Stochastic Lagrangian Transport and Generalized Relative Entropies

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Abstract

We discuss stochastic representations of advection diffusion equations with variable diffusivity, stochastic integrals of motion and generalized relative entropies.

1 Introduction

Recently, Michel, Mischler and Perthame [8] discovered a remarkable property of certain unstable linear equations, in which decay of relative entropies takes place. Their observation was applied to population dynamics models, but the list of applications is growing. Of course, relative entropies have been used for a long time in kinetic theory and conservation laws. However,

the decay of relative entropies, was known before only in stable, self-adjoint situations in which a global attracting steady solution exists and no flow advection is present [9]. The property of decay of relative entropies was slightly generalized to variable diffusion coefficients and applied to Smoluchowski systems in [1]. A stochastic interpretation and proof in the case of constant diffusion coefficients was given in [2]. Here we provide a stochastic interpretation and proof in the case of variable diffusion coefficients. The method of proof and concepts are of more general interest [3, 4].

We consider a linear operator

$$\mathcal{D}\rho = \nu \partial_i(a_{ij}\partial_j\rho) - \mathrm{div}_x(U\rho) + V\rho \tag{1}$$

in $\mathbb{R}^n$, where

$$U(x,t) = (U_j(x,t))_{j=1,\dots n} \tag{2}$$

is a smooth (C^2) function, $V = V(x,t)$ is a continuous and bounded scalar potential and

$$a_{ij}(x,t) = \sigma_{ip}(x,t)\sigma_{jp}(x,t) \tag{3}$$

with the matrix

$$\sigma(x,t) = (\sigma_{ij}(x,t))_{ij} \tag{4}$$

a given smooth (C^2) matrix. We assume that σ is bounded and U and $\nabla_x\sigma$ decay at infinity. We use the shorthand notation $A(D)$ for the operator

$$A(D)\rho = a_{ij}\partial_i\partial_j\rho \tag{5}$$

and use also the non-divergence form

$$\mathcal{D}\rho = \nu A(D)\rho - u \cdot \nabla_x\rho + P\rho \tag{6}$$

where

$$u_j(x,t) = U_j(x,t) - \nu\partial_i(a_{ij}(x,t)) \tag{7}$$

and

$$P = V - \mathrm{div}_x(U). \tag{8}$$

The formal adjoint of the operator $\mathcal{D}$ in $L^2(\mathbb{R}^n)$ is

$$\mathcal{D}^*\phi = \nu\partial_i(a_{ij}\partial_j\phi) + U \cdot \nabla_x\phi + V\phi. \tag{9}$$

The following is the result of Michel, Mischler and Perthame:

Theorem 1 *[7, 8] Let f be a solution of*

$$\partial_t f = \mathcal{D} f \tag{10}$$

and let $\rho > 0$ be a positive solution of the same equation,

$$\partial_t \rho = \mathcal{D} \rho. \tag{11}$$

Let H be a smooth convex function of one variable and let ϕ be a non-negative function obeying pointwise

$$\partial_t \phi + \mathcal{D}^* \phi = 0. \tag{12}$$

Then

$$\frac{d}{dt} \int H\left(\frac{f}{\rho}\right) \phi \rho dx \leq 0. \tag{13}$$

2 Stochastic Lagrangian Flow

In order to represent solutions of equations like (10) we consider the drift

$$v_j(x,t) = u_j + 2\nu(\partial_k \sigma_{jp})\sigma_{kp} = U_j - \nu(\partial_k \sigma_{kp})\sigma_{jp} + \nu(\partial_k \sigma_{jp})\sigma_{kp}. \tag{14}$$

Let $X(a,t)$ be the strong solution of the stochastic differential system

$$dX_j(t) = v_j(X,t)dt + \sqrt{2\nu}\sigma_{jp}(X,t)dW_p \tag{15}$$

with initial data

$$X(a,0) = a. \tag{16}$$

Here W is a standard Brownian process in $\mathbb{R}^n$ starting at time zero from the origin. This process will be fixed throughout the paper and all measurability issues will be with respect to the filtration associated to it and all almost sure statements will be with respect to the probability measure on the standard Wiener space. We will need the following result:

