

GENERALIZED RELATIVE ENTROPIES AND STOCHASTIC REPRESENTATION

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ABSTRACT. We provide a stochastic interpretation of a result of decay of generalized relative entropies that was discovered by Michel, Mischler and Perthame

1. INTRODUCTION

Relative entropies have been used for a long time in kinetic theory and conservation laws. The decay of relative entropies was usually limited to stable situations in which a global attracting steady solution exists ([9]). Recently, Michel, Mischler and Perthame ([8]) discovered a remarkable property of certain unstable linear systems, in which decay of relative entropies can exist under certain circumstances. They applied their observation to population dynamics models but the list of application grows. The author of the present note learned of this remarkable property in a talk given by B. Perthame ([7]) and was struck both by the property and by a statement of the speaker to the effect that the proof is computational and does not reveal the reasons behind the property. A first attempt to make the proof more conceptual ([1]) resulted only in a generalization and a formalization of what was basically the original proof, and to applications to Smoluchowski equations. In this note we present the stochastic underpinning of the phenomenon, and provide a more conceptual understanding. Namely, the property is a consequence of the existence of stochastic integrals of motion. The use of stochastically passive scalars and the existence of stochastic integrals of motion can be used to prove the decay of generalized relative entropies in more complicated situations, when the principal part of the diffusion operator does not have constant coefficients ([2]), but that proof requires a substantial technical treatment. The purpose of this note is to explain the phenomenon in the simplest setting.

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2. GENERALIZED RELATIVE ENTROPIES

We consider a linear operator

$$(1) \quad \mathcal{D}\rho = \Delta_x \rho - \operatorname{div}_x(U\rho) + V\rho$$

in \mathbb{R}^n , where

$$(2) \quad U(x, t) = (U_j)_{j=1, \dots, n}$$

is a smooth function and $V = V(x, t)$ is a smooth scalar potential. We will write \mathcal{D} also in the form

$$(3) \quad \mathcal{D}\rho = \Delta_x \rho - U \cdot \nabla_x \rho + P\rho$$

where

$$(4) \quad P = V - \operatorname{div}_x(U).$$

The formal adjoint of the operator \mathcal{D} in $L^2(\mathbb{R}^n)$ is

$$(5) \quad \mathcal{D}^*\phi = \Delta_x \phi + U \cdot \nabla_x \phi + V\phi.$$

The following is a result of Michel, Mischler and Perthame:

Theorem 1. ([7],[8]) *Let f be a solution of*

$$(6) \quad \partial_t f = \mathcal{D}f$$

and let $\rho > 0$ be a positive solution of the same equation,

$$(7) \quad \partial_t \rho = \mathcal{D}\rho.$$

Let H be a smooth convex function of one variable and let ϕ be a non-negative function obeying pointwise

$$(8) \quad \partial_t \phi + \mathcal{D}^*\phi = 0.$$

Then

$$(9) \quad \frac{d}{dt} \int H\left(\frac{f}{\rho}\right) \phi \rho dx \leq 0.$$

3. STOCHASTIC REPRESENTATION

We take $X(a, t)$ solutions of the SDE ([5]), ([6])

$$dX = U(X, t)dt + \sqrt{2}dW$$

with $X(a, 0) = a$, and dW standard Brownian motion in \mathbb{R}^3 ,

$$W_0 = 0, \quad \langle W_t^i, W_s^j \rangle = \min(t, s) \delta^{ij}.$$

Because the Brownian motion is uniform in space, $\partial_a X$ obeys an ODE, and it follows that

$$(10) \quad \frac{d}{dt} \log \det(\partial_a X) = \operatorname{div}_x(U)_{x=X(a, t)}$$

is true pathwise, (a.s.). We denote the spatial inverse by $A(x, t)$, $A = X^{-1}$. The spatial inverse exists and is smooth ([4]).

