alpha N\beta N}L_{\alpha\beta} + \frac{1}{9}H^3 - \frac{1}{3}TrL^3 - \frac{1}{3}\tilde{\Delta}H \tag{9}$$

For 4-manifolds with boundary, the Chern-Gauss-Bonnet formula is supplemented by

$$\chi(M) = \frac{1}{4\pi^2}\int_M (\frac{|W|^2}{8} + Q)dV + \frac{1}{4\pi^2}\int_{\partial M}(L + T)d\Sigma \tag{10}$$

where $Ld\sigma$ is a pointwise conformal invariant of the boundary.

In order to find geometric interpretation for the fourth order invariant Q, we formulated an analogue ([**CQY1**]) of the Cohn-Vossen inequality for complete surfaces with finite total curvature and derived ([**CQY2**]) a compactification criteria for conformally flat 4-manifold using the curvature invariant Q and the assumption of geometric finiteness.

In general dimensions different from four there is also a natural fourth order operator P, which enjoys the conformal covariance property with respect to conformal changes in metrics. The relation of this operator to the Paneitz operator in dimension four is completely analogous to the relation of the conformal Laplacian to the Laplacian in dimension two. On (M^n, g) when $n \neq 4$, define

$$P = (-\Delta)^2 + \delta(a_n R + b_n Ric)d + \frac{n-4}{2}Q$$

where

$$Q = c_n |Ric|^2 + d_n R^2 - \frac{1}{2(n-1)}\Delta R$$

and $a_n = \frac{(n-2)^2+4}{2(n-1)(n-2)}$, $b_n = -\frac{4}{n-2}$, $c_n = -\frac{2}{(n-2)^2}$, $d_n = \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2}$ are dimensional constants. Then (Branson [**Br**]), writing $g_u = u^{\frac{4}{n-4}}g$, $n \neq 4$ we have

$$(P)_u(\varphi) = u^{-\frac{n+4}{n-4}}P(u\varphi) \tag{11}$$

for all $\varphi \in C^\infty(M^n)$. We also have the analogue for the Yamabe equation:

$$Pu = \frac{n-4}{2}Qu^{\frac{n+4}{n-4}} \qquad \text{on} \quad M^n, \quad n \neq 4 \tag{12}$$

Such semilinear biharmonic equations with critical exponents were first investigated by Pucci-Serrin in [**PuS**], they obtained the analogue of the Brezis-Nirenberg result ([**BN**]) in dimensions $n = 5, 6, 7$ for domains in R^n. There are no global existence results in these dimensions to our knowledge.

It is interesting to note that in dimension three, the equation takes a special form

$$Pu = -\frac{1}{2}Qu^{-7} \tag{13}$$

for the conformal factor $g = u^{-4}g_0$. It is natural to ask whether one can solve the analogue of the Yamabe equation for this operator. In [**XY**] we were able to formulate a criteria for positivity of the operator P in dimension three and obtained some existence result for the equation of prescribing constant Q. The study of this equation is still in a primitive stage, there is much that remains to be developed.

In dimension four, the theory of the fourth order equation can be applied to the study of fully nonlinear equations involving the symmetric functions of the modified Ricci tensor. This set of equations is studied by Viaclovsky [**V**] in his thesis. In dimension four, we can use the fourth order equation as a regularization of the second order equation of prescribing the second elementary symmetric functions $\sigma_2(A)$ where A is the conformal Ricci tensor $A = Rc - \frac{1}{6}Rg$. As a consequence, we were able to give a simple criteria for existence, in a given four dimensional conformal class, of a metric with strongly positive Ricci tensor. The conformal classes in four dimension that satisfy the conformally invariant conditions $\int \sigma_2(A)dV > 0$ and having positive Yamabe invariant, include the 4-sphere, connected sums of up to three copies of $\mathbb{CP}^2$, connected sums of $\mathbb{CP}^2$ with up to eight copies of $\mathbb{CP}^2$ with reversed orientation, and connected sums of up to two copies of $S^2 \times S^2$.

