A posteriori regularity of the three-dimensional Navier-Stokes equations from numerical computations

Sergei I. Chernyshenko,
Aeronautics and Astronautics, School of Engineering Sciences,
University of Southampton,
Highfield, Southampton, SO17 1BJ. UK.

Peter Constantin,
Department of Mathematics, University of Chicago,
5734 University Avenue, Chicago, IL 60637. USA.

James C. Robinson,
Mathematics Institute, University of Warwick, Coventry, CV4 7AL. UK.

Edriss S. Titi
Department of Mathematics
and Department of Mechanical & Aerospace Engineering,
University of California, Irvine, CA 92697-3875. USA.
Also: Department of Computer Science & Applied Mathematics,
Weizmann Institute of Science, Rehovot 76100. Israel.

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Abstract

In this paper we consider the rôle that numerical computations – in particular Galerkin approximations – can play in problems modelled by the 3d Navier-Stokes equations, for which no rigorous proof of the existence of unique solutions is currently available. We prove a robustness theorem for strong solutions, from which we derive an *a posteriori* check that can be applied to a numerical solution to guarantee the existence of a strong solution of the corresponding exact problem.

We then consider Galerkin approximations, and show that *if* a strong solution exists the Galerkin approximations will converge to it; thus if one is prepared to assume that the Navier-Stokes equations are regular one can justify this particular numerical method rigorously.

Combining these two results we show that if a strong solution of the exact problem exists then this can be verified numerically using an algorithm that can be guaranteed to terminate in a finite time.

We thus introduce the possibility of rigorous computations of the solutions of the 3d Navier-Stokes equations (despite the lack of rigorous existence and uniqueness results), and demonstrate that numerical investigation can be used to rule out the occurrence of possible singularities in particular examples.

1 Introduction

The Navier-Stokes equations are the fundamental model of fluid flow. However, no proof of the long-time existence of unique solutions of these equations is currently available. The importance of this long-standing problem has recently been highlighted by its inclusion as one of the Clay Foundation's Millennium Problems.

Despite this mathematical obstacle, the equations are used as the basis of numerical calculations both in theoretical investigations of turbulence and for industrial applications. This paper addresses the validity of such numerical computations given the lack of appropriate rigorous existence and uniqueness results.

One might expect that without the existence of a unique solution there is no hope of guaranteeing that a numerical approximation is really an 'approximation' in any meaningful sense, since it is not clear what is being approximated. However, here we prove three results that enable numerical and exact solutions to be related in a rigorous way.

First we provide an explicit check that can be applied to a numerical solution to guarantee that it is approximating a solution of the exact problem. In particular, this check implies the existence of a unique solution of the exact problem over the same time interval as the calculation. Thus the validity of a numerical solution can be verified rigorously by a simple *a posteriori* condition.

We then turn to the particular example of Galerkin approximations. We show that given the *assumption* of the existence of a sufficiently smooth unique solution of the exact problem, the solutions obtained by the Galerkin method will converge to this exact solution: If one is prepared to take on trust that the Navier-Stokes equations are a meaningful model of fluid flow, numerical experiments (at least those using the Galerkin method) can be justified mathematically.

Finally, these two results can be combined to show that the existence of a sufficiently smooth solution can be verified numerically using an algorithm that can be guaranteed to terminate within a finite time (provided that the solution exists).

Although we concentrate here on spatial discretization (mainly via the spectral Galerkin method), it is relatively straightforward to consider fully discrete methods, and these are also addressed where relevant.

In order to discuss our results in a little more detail, we introduce here the model and its abstract setting. We will consider the 3d Navier-Stokes equations for an incompressible fluid

$$\frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = \tilde{f}(t) \qquad \text{with} \qquad \nabla\cdot u = 0 \tag{1}$$

on a periodic domain $Q = [0, L]^3$ with the additional (convenient) restriction that $\int_Q u = \int_Q \tilde{f} = 0$. However, we would expect similar results to hold for more physically realistic boundary conditions (e.g. Dirichlet boundary conditions on a bounded domain).

In order to recast the equations in their functional form (for full details see the monographs by Temam (1977), Constantin & Foias (1988), or Robinson (2001)) we let $\mathscr{H}$ be the collection of all divergence-free smooth periodic vector-valued functions on Q with zero average, and set

$$\begin{aligned} H &= \text{closure of } \mathscr{H} \text{ in } [L^2(Q)]^3 \\ V &= \text{closure of } \mathscr{H} \text{ in } [H^1(Q)]^3. \end{aligned}$$

We use $|\cdot|$ and $(\cdot,\cdot)$ for, respectively, the norm and inner product in $[L^2(Q)]^3$.

Denoting by Π the orthogonal projection of $[L^2(Q)]^3$ onto H, we apply Π to (1) and obtain

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \nu Au + B(u,u) = f(t), \tag{2}$$

where A is the Stokes operator $Au = -\Pi\Delta u$ (in fact $Au = -\Delta u$ when $u \in D(A)$ for the periodic case which we consider here), $B(u,u) = \Pi[(u\cdot\nabla)u]$, and $f = \Pi\tilde{f}$. The pressure term disappears since gradients are orthogonal (in L^2) to divergence-free fields.

We note that in the space periodic case we have $D(A^{m/2}) = H^m \cap V$; for simplicity we denote this space by V^m, and equip it with the natural norm

$$\|u\|_m = |A^{m/2}u|$$

which is equivalent to the norm in the Sobolev space H^m.

In section 2 we present a very simple ODE lemma that forms the basis of what follows. Section 3 recalls some classical estimates on the nonlinear term $B(u,u)$ in the spaces V^m ($m \geq 3$), and reproduces the proof of a classical regularity result.