Theorem 2 *The inverse of the flow map $a \mapsto X(a,t)$, the stochastic map*

$$x \mapsto A(x,t) \tag{17}$$

exists almost surely and satisfies its defining relations

$$X(A(x,t),t) = x, \quad \forall x \in \mathbb{R}^n, \quad A(X(a,t),t) = a, \quad \forall a \in \mathbb{R}^n, \quad \forall t, \ a.s.$$

3

The map X is smooth and the determinant

$$D(a,t) = \det(\partial_a X(a,t)) \qquad (18)$$

obeys the SDE

$$d(\det(\partial_a X(a,t)) = [\det(\partial_a X(a,t))] \times$$
$$\left\{ [(div_x v)(x,t) + 2\nu E(x,t)]_{|x=X(a,t)}\, dt + \sqrt{2\nu}(\partial_k(\sigma_{kp}))(x,t)_{|x=X(a,t)} dW_p \right\} \qquad (19)$$

with

$$E(x,t) = \sum_{i<j}\sum_p \det(\partial_i \sigma_{jp})_{ij}. \qquad (20)$$

The map $A(x,t)$ satisfies the stochastic partial differential system

$$dA_j + (u \cdot \nabla_x A_j - \nu A(D)A_j)\, dt + \sqrt{2\nu}(\partial_k A_j)\sigma_{kp} dW_p = 0 \qquad (21)$$

with initial data

$$A(x,0) = 0.$$

Remark. In the statement above, $\det(\partial_i \sigma_{jp})_{ij}$ refers to the determinant of the two-by-two matrix $(\partial_r \sigma_{kp})$ with $r,k \in \{i,j\}$ for fixed $i < j$ and p. Theorem 2 was originally proved in [3] for constant coefficients and in [4] for variable coefficients. For completeness, we reproduce the proof (with variable coefficients, as stated above) in Appendix A.

3 Stochastically Passive Scalars and Feynman-Kac Formula

We consider first deterministic smooth time-independent functions f_0 and note that the functions $\theta = \theta_{f_0}(x,t) = f_0(A(x,t))$ are stochastically passive in the sense that they obey the equation

$$d\theta + (u \cdot \nabla_x \theta - \nu A(D)\theta)\, dt + \sqrt{2\nu}\partial_k \theta \sigma_{kp} dW_p = 0 \qquad (22)$$

with initial data

$$\theta(x,0) = f(x). \qquad (23)$$

Solutions of the SPDE (22) form an algebra; in particular, products of solutions are solutions, a nontrivial fact due to the presence of the stochastic

term. The expected values of these scalars obey advection-diffusion equations and do not form an algebra in general, if $\nu > 0$. We consider now the function

$$I(a,t) = \exp\left\{\int_0^t P(X(a,s),s)ds\right\} \tag{24}$$

where $P(x,t)$ is given in (8) and consider the function

$$\psi = \psi_{f_0}(x,t) = \theta_{f_0}(x,t)I(A(x,t),t) \tag{25}$$

We have

Theorem 3 *The process $\psi = \psi_{f_0}$ given by*

$$\psi(x,t) = f_0(A(x,t))\exp\left\{\int_0^t P(X(a,s),s)ds_{|a=A(x,t)}\right\} \tag{26}$$

solves

$$d\psi - (\mathcal{D}\psi)\,dt + \sqrt{2\nu}\nabla_x\psi\sigma dW = 0 \tag{27}$$

with initial datum $\psi(x,0) = f_0(x)$.