We give an outline of the rest of the paper. In section two we study the fourth order equation on 3-manifolds. We discuss the uniqueness question for the equation (12) in Euclidean 3-space. We formulate a criteria for existence result for prescribing constant Q for a class of 3-manifolds. In section three, we consider the fourth order equation on conformally flat 4-manifolds and report on the compactification criteria of [**CQY2**]. Finally in section four we discuss the fully nonlinear equations for prescribing the elementary symmetric functions of the conformal Ricci tensor on a 4-manifold.

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2. The fourth order operator in dimension three

For the P operator in dimension three we have

$$P = (-\Delta)^2 + \delta(\frac{5}{4}Rg - 4Rc)d - \frac{1}{2}Q \tag{14}$$

where

$$Q = -2|Rc|^2 + \frac{23}{32}R^2 - \frac{1}{4}\Delta R \tag{15}$$

The Q curvature equation is given by

$$Pu = -\frac{1}{2}Qu^{-7} \tag{16}$$

The analogue of the Yamabe problem in this setting would be to solve equation (16) with Q given by a constant. This is naturally the Euler equation of the variational functional

$$F[u] = (\int_M u^{-6}dV)^{1/3} \int_M Pu \cdot udV \tag{17}$$

The direct method would be to minimize the functional over the class of positive functions in the Sobolev space $W^{2,2}$. The negative exponent in the integral means that the analytic difficulty is associated with the conformal factor touching zero. The negative sign of the coefficient for the Q curvature term in equation (16) makes a sharp contrast with the Yamabe equation. For example, among the eight standard geometries, only in the case of the sphere and hyperbolic 3-manifolds the Q curvature is positive. There is some preliminary result in this direction.

In studying a nonlinear equation involving a critical exponent, it will be important to have an understanding of the blowup solutions. Thus one is interested in global positive solutions in Euclidean 3-space of the equation

$$\Delta^2 u = -\frac{15}{16} u^{-7} \tag{18}$$

Assuming the solution actually came from a positive solution of the corresponding equation on S^3 via the stereographic projection, it would have the natural asymptotic behavior: $\frac{u(x)}{|x|}$ tends to a positive constant as $|x|$ tends to infinity. Adapting the method of moving planes, Choi and Xu ([**CX**]) has classified such entire solutions: after translations and dilations u is of the form $u(x) = 2^{-\frac{1}{2}}(1+|x|^2)^{\frac{1}{2}}$. In the same article, they also showed that the same assertion holds if, instead of the asymptotic condition at infinity, the scalar curvature of the metric is assumed to be non-negative at infinity.

The question of existence turns out to be simplest when the operator P is positive and the manifold (M^3, g_0) is in the positive Yamabe class. We have

Theorem 2.1. — [**XY**] *If (M^3, g_0) has positive scalar curvature and the operator P is positive, then the functional F achieves a positive minimum at a positive smooth function u.*

Remark 2.1. — 1. The positivity of the operator P does not follow from the positivity of the scalar curvature. In fact on the standard 3-sphere the operator P has a negative eigenvalue due to the fact $Q_0 = \frac{15}{8}$. A simple criteria for positivity of the operator P on (M^3, g) is that there is a conformal metric in which $Q < 0$ and $R > 0$. The class of conformal structures satisfying the these conditions includes the standard product structures on $S^1 \times S^2$ and their connected sums. In view of Yau's conjecture [**SY**], it is quite likely that the only possible topology supporting conformal structures with these positivity conditions are those listed.

2. In a recent article, Djadli-Hebey-Ledoux [**DHL**] studied the best constants in a Sobolev inequality related to the Paneitz equation in dimensions $n \geq 5$.