Section 4.1 proves the robustness of strong solutions: We show in theorem 3 for $m \geq 3$ that if the initial data $u_0 \in V^m$ and forcing $f(t) \in L^2(0,T;V^{m-1}) \cap L^1(0,T;V^m)$ give rise to a strong solution, then so do 'nearby' initial data $v_0 \in V^m$ and forcing functions $g \in L^2(0,T;V^{m-1})$, under the explicit condition that

$$\|u_0 - v_0\|_m + \int_0^T \|f(s) - g(s)\|_m \, \mathrm{d}s < \frac{1}{c_m T} \exp\left[-c_m \int_0^T \left(\|u(s)\|_m + \|u(s)\|_{m+1}\right) \mathrm{d}s\right]$$

(note that this depends only on the solution u). A similar result holds for the Euler equations under slightly modified hypotheses, and this is shown in section 4.2.

As corollary 5 in section 5 we use the observation that any sufficiently smooth numerically computed solution u is the exact solution of the Navier-Stokes equation for some appropriate forcing function to turn the previous robustness result into an *a posteriori* test for the existence of a strong solution based on numerical observations. Leaving the precise definition of 'sufficiently smooth' to the formal statement of the result, we show that if u is a 'good' approximation to the solution of

$$\frac{\mathrm{d}v}{\mathrm{d}t} + \nu A v + B(v,v) = f(t) \qquad \text{with} \qquad v(0) = v_0 \in V^m \tag{3}$$

in the sense that

$$\|u(0) - v_0\|_m + \int_0^T \left\| \frac{\mathrm{d}u}{\mathrm{d}t}(s) + \nu A u(s) + B(u(s),u(s)) - f(s) \right\|_m \mathrm{d}s < \frac{1}{c_m T} \exp\left[-c_m \int_0^T \left(\|u(s)\|_m + \|u(s)\|_{m+1}\right) \mathrm{d}s\right]$$

then $v(t)$ must be a strong solution of (3) on $[0,T]$ with $v \in L^\infty(0,T;V^m) \cap L^2(0,T;V^{m+1})$. Crucially, this condition depends only on the numerical solution u (and the given initial data and forcing, v_0 and $f(t)$).

In section 6 we consider the convergence of approximations to solutions of the Navier-Stokes obtained via a Fourier Galerkin method. Here we *assume* the existence of a sufficiently regular exact solution (in fact we take $u_0 \in V^m$ and $f \in L^2(0,T;V^m)$ and assume the existence of a strong solution), and demonstrate in theorem 6 that under this condition, its Galerkin approximations u_n do indeed converge to the correct limit in both $L^\infty(0,T;V^m)$ and $L^2(0,T;V^{m+1})$.

Finally we combine our *a posteriori* test and the convergence of Galerkin approximations in theorem 8 to show that if, for a given choice of initial data $u_0 \in V^m$ and forcing $f \in L^1(0,T;V^m) \cap L^2(0,T;V^{m-1})$ (where $m \geq 3$), a strong solution does exist for some time interval $[0,T]$, then this can be verified computationally in a finite number of steps. (Essentially we show that for a sufficiently large Galerkin calculation we can ensure that the *a posteriori* test of corollary 5 must be satisfied.)

It should be noted that we do not aim here to prove results that are optimal with regard to the regularity of solutions, but rather to take sufficient regularity (initial conditions in the Sobolev space H^m with $m \geq 3$) that the arguments are at their most straightforward. Similar results for initial conditions in H^1 and H^2 are possible, and will be presented elsewhere.

2 A simple ODE lemma

The following simple lemma, after that in Constantin (1986), is central to all the results that follow. Although all solutions of the equation

$$\dot{y} = \delta(t) + \alpha y^2 \qquad \text{with} \qquad y(0) = y_0 > 0 \quad \text{and} \quad \alpha > 0$$

blow up in a finite time, if y_0 and $\delta(t)$ are sufficiently small (in appropriate senses) then the solution can be guaranteed to exist on $[0,T]$.

Lemma 1. *Let $T > 0$ and $\alpha > 0$ be constants, and let $\delta(t)$ be a non-negative continuous function on $[0,T]$. Suppose that y satisfies the differential inequality*

$$\frac{\mathrm{d}y}{\mathrm{d}t} \leq \delta(t) + \alpha y^2 \qquad \textit{with} \qquad y(0) = y_0 \geq 0 \tag{4}$$

and define

$$\eta = y_0 + \int_0^T \delta(s)\,\mathrm{d}s.$$

Then

$$y(t) \leq \frac{\eta}{1-\alpha\eta t} \tag{5}$$

while $0 \leq t \leq T$ *and the right-hand side is finite. In particular if*

$$\alpha\eta T < 1 \tag{6}$$

then $y(t)$ *remains bounded on* $[0,T]$, *and clearly* $y(t) \to 0$ *uniformly on* $[0,T]$ *as* $\eta \to 0$.

Proof. Observe that $y(t)$ is bounded by $Y(t)$ where

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = \alpha Y^2 \qquad \text{with} \qquad Y(0) = y_0 + \int_0^T \delta(s)\,\mathrm{d}s,$$

which yields (5) and the remainder of the lemma follows immediately. □

3 A classical regularity result

In this section we prove a slight variant of a classical regularity result which can be found as theorem 10.6 in Constantin & Foias, 1988.