The proof of this result follows using stochastic calculus [5], [6]. Indeed, the function $I(a,t)$ obeys

$$\partial_t I(a,t) = P(X(a,t),t)I(a,t) \tag{28}$$

pathwise (almost surely). Then, a calculation using (21) (see [3], [4]) shows that the function

$$J(x,t) = I(A(x,t),t) \tag{29}$$

solves

$$dJ + (u \cdot \nabla_x J - PJ - \nu A(D)J)dt + \sqrt{2\nu}\nabla_x J\sigma dW = 0. \tag{30}$$

The function ψ_{f_0} is the product

$$\psi_{f_0} = \theta_{f_0}J,$$

and therefore, from Itô's formula

$$d\psi_{f_0} = Jd\theta + \theta dJ + d\langle J,\theta\rangle$$

5

and the equations obeyed by J, θ, we have

$$d\psi_{f_0} =$$
$$(-u \cdot \nabla_x \psi_{f_0} + P\psi_{f_0} + \nu J A(D)\theta + \nu \theta A(D)J + 2\nu(\partial_k J)\sigma_{kp}(\partial_j \theta)\sigma_{jp})dt$$
$$- \sqrt{2\nu}\nabla_x \psi_{f_0}\sigma dW.$$

This means

$$d\psi = (-u \cdot \nabla_x \psi + P\psi + \nu A(D)\psi)dt - \sqrt{2\nu}\nabla_x \psi_{f_0}\sigma \ dW.$$

Because of (6) we have (27).

4   Stochastic Integrals of Motion.

Proposition 1 *Consider a deterministic function ϕ that solves (12). Then the function*

$$M(a,t) = \phi(X(a,t),t) \det(\partial_a X(a,t)) \exp\left\{\int_0^t P(X(a,s),s)ds\right\} \qquad (31)$$

is a martingale.

Proof. We start by writing

$$M(a,t) = \Phi(a,t)I(a,t)D(a,t)$$

with

$$\Phi(a,t) = \phi(X(a,t),t),$$

I given above in (24) and D given in (18). Next, we compute the equation obeyed by ΦI. In view of (28) and using Itô's formula we have

$$d(\Phi I) = I\left\{(\partial_t \phi(X(a,t),t) + P\phi(X(a,t),t))\, dt + \right.$$
$$\left. + \nabla_x \phi_{|X(a,t)} \cdot dX + \tfrac{1}{2}\partial_i \partial_j \phi_{|X(a,t)}d\langle X_i, X_j\rangle\right\},$$

which gives, in view of (15)

$$d(\Phi I) =$$
$$= \{\partial_t \phi + P\phi + v \cdot \nabla_x \phi + \nu A(D)\phi\}_{|X(a,t)}\, dt + \sqrt{2\nu}I((\partial_i \phi)\sigma_{ip})_{|X(a,t)}dW_p.$$

Using (12) and (14) we have

$$d(\Phi I) = I\left\{-2\nu(\partial_k(\sigma_{kp}))\sigma_{jp}(\partial_j\phi) - (\mathrm{div}_x U)\phi\right\}_{|X(a,t)} dt+$$
$$+\sqrt{2\nu}I((\partial_j\phi)\sigma_{jp})_{|X(a,t)}dW_p. \tag{32}$$

Now, by Itô,

$$dM = Dd(\Phi I) + \Phi ID + d\langle D, \Phi I\rangle.$$

In view of (19) and (32) we have

$$d\langle D, \Phi I\rangle = 2\nu DI\left\{(\partial_k\sigma_{kp})\sigma_{jp}(\partial_j\phi)\right\}_{|X(a,t)} dt$$

and consequently the terms $\pm 2\nu DI(\partial_k\sigma_{kp})\sigma_{jp}(\partial_j\phi)dt$ cancel and we obtain

$$dM = ID\left\{-(\mathrm{div}_x U)\phi + (\mathrm{div}_x v + 2\nu E)\right\}_{|X(a,t)} dt+$$
$$+\sqrt{2\nu}ID\left\{\sigma_{jp}(\partial_j\phi) + (\partial_k\sigma_{kp})\right\}_{|X(a,t)} dW_p.$$