3. An extension of the Cohn-Vossen inequality

We recall the Cohn-Vossen ([**CV**]) inequality for complete surfaces. Suppose (M, g) is a complete surface with Gauss curvature K in L^1, then

$$\int_M K dA \leq 2\pi\chi \tag{19}$$

In fact, Huber ([**Hu**]) has shown that such a surface has a conformal compactification $M = \tilde{M} \backslash \{P_1, \ldots, P_n\}$ where $\tilde{M}$ is a compact Riemann surface. At each puncture P_i by inverting a conformal disc $D_i \backslash \{P_i\}$, Finn ([**Fn**]) has considered the isoperimetric ratio $\nu_i = \lim_{r\to\infty} \frac{(Length(\partial D_r))^2}{4\pi Area(D_r)}$,and accounted for the deficit in the inequality above:

$$\chi(M) - \frac{1}{\pi}\int_M K dA = \sum_{i=1}^{n} \nu_i \tag{20}$$

A completely analogous situation holds in dimension four provided we restrict ourselves to conformally flat 4-manifolds of positive scalar curvature. Let us first recall that Schoen-Yau ([**SY**]) has demonstrated that for such manifolds, the holonomy cover of such manifolds embed conformally as domain $\tilde{M}$ in S^4 with a boundary which has Hausdorff dimension less than one. Thus by going to a covering of such manifolds we may assume that we are dealing with domains in $\mathbb{R}^4$.

Theorem 3.1. — [**CQY1**] *Let $e^{2w}|dx|^2$ be a complete metric on $\Omega = \mathbb{R}^4 \setminus \{P_1, \dots, P_n\}$ with nonnegative scalar curvature near the punctures. Suppose in addition that Q is integrable. Then we have*

$$\chi(\Omega) - \frac{1}{4\pi^2}\int_\Omega Q dV = \sum_{i=1}^{n} \nu_i \tag{21}$$

where at each puncture P_i a conformal disk $D_i \setminus \{P_i\}$ is inverted and

$$\nu_i = \lim_{r\to\infty} \frac{(vol(\partial B_r))^{4/3}}{4(2\pi^2)^{1/3} vol(Br)} \tag{22}$$

To give some idea of the proof of Theorem 3.1, we explain the situation on $\mathbb{R}^4$. The proof is based on an idea of Finn, to compare the conformal factor with the biharmonic potential derived from the measure QdV. The positivity of the scalar curvature at infinity implies that the conformal factor agrees with the potential up to a constant. Working then with the expression of the potential as a logarithmic integral, a delicate analysis shows that the isoperimetric ratio ν can be compared with that of the symmetrized potential. In the latter case the required identity follows from an analysis of the resultant ODE.

The finiteness of the Q integral together with the embedding result of Schoen-Yau has strong implication for the underlying topology:

Theorem 3.2. — [**CQY2**] *Let (M^4, g) be a simply connected complete conformally flat manifold satisfying scalar curvature $R \geq c > 0$, $Ric \geq -c$, and $\int |Q| dv < \infty$; then M is conformally equivalent to $\mathbb{R}^4 \setminus \{P_1, \dots, P_k\}$. In case M^4 is not assumed simply connected, under the additional assumption that M^4 is geometrically finite as a Kleinian manifold, then M is conformally equivalent to $\tilde{M} \setminus \{P_1, \dots, P_k\}$, where $\tilde{M}$ is a compact conformally flat manifold. In addition, we have*

$$\chi(M) = \frac{1}{4\pi^2}\int_M Q dV + k$$

Remark 3.1. — 1. As a consequence of this finiteness criteria, we can classify the complete conformal metrics defined on domains in S^4, which satisfy the curvature conditions in the statement of Theorem 3.2, and in addition has constant Q curvature which are integrable. There are only three such metrics: the standard metric on S^4, the flat metric on $\mathbb{R}^4$ and the cylindrical metric on $\mathbb{R}^4 \setminus \{0\}$.

2. The notion of geometric finiteness is a natural one that allows good control of the ends of the associated hyperbolic manifold. The question which Kleinian groups are geometrically finite has been intensively studied in dimension two. For example, Bishop-Jones [**BJ**] has shown that in dimension two, a finitely generated Kleinian group is geometrically finite if and only if the limit set has Hausdorff dimension strictly less than two. In a preliminary study of the situation in higher dimensions,

we ([**CQY3**]) were able to show that if the Kleinian manifold is compact, has positive Yamabe invariant, then the group is geometrically finite.