First, we recall the following bounds on the nonlinear term, which we will use repeatedly (see Constantin & Foias (1988) for a proof):

$$\|B(u,v)\|_m \leq c_m \|u\|_m \|v\|_{m+1} \qquad m \geq 2, \tag{7}$$

$$|(B(w,v), A^m w)| \leq c_m \|v\|_{m+1} \|w\|_m^2 \qquad m \geq 2, \tag{8}$$

$$|(B(v,w), A^m w)| \leq c_m \|v\|_m \|w\|_m^2 \qquad m \geq 3. \tag{9}$$

Theorem 2. *Let* $u \in L^\infty(0,T;V) \cap L^2(0,T;V^2)$ *be a strong solution of the 3d Navier-Stokes equations with* $u_0 \in V^m$ *and* $f \in L^2(0,T;V^{m-1})$. *Then in fact*

$$u \in L^\infty(0,T;V^m) \cap L^2(0,T;V^{m+1})$$

and

$$\mathrm{d}u/\mathrm{d}t \in L^2(0,T;V^{m-1}). \tag{10}$$

Proof. We give a formal argument which can be made rigorous using the Galerkin procedure. The proof is inductive, supposing initially that $u \in L^2(0,T;V^k)$ for some $k \le m$. Taking the inner product of the equation with $A^k u$ we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_k^2 + \nu\|u\|_{k+1}^2 \le |(B(u,u),A^k u)| + (f,A^k u)$$

and so using (8) (valid here for $k \ge 2$)

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_k^2 + \nu\|u\|_{k+1}^2 \le c_k\|u\|_k^2\|u\|_{k+1} + \|f\|_{k-1}\|u\|_{k+1}.$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_k^2 + \nu\|u\|_{k+1}^2 \le \frac{c_k^2}{\nu}\|u\|_k^4 + \frac{\|f\|_{k-1}^2}{\nu}, \tag{11}$$

Dropping the term $\nu\|u\|_{k+1}^2$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_k^2 \le \left(\frac{c_k^2\|u\|_k^2}{\nu}\right)\|u\|_k^2 + \frac{\|f\|_{k-1}^2}{\nu};$$

It now follows from the Gronwall inequality that our assumption $u \in L^2(0,T;V^k)$ implies that $u \in L^\infty(0,T;V^k)$.

Returning to (11) and integrating between 0 and T we obtain

$$\nu\int_0^T \|u(s)\|_{k+1}^2\,\mathrm{d}s \le \|u(0)\|_k^2 + \frac{c_k^2}{\nu}\int_0^T \|u(s)\|_k^4\,\mathrm{d}s + \frac{1}{\nu}\int_0^T \|f(s)\|_{k-1}^2\,\mathrm{d}s, \tag{12}$$

which shows in turn that $u \in L^2(0,T;V^{k+1})$.

Since by assumption $u \in L^2(0,T;V^2)$, the first use of the induction requires $k=2$, for which inequality (8) is valid: we can therefore conclude by induction that $u \in L^\infty(0,T;V^m) \cap L^2(0,T;V^{m+1})$. Finally, since

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\nu A u - B(u,u) + f$$

and each term on the right-hand side is contained in $L^2(0,T;V^{m-1})$, the bound on the time derivative in (10) follows. □

We note here that it follows from this theorem that if u_0 and f are smooth (in V^m for all m) then so is the solution. Different techniques (due to Foias & Temam, 1989) can be used to show that the solution is analytic in the space variable (i.e. in a certain Gevrey class) provided that the data is.

4 Robustness of strong solutions

4.1 The Navier-Stokes equations

Using lemma 1 we show that if the 3d Navier-Stokes equations have a sufficiently smooth strong solution for given initial data u_0 and forcing f then they also have a strong solution for close enough data. The argument is based closely on that in Constantin (1986) which, given the existence of a strong solution of the Euler equations, deduces the existence of strong solutions for the Navier-Stokes equations for small enough $\nu > 0$. (The same argument is used in Chapter 11 of the monograph by Constantin & Foias (1988)).

We choose to state our primary result for sufficiently smooth strong solutions, namely those corresponding to initial data in V^m with $m \geq 3$. This enables us to use all the inequalities (7–9) and thereby obtain a relatively simple 'closeness' condition in (13). A similar approach works with strong solutions that have the minimal required regularity ($u \in L^\infty(0,T;V) \cap L^2(0,T;V^2)$) but the results are less elegant; these results will be presented in a future paper.

Note that while here we concentrate on the robustness of solutions defined on finite time intervals, a result valid for all $t \geq 0$ given a solution u for which $\int_0^\infty \|u(s)\|_1^4 \, \mathrm{d}s < \infty$ (i.e. which decays appropriately as $t \to \infty$) has been obtained by Ponce et al. (1993) for the particular case $f = 0$.

Theorem 3. *Let $m \geq 3$ and let u be a strong solution of the 3d Navier-Stokes equations*

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \nu A u + B(u,u) = f(t) \qquad \textit{with} \qquad u(0) = u_0 \in V^m$$

and $f \in L^2(0,T;V^{m-1}) \cap L^1(0,T;V^m)$. Then if $g \in L^2(0,T;V^{m-1})$ and

$$\|u_0 - v_0\|_m + \int_0^T \|f(s) - g(s)\|_m \, \mathrm{d}s < \frac{1}{c_m T} \exp\left[-c_m \int_0^T \left(\|u(s)\|_m + \|u(s)\|_{m+1}\right) \mathrm{d}s\right] \tag{13}$$

the solution v of

$$\frac{\mathrm{d}v}{\mathrm{d}t} + \nu A v + B(v,v) = g(t) \qquad \textit{with} \qquad v(0) = v_0 \in V^m$$

is a strong solution on $[0, T]$ *and is as regular as* u*.*

We remark here that throughout this paper we consider only Leray-Hopf weak solutions, i.e. weak solutions satisfying the energy inequality starting from almost every time.

Proof. Standard existence results guarantee that v is a strong solution on some time interval $[0, T^*)$. If T^* is maximal then $\|u(t)\|_1 \to \infty$ as $t \to T^*$; clearly we also have $\|u(t)\|_m \to \infty$ as $t \to T^*$. We suppose that $T^* \leq T$ and deduce a contradiction.