Theorem 2. *We consider the stochastic function*

$$(11) \quad \psi_{f_0}(x, t) = f_0(A(x, t)) \exp \left\{ \int_0^t V(X(a, s), s) ds_{|a=A(x,t)} \right\} \det(\nabla_x A)(x, t).$$

Then $\psi = \psi_{f_0}$ solves

$$(12) \quad d\psi + (\nabla_x \cdot (U\psi) - \Delta_x \psi - V\psi) dt + \sqrt{2} \nabla_x \psi \cdot dW = 0$$

with initial datum $\psi(x, 0) = f_0(x)$.

The proof follows from the Itô formula and the equation ([3]), ([4])

$$(13) \quad dA + (U \cdot \nabla_x A - \Delta_x A) dt + \sqrt{2} \nabla_x A \cdot dW = 0$$

obeyed by A . We note first that

$$(14) \quad \psi_{f_0} = f_0(A) \exp \left\{ \int_0^t P(X(a, s), s) ds_{|a=A(x,t)} \right\}$$

holds. Indeed, (14) follows by integrating (10) in time and using

$$[\det(\partial_a X)]_{|a=A(x,t)}^{-1} = \det(\nabla_x A)(x, t),$$

that holds pathwise. Now, from (13) it follows that

$$\chi = f_0(A)$$

is a stochastically passive scalar, by which we mean a solution of the stochastic Lagrangian transport equation

$$(15) \quad d\chi + (U \cdot \nabla_x \chi - \Delta \chi) dt + \sqrt{2} \nabla_x \chi \cdot dW = 0.$$

Stochastically passive scalars form an algebra: sums and products of stochastically passive scalars are stochastically passive. The fact that the product of stochastically passive scalars is stochastically passive is nontrivial and follows from a cancellation induced by the quadratic variation of the martingale part of the second order equations obeyed by them.

From (14), using (15) and stochastic calculus ([6]) we deduce now (12). Indeed, the function

$$I(a, t) = \exp \left\{ \int_0^t P(X(a, s), s) ds \right\}$$

obeys

$$\partial_t I(a, t) = P(X(a, t), t) I(a, t)$$

pathwise. Then, a calculation ([3]), ([4]) shows that the function

$$E(x, t) = I(A(x, t), t)$$

solves

$$dE + (U \cdot \nabla E - PE - \Delta E)dt + \sqrt{2}\nabla E \cdot dW = 0.$$

Because $P = V - \operatorname{div}_x(U)$ we have

$$dE + (\operatorname{div}_x(UE) - VE - \Delta E)dt + \sqrt{2}\nabla E \cdot dW = 0.$$

The function ψ_{f_0} is the product

$$\psi_{f_0} = \chi E,$$

and therefore, from Itô's formula

$$d\psi_{f_0} = Ed\chi + \chi dE + d\langle E, \chi \rangle$$

and the equations obeyed by E, χ , we have

$$\begin{aligned} d\psi_{f_0} = & \\ & (-\operatorname{div}_x(U\psi_{f_0}) + V\psi_{f_0} + E\Delta_x\chi + \chi\Delta_xE + 2\nabla_xE \cdot \nabla_x\chi)dt \\ & - \sqrt{2}\nabla_x\psi_{f_0} \cdot dW. \end{aligned}$$

This gives (12), and finishes the calculation.

4. STOCHASTIC INTEGRALS OF MOTION.

Proposition 1. *Consider a deterministic function ϕ that solves (8). Then the function*

$$(16) \quad M(a, t) = \phi(X(a, t), t) \exp \left\{ \int_0^t V(X(a, s), s) ds \right\}$$

is a martingale.