We will now indicate some ideas used in the proof of Theorem 3.2 in the case when M^4 is simply connected. Suppose Ω is a domain in $\mathbb{R}^4$ on which there is a conformal metric $g = u^2|dx|^2 = e^{2w}|dx|^2$ satisfying the assumptions of Theorem 3.2. One of the key ingredients in the proof of Theorem 3.2 is to establish the following size estimate of the conformal factor $u(x)$ for $x \in \Omega$ in terms of the Euclidean distance $d(x) = distance(x, \partial\Omega)$.

Lemma 3.3. — *Suppose $M = (\Omega, u^2|dx|^2)$ is a complete manifold which satisfies the curvature assumptions as in Theorem 3.2. Then there exists some constant C so that*

$$\frac{1}{C} d(x)^{-1} \leq u(x) \leq C d(x)^{-1} \quad \textit{for all } x \in \Omega \tag{23}$$

We remark that the left hand side of 23 follows from some estimate of Schoen-Yau ([**SY**], Theorem 2.12, Chapter VI). The estimate of the right hand side of (23) is derived via a blow up argument for the Paneitz equation, together with the following uniqueness result.

Lemma 3.4. — *On $(\mathbb{R}^4, u^2|dx|^2)$, the only metric with $Q \equiv 0$ and $R \geq 0$ at infinity is isometric to $(\mathbb{R}^4, |dx|^2)$.*

We now consider the sets

$$U_\lambda = \{x : u(x) \leq \lambda\} \text{ and } S_\lambda = \{x : u(x) = \lambda\},$$

for large values of λ. Apply the Chern-Gauss-Bonnet formula (10) for the domain U_λ, we obtain

$$C \geq \lambda \frac{d}{d\lambda} V(\lambda) \tag{24}$$

where

$$V(\lambda) = \int_{S_\lambda} (\partial_n w)^3 d\sigma + \int_{S_\lambda} J(\partial_n w) e^{2w} d\sigma + 2 \int_{U_\lambda} J|\nabla u|^2 dx$$

The positivity of the scalar curvature then implies that

$$V(\lambda) \geq C \int_{U_\lambda} u^4 dxu \tag{25}$$

Then the estimate (23) in Lemma 3.3 together with (24) and (25) allow us to use a covering argument to show that Λ consists of a finite number of points.

4. Construction of Strongly Positive Ricci Curvature Metrics

In the thesis of J. Viaclovsky ([**V**]), a family of fully nonlinear differential equations are introduced as generalizations of the Yamabe equation that pertain to the conformal structure of a Riemannian manifold. Consider the conformal Ricci tensor: $A = Rc - \frac{1}{2(n-1)}Rg$. The k-th elementary symmetric function of the eigenvalues of the matrix A is denoted by $\sigma_k(A)$. They constitute natural invariants of the Ricci tensor. In particular σ_1 is a multiple of the scalar curvature. In even dimensions $n = 2k$ the integral $\int \sigma_k dV$ is in fact a conformal invariant of the manifold. In particular, in dimension four,

$$\sigma_2 = -\frac{1}{2}|E|^2 + \frac{1}{24}R^2 \tag{26}$$

is part of the Gauss-Bonnet integrand that is related to the fourth order curvature invariant

$$Q = -\frac{1}{12}\Delta R + \frac{1}{2}\sigma_2 \tag{27}$$

In low dimensions the sign of the quantity $\sigma_2(A)$ implies very strong restrictions on the curvature tensor. In dimension three, this is discussed in the article of Gursky in this volume. In dimension four, the positivity of $\sigma_2(A)$ implies first of all that the scalar curvature R cannot change sign, and more importantly, the Ricci curvature has the same sign as R. In case $R > 0$, an elementary algebraic argument shows that $(\frac{1}{2}R - \frac{3}{R}\sigma_2)g > Rc \geq \frac{3}{R}\sigma_2 g$. Thus the Ricci tensor is strongly positive in this sense. It would be interesting to find condition on the conformal class in which we can find a metric with positive $\sigma_2(A)$. A natural set of condition would be that $\int \sigma_2(A)dV > 0$ and that the conformal structure is in the positive Yamabe class.