While v remains strong, the assumption that $v_0 \in V^m$ and that $g \in L^2(0, T; V^{m-1})$ allows one to use the regularity results of theorem 2 to deduce that

$$v \in L^\infty(0, T'; V^m) \cap L^2(0, T'; V^{m+1})$$

for any $T' < T^*$; we also have $\mathrm{d}v/\mathrm{d}t \in L^2(0, T'; V^{m-1})$ (cf. the argument in the proof of theorem 2). It also follows from theorem 2 that the solution u enjoys similar regularity on $[0, T]$.

The difference $w = u - v$ satisfies

$$\frac{\mathrm{d}w}{\mathrm{d}t} + \nu Aw + B(u, w) + B(w, u) + B(w, w) = f - g \tag{14}$$

with $w(0) = w_0 = u_0 - v_0$. On $[0, T^*)$ we know that w is sufficiently regular that

$$\langle A^{m/2} \frac{\mathrm{d}w}{\mathrm{d}t}, A^{m/2} w \rangle = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} |A^{m/2} w|^2. \tag{15}$$

We can therefore take the inner product with $A^m w$ and obtain, using (8) and (9),

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|w\|_m^2 \leq c_m \|u\|_m \|w\|_m^2 + c_m \|u\|_{m+1} \|w\|_m^2 + c_m \|w\|_m^3 + \|f - g\|_m \|w\|_m. \tag{16}$$

Dividing by $\|w\|_m$ yields[1]

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_m &\leq \|f-g\|_m + c_m\|u\|_m\|w\|_m + c_m\|u\|_{m+1}\|w\|_m + c_m\|w\|_m^2 \\ &\leq \|f-g\|_m + c_m(\|u\|_m + \|u\|_{m+1})\|w\|_m + \|w\|_m^2.\end{aligned}$$

We multiply by $\exp(-c_m\int_0^t \|u(s)\|_m + \|u(s)\|_{m+1}\,\mathrm{d}s)$ and consider

$$y(t) = \|w(t)\|_m \exp\left[-c_m\int_0^t (\|u(s)\|_m + \|u(s)\|_{m+1})\ \mathrm{d}s\right],$$

for which we obtain the inequality

$$\frac{\mathrm{d}y}{\mathrm{d}t} \leq \|f-g\|_m + \alpha y^2 \qquad \text{with} \qquad y(0) = \|u_0 - v_0\|_m,$$

where

$$\alpha = c_m \exp\left[c_m\int_0^T (\|u(s)\|_m + \|u(s)\|_{m+1})\ \mathrm{d}s\right]. \tag{17}$$

In this case the condition (6) from lemma 1 becomes (13). If this is satisfied then $y(t)$ is uniformly bounded on $[0,T^*)$; the solution can therefore be extended as a strong solution beyond $t = T^*$, contradicting the fact that $T^* \leq T$. It follows that $v(t)$ is a strong solution on $[0,T]$ with the same regularity as u. □

4.2 The Euler equations ($\nu = 0$)

We note here that the dissipative term νAu plays no direct rôle in the proof of theorem 3; however, it does enter indirectly via the regularity results of theorem 2 that are required to justify the equality in (15). It is possible to circumvent this via an appropriate mollification (cf. Constantin, E, & Titi, 1994) and obtain the above result (and those that follow) for solutions of the Euler equations ($\nu = 0$) as well as those of the Navier-Stokes equations.

[1]Of course, one has to worry here whether $\|w(t)\|_m$ is zero. However, if $\|w(t_0)\|_m = 0$ for some t_0 then $w(t_0) = 0$, and then the uniqueness of strong solutions in class of Leray-Hopf weak solutions implies that $u(t) = v(t)$ for all $t \geq t_0$, i.e. $w(t) = 0$ for all $t \in [t_0, T]$, a contradiction.

Theorem 4. *Let $m \geq 3$ and let $u \in L^2(0,T;V^{m+3})$ be a solution of the 3d Euler equations*

$$\frac{\mathrm{d}u}{\mathrm{d}t} + B(u,u) = f(t) \qquad \text{with} \qquad u(0) = u_0 \in V^{m+3}$$

where $f \in L^2(0,T;V^{m-1}) \cap L^1(0,T;V^m)$. If $g \in L^2(0,T;V^{m-1})$ and

$$\|u_0 - v_0\|_m + \int_0^T \|f(s) - g(s)\|_m \,\mathrm{d}s < \frac{1}{c_m T} \exp\left[-c_m \int_0^T (\|u(s)\|_m + \|u(s)\|_{m+1})\,\mathrm{d}s\right] \tag{18}$$

then the equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} + B(v,v) = g(t) \qquad \text{with} \qquad v(0) = v_0 \in V^{m+3},$$

has a solution $v \in L^\infty(0,T;V^m)$.

Proof. Suppose that $\|v(t)\|_m < \infty$ for all $t \in [0,T^*)$ but that $\|v(t)\|_m \to \infty$ as $t \to T^*$. As in the proof of Theorem 3 we assume that $T^* \leq T$ and obtain a contradiction. First, observe that taking $T' < T^*$ it follows from the regularity theory of Beale, Kato, & Majda (1984) that $v \in L^2(0,T';V^{m+3})$.

Now denote by $\varphi \in C_0^\infty(\mathbb{R}^3)$ a standard mollifier supported in the unit ball, set $\varphi^\epsilon(x) = \epsilon^{-3}\varphi(x/\epsilon)$, and write u^ϵ for the mollification of u by convolution with φ^ϵ:

$$u^\epsilon(x) = \int_{\mathbb{R}^3} \varphi^\epsilon(y) u(x-y)\,\mathrm{d}y.$$

(Note that the convolution is taken over all of $\mathbb{R}^3$ with u extended periodically, but the norms and inner products are still taken over Q.)