Indeed, by Itô

$$\begin{aligned} dM(a, t) = & \exp \left\{ \int_0^t V(X(a, s), s) \right\} \\ & \left\{ \begin{aligned} & \nabla_x \phi(X(a, t), t) \cdot dX + \phi(X(a, t), t) V(X(a, t), t) dt \\ & + \frac{1}{2} \nabla_x \nabla_x \phi(X(a, t), t) d\langle X, X \rangle + \partial_t \phi(X(a, t), t) dt \end{aligned} \right\} \end{aligned}$$

holds because because $\exp \left\{ \int_0^t V(X(a, s), s) ds \right\}$ is BV. Now, using the equation (8) we have

$$\begin{aligned} dM(a, t) = & \\ & \sqrt{2}\nabla_x \phi(X(a, t), t) \exp \left\{ \int_0^t V(X(a, s), s) ds \right\} dW \end{aligned}$$

that is, M is the martingale

$$\begin{aligned} M(a, t) = & \\ & \phi(a, 0) + \int_0^t \sqrt{2}\nabla_x \phi(X(a, s), s) \exp \left\{ \int_0^s V(X(a, \tau), \tau) d\tau \right\} dW_s. \end{aligned}$$

Using this fact and the representation (11), it follows that

$$\begin{aligned} & \psi_{\rho_0}(x, t) \phi(x, t) H \left(\frac{\psi_{f_0}(x, t)}{\psi_{\rho_0}(x, t)} \right) = \\ & \rho_0(A(x, t)) H \left(\frac{f_0(A(x, t))}{\rho_0(A(x, t))} \right) M(A(x, t), t) \det(\nabla_x A) \end{aligned}$$

holds. Thus, the quantity of interest, $\psi_{\rho_0} \phi H \left(\frac{\psi_{f_0}}{\psi_{\rho_0}} \right)$, is the product of a stochastically passive scalar, a martingale composed with A and the Jacobian $\det(\nabla_x A)$. Consequently, we have almost surely

$$(17) \quad \int \psi_{\rho_0}(x, t) H \left(\frac{\psi_{f_0}(x, t)}{\psi_{\rho_0}(x, t)} \right) \phi(x, t) dx = \int \rho_0(a) H \left(\frac{f_0(a)}{\rho_0(a)} \right) M(a, t) da.$$

The expected value is then constant in time:

$$(18) \quad \frac{d}{dt} \mathbb{E} \left\{ \int \psi_{\rho_0} H \left(\frac{\psi_{f_0}}{\psi_{\rho_0}} \right) \phi dx \right\} = 0.$$

5. PROOF OF DECAY

If we denote

$$(19) \quad f(x, t) = \mathbb{E} \psi_{f_0}(x, t)$$

and

$$(20) \quad \rho(x, t) = \mathbb{E} \psi_{\rho_0}(x, t)$$

we have from (12) that f solves (6), $\rho > 0$ solves (7). We prove that we have (9).

The starting point is (18). In view of (19) and (20), the statement that needs to be proved is

$$(21) \quad \int \mathbb{E}(\psi_{\rho_0}) H \left(\frac{\mathbb{E}(\psi_{f_0})}{\mathbb{E}(\psi_{\rho_0})} \right) \phi dx \leq \mathbb{E} \left\{ \int \psi_{\rho_0} H \left(\frac{\psi_{f_0}}{\psi_{\rho_0}} \right) \phi dx \right\}$$

The conservation (18) works for any H , but we expect (21) to hold only for convex H . Indeed, (21) can be reduced to a Jensen inequality. We claim more, that for all x, t we have

$$(22) \quad \mathbb{E}(\psi_{\rho_0}) H \left(\frac{\mathbb{E}(\psi_{f_0})}{\mathbb{E}(\psi_{\rho_0})} \right) \leq \mathbb{E} \left\{ \psi_{\rho_0} H \left(\frac{\psi_{f_0}}{\psi_{\rho_0}} \right) \right\}$$

Considering the functions

$$(23) \quad g = \frac{\psi_{\rho_0}}{\mathbb{E}(\psi_{\rho_0})}$$

and

$$(24) \quad v = \frac{\psi_{f_0}}{\mathbb{E}(\psi_{\rho_0})}$$

we see that (22) becomes

$$(25) \quad H(\mathbb{E}(v)) \leq \mathbb{E} \left\{ gH \left(\frac{v}{g} \right) \right\}.$$