Theorem 4.1. — [**CGY2**] *On a compact 4-manifold (M, g_0) with positive Yamabe invariant, if the conformal invariant $\int \sigma_2(A)dV$ is positive, there is a metric conformal to g_0 for which $\sigma_2(A)$ is pointwise positive.*

To give a brief idea of the proof, we first remark that the variational approach to the equation $\sigma_2(A) =$ constant is difficult due to the conformal invariance of the integral. However, it is possible to regularize the equation as the limiting equation of a family of fourth order equations that we had studied earlier ([**CY1**]):

$$\gamma_1|\eta|^2 + Q - \frac{1}{24}(3\delta - 2)\Delta R = 0 \tag{28}$$

where η is any fixed non-vanishing section of $S^2(T^*(M))$ i.e. a symmetric bilinear form on the tangent vectors, and γ_1 is chosen by the normalization

$$\gamma_1 = \frac{-\int QdV}{\int |\eta|^2}dV.$$

This equation is then equivalent to

$$\sigma_2(A) = \frac{\delta}{4}\Delta R - 2\gamma_1|\eta|^2.$$

The parameters are chosen so that when $\delta = 1$, the existence of solution is proved in our earlier paper ([**CY1**]). The regularity of the solution is provided in the article ([**CGY1**]). We then used a continuity argument in ([**CGY2**]) to run the parameter δ in the range $0 < \delta \leq 1$. The a priori estimates that are available shows there is a weak limit in $C^{1,\alpha}$ as δ tends to zero.

Unfortunately, that is not strong enough to conclude it is a strong solution of the equation $(*)_0$. By using the Yamabe flow applied to the solutions g_δ we were able to prove the limiting metric for a fixed small time t is smooth and satisfied the positivity condition $\sigma_2(A) > 0$.

Remark 4.1. — 1. There are topological constraints on a 4-manifold implied by the conditions of Theorem 4.1. The Gauss Bonnet formula

$$\chi = \frac{1}{8\pi^2} \int |W_+|^2 + |W_-|^2 + \sigma_2,$$

and the index formula

$$\tau = \frac{1}{12\pi^2} \int |W_+|^2 - |W_-|^2$$

combine to give the constraint $2\chi + 3\tau > 0$ as well as $2\chi - 3\tau > 0$. Since the positivity of Ricci curvature implies the finiteness of fundamental group, the universal cover of the manifolds in question still satisfy the same conditions. According to the results of Freedman and Donaldson, the class of simply connected 4-manifolds carrying a conformal structure satisfying the conditions of Theorem 4.1 must be of the form $k(\mathbb{CP}^2)\#l(-\mathbb{CP}^2)$ where $l < k$ and $4 + 5l > k$ or of the form $k(S^2 \times S^2)$. Here $-\mathbb{CP}^2$ is the complex projective plane taken with the opposite orientation. Among these it is easy to check that the 4-sphere, connected sums of up to three copies of $\mathbb{CP}^2$, connected sums of $\mathbb{CP}^2$ with up to eight copies of $-\mathbb{CP}^2$, and connected sums of up to 2 copies of $S^2 \times S^2$ do carry such conformal structures.

2. In the study of fully nonlinear second order elliptic equations, many authors look for solutions of the equations prescribing the elementary symmetric functions of the hessian. It is usual to assume some boundary conditions that assure the existence of functions whose Hessian lie in the positive cone defined to be the connected component of square matrices that satisfy the constraint $\sigma_k(A) > 0$ and contain the identity matrix. Our result may be viewed as supplying a criteria for the existence of functions for the $\sigma_2(A)$ equation.

3. The regularization procedure used in dimension four can be used formally to regularize the $\sigma_2(A)$ equation in other dimensions as well. Namely by adding, to the functional which computes the Sobolev quotient in dimensions three and beyond four, a term which calculates the integral $\int R^2 dV$ of the conformal metric with an appropriately chosen coefficient, it is possible to simultaneously cancel the fourth order term ΔR as well as to rearrange the remaining quadratic term in the Ricci tensor to be a multiple of $\sigma_2(A)$. This possibility makes the study of fourth order equations (12) all the more interesting. Suffice it to say, there is much that remains to be developed.

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