We return to (14) with ν set to zero,

$$\frac{\mathrm{d}w}{\mathrm{d}t} + B(u,w) + B(w,u) + B(w,w) = f - g,$$

mollify the equation,

$$\frac{\mathrm{d}w^\epsilon}{\mathrm{d}t} + B(u,w)^\epsilon + B(w,u)^\epsilon + B(w,w)^\epsilon = (f-g)^\epsilon,$$

and take the inner product with w^ϵ in V^m. In this way we obtain (16) with all quantities replaced by their mollified counterparts, but with the addition of three error terms,

$$\delta_{uww} + \delta_{wuw} + \delta_{www},$$

where

$$\delta_{uvw} = (B(u,v)^\epsilon, w^\epsilon)_m - (B(u^\epsilon, v^\epsilon), w^\epsilon)_m.$$

Now, note that we have the pointwise identity (cf. Constantin, E, & Titi, 1994)

$$[(u\cdot\nabla)v]^\epsilon = (u^\epsilon\cdot\nabla)v^\epsilon + r_\epsilon(u,v) - [((u-u^\epsilon)\cdot\nabla)(v-v^\epsilon)]$$

where

$$r_\epsilon(u,v)(x) = \int \varphi^\epsilon(y)[(u_y(x)\cdot\nabla)v_y(x)]\,\mathrm{d}y \qquad \text{with} \qquad u_y(x) = u(x-y)-u(x).$$

It follows that

$$\|[(u\cdot\nabla)v]^\epsilon - (u^\epsilon\cdot\nabla)v^\epsilon\|_m \le \|r_\epsilon(u,v)\|_m + \|((u-u^\epsilon)\cdot\nabla)(v-v^\epsilon)\|_m.$$

Since in 3d $|u(x)-u(y)| \le C\|u\|_{H^2}|x-y|^{1/2}$, it follows that

$$\|u-u^\epsilon\|_{H^m} \le c\epsilon^{1/2}\|u\|_{H^{m+2}} \qquad \text{and} \qquad \|r_\epsilon(u,v)\|_{H^m} \le \left(c^2\|u\|_{H^{m+2}}\|v\|_{H^{m+3}}\right)\epsilon,$$

and so we have

$$|\delta_{uvw}| \le \left(k\|u^\epsilon\|_{H^{m+2}}\|v^\epsilon\|_{H^{m+3}}\|w^\epsilon\|_{H^m}\right)\epsilon.$$

Now, use the fact that $\|u^\epsilon\|_{H^s} \le \|u\|_{H^s}$ for $\epsilon < L/2$, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|w^\epsilon\|_m \;\le\;& c_m\|u\|_m\|w^\epsilon\|_m + c_m\|u\|_{m+1}\|w^\epsilon\|_m + c_m\|w^\epsilon\|_m^2 + \|f-g\|_m \\ & + k'\Big(\|u\|_{m+2}\|w^\epsilon\|_{m+3} + \|w^\epsilon\|_{m+2}\|u\|_{m+3} + \|w^\epsilon\|_{m+2}\|w^\epsilon\|_{m+3}\Big)\epsilon\end{aligned}$$

for each $0 < \epsilon < L/2$.

Now define (cf. (17))

$$\alpha_t = c_m \exp\left(c_m \int_0^t \|u(s)\|_m + \|u(s)\|_{m+1}\,\mathrm{d}s\right)$$

to obtain

$$\|w^\epsilon(t)\|_m \le \frac{\eta_{T',\epsilon}}{1-\alpha_{T'}\eta_{T',\epsilon}T'} \qquad \text{for all} \qquad t\in[0,T']$$

provided that

$$\begin{aligned}\eta_{T',\epsilon} &= \|u_0-v_0\|_m + \int_0^{T'} \|f(s)-g(s)\|_m \,\mathrm{d}s \\ &\quad +\epsilon\, k' \int_0^{T'} \Big(\|u\|_{m+2}\|w^\epsilon\|_{m+3} + \|w^\epsilon\|_{m+2}\|u\|_{m+3} + \|w^\epsilon\|_{m+2}\|w^\epsilon\|_{m+3}\Big)\,\mathrm{d}s \\ &< \frac{1}{c_m T'} \exp\left[-c_m \int_0^{T'} (\|u(s)\|_m + \|u(s)\|_{m+1})\,\mathrm{d}s\right].\end{aligned}$$

(cf. (13)). Since both u and v are regular solutions on $[0,T']$, it follows that

$$\int_0^{T'} \|u\|_{m+2}\|w\|_{m+3} + \|w\|_{m+2}\|u\|_{m+3} + \|w\|_{m+2}\|w\|_{m+3}\,\mathrm{d}s$$

is finite. Given that $\|w^\epsilon\|_s \le \|w\|_s$ for all $\epsilon < L/2$ we can therefore let $\epsilon \to 0$ and obtain the bound

$$\|w^\epsilon(t)\|_m \le \frac{\eta_{T'}}{1-\alpha_{T'}\eta_{T'}T'} \qquad \text{for all} \qquad t\in[0,T'] \tag{19}$$

where now

$$\eta_{T'} = \|u_0-v_0\|_m + \int_0^{T'} \|f(s)-g(s)\|_m,$$

provided that

$$\eta_{T'} < \frac{1}{c_m T'} \exp\left[-c_m \int_0^{T'} (\|u(s)\|_m + \|u(s)\|_{m+1})\,\mathrm{d}s\right]. \tag{20}$$

Since (18) holds, it follows that (20) is verified for all T' sufficiently close to T. We can therefore deduce from (19) that

$$\|w(t)\|_m \le \frac{\eta_T}{1-\alpha_T\eta_T T} \qquad \text{for all} \qquad t\in[0,T),$$

and we have obtained a contradiction. It follows that v is a regular solution on $[0,T]$. □

Similar techniques should be applicable to extend the other results of this paper to the Euler case, but from now on we treat only the Navier–Stokes equations.