This, however, is nothing but Jensen's inequality for the probability measure

$$\begin{aligned} Ph &= \mathbb{E}(gh), \\ H \left(P \left(\frac{v}{g} \right) \right) &\leq PH \left(\frac{v}{g} \right). \end{aligned}$$

6. APPENDIX: DIRECT PROOF

For the sake of completeness, we present here a direct proof. This is a calculation ([1]), similar to the original proof of ([8]), but organized somewhat differently. We associate to \mathcal{D} and to the scalar positive function ρ the operator \mathcal{D}_ρ defined by

$$(26) \quad \mathcal{D}_\rho h = \frac{1}{\rho^2} \partial_i (\rho^2 \partial_i h) - U \cdot \nabla_x h.$$

The proof has two ingredients, the first of which is a pointwise inequality:

Lemma 1. *Let $h = H \left(\frac{f}{\rho} \right)$ with H convex, let f solve (6) and let $\rho > 0$ solve (7). Then*

$$(27) \quad \partial_t h - \mathcal{D}_\rho h \leq 0$$

The lemma is easily verified. In fact, the identity

$$(28) \quad \begin{aligned} &\partial_t h - \mathcal{D}_\rho h = \\ &-H'' \left(\frac{f}{\rho} \right) \left| \nabla_x \left(\frac{f}{\rho} \right) \right|^2 + \left\{ \frac{1}{\rho} (\partial_t f - \mathcal{D}f) - \frac{f}{\rho^2} (\partial_t \rho - \mathcal{D}\rho) \right\} H' \left(\frac{f}{\rho} \right) \end{aligned}$$

holds for all smooth functions f, ρ where $\rho \neq 0$. The second ingredient concerns the formal adjoint of \mathcal{D}_ρ :

Lemma 2. *If \mathcal{D}_ρ is associated to $\rho > 0$, then*

$$(29) \quad \mathcal{D}_\rho^*(\phi\rho) = \rho\mathcal{D}^*\phi - \phi\mathcal{D}\rho$$

holds pointwise for any smooth function ϕ .

Indeed,

$$(30) \quad \mathcal{D}_\rho^*(\rho\phi) = \partial_j \left(\rho^2 \partial_j \left(\frac{\phi}{\rho} \right) \right) + \operatorname{div}_x (U\phi\rho)$$

and therefore we obtain using (5) and (3))

$$\begin{aligned}\mathcal{D}_\rho^*(\rho\phi) &= \rho(\Delta_x\phi + \operatorname{div}_x(U\phi)) - \phi(\Delta_x\rho - U \cdot \nabla_x\rho) = \\ &= \rho\mathcal{D}^*\phi - \phi\mathcal{D}\rho.\end{aligned}$$

The proof of the theorem follows now from the two lemmas. Using the notation $h = H\left(\frac{f}{\rho}\right)$ as above we have

$$\frac{d}{dt} \int \phi\rho h dx = \int \{(\phi\partial_t\rho + \rho\partial_t\phi)h + \phi\rho[\mathcal{D}_\rho h]\} dx + \int \phi\rho(\partial_t h - \mathcal{D}_\rho h) dx$$

Using the first lemma we have

$$\frac{d}{dt} \int \phi\rho h dx \leq \int [\phi\partial_t\rho + \rho\partial_t\phi + \mathcal{D}_\rho^*(\phi\rho)] h dx$$

and using the second lemma we conclude

$$\frac{d}{dt} \int \phi\rho h dx \leq \int (\partial_t\phi + \mathcal{D}^*\phi)\rho h dx + \int (\partial_t\rho - \mathcal{D}\rho)\phi h dx \leq 0.$$

Note that if $H \geq 0$ then $\phi_t + \mathcal{D}^*\phi \leq 0$ is sufficient. This proof generalizes easily to variable diffusion coefficients ([1]).

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