5 Deducing the existence of a strong solution via numerics

As an application of theorem 3 – and this may be the most significant result in this paper – we give an *a posteriori* test to determine whether or not a numerical solution of the 3d Navier-Stokes equations is meaningful. That is, we give a criterion *depending only on the numerically computed solution* that, if satisfied, guarantees that the exact equation being approximated possesses a strong (and hence unique) solution. Note that in the statement of the theorem the provenance of the function u is irrelevant.

Corollary 5. *Let* $f \in L^2(0,T;V^m)$ *and* $u \in C^0([0,T];V^m) \cap L^2(0,T;V^{m+1})$ *with*

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \nu Au + B(u,u) \ \in \ L^1(0,T;V^m) \cap L^2(0,T;V^{m-1}).$$

for some $m \geq 3$. *If* $v_0 \in V^m$ *and*

$$\|u(0) - v_0\|_m + \int_0^T \left\| \frac{\mathrm{d}u}{\mathrm{d}t}(s) + \nu Au(s) + B(u(s),u(s)) - f(s) \right\|_m \mathrm{d}s$$
$$< \frac{1}{c_m T} \exp\left[-c_m \int_0^T (\|u(s)\|_m + \|u(s)\|_{m+1}) \, \mathrm{d}s \right] \tag{21}$$

then the solution of the Navier-Stokes equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} + \nu Av + B(v,v) = f(t) \qquad \textit{with} \qquad v(0) = v_0 \in V^m \tag{22}$$

is a strong solution on $[0,T]$ *with* $v \in L^\infty(0,T;V^m) \cap L^2(0,T;V^{m+1})$.

Proof. The function u is regular enough that it is the (unique) strong solution of the Navier-Stokes equation (for the unknown $\tilde{u}$)

$$\frac{\mathrm{d}\tilde{u}}{\mathrm{d}t} + \nu A\tilde{u} + B(\tilde{u},\tilde{u}) = \underbrace{\frac{\mathrm{d}u}{\mathrm{d}t} + \nu Au + B(u,u)}_{\text{notional forcing}} \quad \text{with} \quad \tilde{u}(0) = u_0;$$

note that the conditions on u ensure that the right-hand side is an element of $L^2(0,T;V^{m-1})$. We now use theorem 3 to compare $\tilde{u}$ with the solution of (22): in this case the condition (13) to guarantee that v is a strong solution is precisely (21). □

If u comes from a discrete time-stepping algorithm, so that its approximate values u_n are only specified at times t_n (with $t_{n+1} > t_n$) then we can define a continuous function u via linear interpolation,

$$u(t) = \frac{t_{n+1} - t}{t_{n+1} - t_n} u_n + \frac{t - t_n}{t_{n+1} - t_n} u_{n+1} \qquad \text{for} \qquad t \in (t_n, t_{n+1}]. \tag{23}$$

If $u_n \in V^{m+2}$ then certainly $u \in C^0([0,T]; V^{m+2})$ and $\mathrm{d}u/\mathrm{d}t \in L^2(0,T; V^{m+2})$. One can now apply the test of corollary 5 to this function u; if the test is satisfied this again proves the existence of a strong solution for (22).

6 Convergence of the Galerkin approximations

We now turn our attention to one particular form of numerical solution, namely the Galerkin approximation. We show that given the existence of a suitably smooth strong solution, this numerical method is meaningful in that the Galerkin approximations can be guaranteed to converge to the strong solution. Similar results – convergence given the assumption that a strong solution exists – are given for finite element methods by Heywood & Rannacher (1982), for a Fourier collocation method by E (1993), and for a nonlinear Galerkin method by Devulder, Marion, & Titi (1993).

In some sense the result of Heywood (1982) that the Galerkin approximations of a stable solution of the equations converge uniformly on the whole time interval is in a similar spirit: properties of the Galerkin method are deduced from an assumption on the full equation. Of course, in this context it is perhaps more natural to seek conditions under which one can guarantee the existence of such a solution given properties of the Galerkin approximations, see for example Constantin, Foias, & Temam (1984) and Titi (1987). For similar results for time-periodic solutions see Titi (1991).

We should emphasise again that, in contrast to some related analyses of the Galerkin method (e.g. Rautmann, 1980) no assumption is made on the regularity of the Galerkin approximations themselves.

6.1 The Galerkin approximation

We denote by P_n the orthogonal projection in H onto the first n eigenfunctions of the Stokes operator, and by Q_n its orthogonal complement. Denoting these eigenfunctions by $\{w_j\}_{j=1}^{\infty}$, and their corresponding eigenvalues by $0 < \lambda_1 \le \lambda_2 \le \cdots$, we have

$$P_n u = \sum_{j=1}^{n} (u, w_j) w_j \qquad \text{and} \qquad Q_n u = \sum_{j=n+1}^{\infty} (u, w_j) w_j.$$

Note that if $u \in V^m$ then

$$\|Q_n u\|_m^2 = \sum_{j=n+1}^{\infty} \lambda_j^m |(u, w_j)|^2 \le \sum_{j=1}^{\infty} \lambda_j^m |(u, w_j)|^2 = \|u\|_m^2$$

and clearly $Q_n u \to 0$ in V^m as $n \to \infty$.

The Galerkin approximation of (2) is obtained by projecting all terms onto the space $P_n H$:

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} + \nu A u_n + P_n B(u_n, u_n) = P_n f(t) \qquad \text{with} \qquad u_n(0) = P_n u_0. \tag{24}$$

6.2 Convergence of the Galerkin approximation

Again we present our result for sufficiently strong solutions. With some care one can combine the approach of Devulder et al. (1993) with that used here to give a proof for strong solutions with minimal regularity; this will be presented elsewhere.

Theorem 6. *Let $u_0 \in V^m$ with $m \ge 3$, $f \in L^2(0,T;V^m)$, and let $u(t)$ be a strong solution of the Navier-Stokes equations*

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \nu A u + B(u, u) = f(t) \qquad \textit{with} \qquad u(0) = u_0. \tag{25}$$

Denote by u_n the solution of the Galerkin approximation (24). Then $u_n \to u$ strongly in both $L^\infty(0,T;V^m)$ and $L^2(0,T;V^{m+1})$ as $n \to \infty$.

Proof. The key, as with the robustness theorem, is to arrange $B(u,u) - B(u_n,u_n)$ so that it only involves u_n in the form $u_n - u$. Writing $w_n = u - u_n$ yields the equation (cf. (14))

$$\frac{\mathrm{d}w_n}{\mathrm{d}t} + \nu A w_n + P_n B(u,w_n) + P_n B(w_n,u) + P_n B(w_n,w_n) = Q_n f - Q_n B(u,u).$$

Taking the inner product of this equation with $A^m w_n$ we obtain

$$\begin{aligned} \tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_n\|_m^2 + \nu\|w_n\|_{m+1}^2 \;\; \leq \;\; & c_m\|u\|_m\|w_n\|^2 + c_m\|u\|_{m+1}\|w_n\|_m^2 + c_m\|w_n\|_m^3 \\ & + \|Q_n f - Q_n B(u,u)\|_m\|w_n\|_m. \end{aligned} \tag{26}$$

Dropping the term $\nu\|w_n\|_{m+1}^2$ and dividing by $\|w_n\|_m$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w_n\|_m \leq c_m(\|u\|_m + \|u\|_{m+1})\|w_n\| + c_m\|w_n\|_m^2 + \|Q_n f - Q_n B(u,u)\|_m.$$

Setting

$$y_n(t) = \|w_n(t)\|_m \exp\left[-c_m\int_0^t (\|u(s)\|_m + \|u(s)\|_{m+1})\,\mathrm{d}s\right]$$

we obtain

$$\dot{y}_n \leq \|Q_n f - Q_n B(u,u)\|_m + \alpha y_n^2 \qquad \text{with} \qquad y_n(0) = \|Q_n u_0\|_m,$$

where as in the proof of theorem 3

$$\alpha = c_m \exp\left[c_m\int_0^T (\|u(s)\|_m + \|u(s)\|_{m+1})\,\mathrm{d}s\right].$$

Noting that α is independent of n, and that $y_n(t)$ is proportional to $\|w_n(t)\|_m$, convergence of the Galerkin solutions will follow from convergence of $y_n(t)$ to zero. Using lemma 1 this will follow from

$$\|Q_n u_0\|_m + \int_0^T \|Q_n[f(s) - B(u(s),u(s))]\|_m\,\mathrm{d}s \to 0 \quad \text{as} \quad n \to \infty,$$

which we now demonstrate.

That $\|Q_n u_0\|_m \to 0$ as $n \to \infty$ is immediate from the definition of Q_n. To show convergence of the integral term, observe that since

$$\|B(u,u)\|_m \leq c_m \|u\|_m \|u\|_{m+1} \tag{27}$$

and the regularity result of theorem 2 guarantees that $u \in L^2(0,T;V^{m+1})$, it follows that

$$f(s) - B(u(s),u(s)) \in V^m \qquad \text{for} \qquad \text{a.e. } s \in [0,T].$$

We therefore know that $\|Q_n[f(s) - B(u(s),u(s))]\|_m$ converges pointwise to zero for a.e. $s \in [0,T]$, while it is clear that

$$\|Q_n[f(s) - B(u(s),u(s)]\|_m \leq \|f(s) - B(u(s),u(s))\|_m \quad \text{for} \quad \text{a.e. } s \in [0,T],$$

and the right-hand side is an element of $L^1(0,T)$. It follows from the Lebesgue dominated convergence theorem that

$$\int_0^T \|Q_n[f(s) - B(u(s),u(s))]\|_m \, \mathrm{d}s \to 0 \quad \text{as} \quad n \to \infty. \tag{28}$$

Returning to (26) and integrating between 0 and T one can easily show the convergence of w_n to zero in $L^2(0,T;V^{m+1})$. □

It is clear that if the Galerkin approximations converge then one can devise a fully discrete method that converges in a similar sense, and this can easily be made precise. In the following theorem one can take $u_{n,\delta}$ to be the linear interpolant of a discrete set of values $u_{n,\delta}(t_j) \in V^m$ as in (23) .

Corollary 7. *Let $u_0 \in V^m$ with $m \geq 3$, $f \in L^2(0,T;V^m)$, and let $u(t)$ be a strong solution of the Navier-Stokes equations (25). For each n, let $\{u_{n,\delta}\}_{\delta>0}$ be a collection of functions from $[0,T]$ into P_nH with the property that*

$$u_{n,\delta}(t) \to u_n(t) \quad \text{in} \quad L^\infty(0,T;V^m) \quad \text{as} \quad \delta \to 0, \tag{29}$$

where u_n is the solution of the Galerkin approximation (24). Then there exist $\delta_n > 0$ such that

$$\sup_{0<\delta<\delta_n} \|u_{n,\delta} - u\|_{L^\infty(0,T;V^m)} + \|u_{n,\delta} - u\|_{L^2(0,T;V^{m+1})} \to 0$$

as $n \to \infty$.

Proof. Since all norms on any finite-dimensional space are equivalent (in particular those on $P_nH = P_nV^m = P_nV^{m+1}$) the convergence in (29) also implies convergence in $L^\infty(0,T;V^{m+1})$, and so in $L^2(0,T;V^{m+1})$. Otherwise the result is immediate. □

7 Guaranteed numerical verification of the existence of a strong solution

Our final result combines corollary 5 and theorem 6 to show that the existence of a smooth strong solution can be verified algorithmically *in a finite time* by a sufficiently refined numerical computation. Namely, we show that for n sufficiently large the Galerkin solution *will* satisfy the regularity test (21) from corollary 5.

Theorem 8. *Suppose that for some $m \geq 3$, $u_0 \in V^m$, $f \in L^1(0,T;V^m) \cap L^2(0,T;V^{m-1})$, and that u is a strong solution of the Navier-Stokes equations (25). Then there exists an N such that the solution u_n of the Galerkin approximation (24) satisfies condition (21) for every $n \geq N$, i.e. will pass the* a posteriori *test for the existence of a strong solution with data (u_0, f).*

Proof. First, we note that the convergence of the Galerkin approximations to u that is provided by theorem 6 shows that $\int_0^T (\|u_n(s)\|_m + \|u_n(s)\|_{m+1})\,\mathrm{d}s$ is bounded independently of n.

We need, therefore, only show that

$$\|Q_nu_0\|_m + \int_0^T \left\|\frac{\mathrm{d}u_n}{\mathrm{d}t}(s) + \nu Au_n(s) + B(u_n(s),u_n(s)) - f(s)\right\|_m \mathrm{d}s$$

(the left-hand side of (21)) tends to zero as $n \to \infty$. The requirement on the initial condition is trivially satisfied, so we consider here only the 'remainder term'

$$\int_0^T \left\|\frac{\mathrm{d}u_n}{\mathrm{d}t}(s) + \nu Au_n(s) + B(u_n(s),u_n(s)) - f(s)\right\|_m \mathrm{d}s.$$

Now, since u_n satisfies the Galerkin approximation (24) we have

$$\frac{\mathrm{d}u_n}{\mathrm{d}t}(s) + \nu Au_n(s) + B(u_n(s),u_n(s)) - f(s) = Q_n[B(u_n(s),u_n(s)) - f(s)],$$

and so the remainder term is in fact equal to

$$\int_0^T \|Q_n[B(u_n(s), u_n(s)) - f(s)]\|_m \, \mathrm{d}s. \tag{30}$$

In particular this shows that solutions of the Galerkin approximation have the regularity required to ensure that the integral on the left-hand side of (21) is well-defined.

The integral in (30) coincides with the expression in (28), whose convergence to zero we showed above in the proof of theorem 6, except that the argument of B is u_n rather than u. However,

$$B(u_n(s), u_n(s)) - B(u(s), u(s)) = B(u_n(s) - u(s), u_n(s)) + B(u(s), u_n(s) - u(s))$$

and so

$$\begin{aligned}
&\int_0^T \|Q_n[B(u_n(s), u_n(s)) - B(u(s), u(s))\|_m \\
&\quad \leq \int_0^T \|B(u_n(s), u_n(s)) - B(u(s), u(s))\|_m \, \mathrm{d}s \\
&\quad \leq \int_0^T \|B(u_n(s) - u(s), u_n(s))\|_m \, \mathrm{d}s + \int_0^T \|B(u(s), u_n(s) - u(s))\|_m \, \mathrm{d}s \\
&\quad \leq c_m \int_0^T \|u_n(s) - u(s)\|_m \|u_n(s)\|_{m+1} \, \mathrm{d}s \\
&\qquad\qquad + c_m \int_0^T \|u(s)\|_m \|u_n(s) - u(s)\|_{m+1} \, \mathrm{d}s
\end{aligned}$$

Since $u_n \to u$ strongly in both $L^\infty(0, T; V^m)$ and $L^2(0, T; V^{m+1})$ the result follows. □

Once more it is possible to treat fully discrete schemes within a similar framework. Supposing as above that a scheme gives rise to a discrete set of values $u_{n,\delta}(t_j)$, one can define a linear interpolation $u_{n,\delta}$. We then have the following result:

Corollary 9. *Suppose that for some $m \geq 3$, $u_0 \in V^m$, $f \in L^1(0, T; V^m) \cap L^2(0, T; V^{m-1})$, and that u is a strong solution of the Navier-Stokes equations*

(25). For each n, let $\{u_{n,\delta}\}_{\delta>0}$ be a collection of functions from $[0,T]$ into P_nH with the property that

$$\mathrm{d}u_{n,\delta}(t)/\mathrm{d}t \to \mathrm{d}u_n(t)/\mathrm{d}t \quad \text{in} \quad L^\infty(0,T;V^m) \quad \text{as} \quad \delta \to 0, \tag{31}$$

where u_n is the solution of the Galerkin approximation (24). Then there exists an N and a sequence δ_n such that the interpolant $u_{n,\delta}$ satisfies condition (21) for every $n \geq N$ and $\delta < \delta_n$, i.e. the fully discrete numerical solution will pass the a posteriori *test for the existence of a strong solution with data (u_0, f).*

Proof. The interpolant $u_{n,\delta}$ satisfies the Galerkin approximation (24) except for an error

$$r_{n,\delta} = \frac{\mathrm{d}u_n}{\mathrm{d}t} - \frac{\mathrm{d}u_{n,\delta}}{\mathrm{d}t};$$

this error converges to zero in $L^\infty(0,T;V^m)$ because of (31) (cf. Higham & Stuart, 1998). It follows that we now obtain

$$\int_0^T \|Q_n[B(u_n(s),u_n(s)) - f(s) + r_{n,\delta}(s)]\|_m \,\mathrm{d}s.$$

rather than (30), where $r_{n,\delta}$ converges to zero in $L^\infty(0,T;V^m)$. This is clearly sufficient to follow the argument in the proof of theorem 8. □

8 Conclusion

Despite the lack of a guarantee that unique solutions exist for the three-dimensional Navier-Stokes equations, we have shown that it is possible to perform 'rigorous' numerical experiments. In particular, we have given an *a posteriori* test that, if satisfied by a numerical solution, guarantees that it approximates a true strong solution of the Navier-Stokes equations. Remarkably, the existence of such a solution can be verified using such numerical computations in a finite time; some computations along these lines will be reported in a future publication.

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