Now, in view of (14) we have that

$$(\mathrm{div}_x v) - (\mathrm{div}_x U) = \nu\partial_j[(\partial_k\sigma_{jp})\sigma_{kp}] - \nu\partial_j[(\partial_k\sigma_{kp})\sigma_{jp}]$$

and therefore the coefficient of dt in dM is

$$DI\left\{2\nu E + \nu\partial_j[(\partial_k\sigma_{jp})\sigma_{kp}] - \nu\partial_j[(\partial_k\sigma_{kp})\sigma_{jp}]\right\}$$

Now

$$DI\left\{\nu\partial_j[(\partial_k\sigma_{jp})\sigma_{kp}] - \nu\partial_j[(\partial_k\sigma_{kp})\sigma_{jp}]\right\} =$$
$$DI\left\{\nu(\partial_k\sigma_{jp})(\partial_j\sigma_{kp}) - \nu(\partial_k\sigma_{kp})(\partial_j\sigma_{jp})\right\} =$$
$$= DI2\sum_{k<j}\sum_p\left\{\nu(\partial_k\sigma_{jp})(\partial_j\sigma_{kp}) - \nu(\partial_k\sigma_{kp})(\partial_j\sigma_{jp})\right\} =$$
$$= -2\nu E$$

and therefore the coefficient of dt in dM vanishes. We obtained

$$dM = \sqrt{2\nu}ID\left\{\sigma_{jp}(\partial_j\phi) + (\partial_k\sigma_{kp})\right\}_{|X(a,t)} dW_p, \tag{33}$$

that is, M is the martingale

$$M(a,t) =$$
$$\phi(a,0) + \sqrt{2\nu}\int_0^t I(a,s)D(a,s)\left\{\sigma_{jp}(\partial_j\phi) + (\partial_k\sigma_{kp})\right\}_{|X(a,s)} dW_p(s)$$

7

Theorem 4 *Let h_0 and ρ_0 be smooth time independent deterministic functions. Consider the stochastically passive scalar $\theta_{h_0}(x,t) = h_0(A(x,t))$ and the process ψ_{ρ_0} of (26) with initial datum ρ_0. Consider also $\phi(x,t)$, a deterministic solution of (12). Then the random variable*

$$\mathcal{E}(t) = \int_{\mathbb{R}^n} \phi(x,t)\psi_{\rho_0}(x,t)\theta_{h_0}(x,t)dx \tag{34}$$

is a martingale. In particular

$$\mathbb{E}(\mathcal{E}(t)) = \int_{\mathbb{R}^n} \phi(a,0)\rho_0(a)h_0(a)da \tag{35}$$

holds.

Proof. In view of the change of variables formula and the definition of ψ_{ρ_0} we have that

$$\mathcal{E}(t) = \int_{\mathbb{R}^n} M(a,t)\rho_0(a)h_0(a)da \tag{36}$$

with M given in (31). The result follows then from the previous proposition. More precisely

$$d\mathcal{E} =$$
$$\sqrt{2\nu}\left\{\int_{\mathbb{R}^n} \exp\left\{\int_0^s P(X(a,\tau),\tau)d\tau_{|a=A(x,s)}\right\}\left\{\sigma_{jp}(\partial_j\phi) + (\partial_k\sigma_{kp})\right\}dx\right\}dW_p \tag{37}$$

gives explicitly the SDE obeyed by $\mathcal{E}$.

5   Generalized Relative Entropies

We take now a smooth deterministic, time independent function H of one variable, a deterministic solution of (12), two smooth deterministic, time independent functions f_0 and ρ_0, of which ρ_0 is strictly positive. We form the processes ψ_{ρ_0} and ψ_{f_0} given by the expressions (26). Then it t follows that

$$\psi_{\rho_0}(x,t)\phi(x,t)H\left(\frac{\psi_{f_0}(x,t)}{\psi_{\rho_0}(x,t)}\right) =$$
$$\psi_{\rho_0}(x,t)\phi(x,t)H\left(\frac{f_0(A(x,t))}{\rho_0(A(x,t))}\right)$$

8

holds. Thus, the quantity of interest, $\psi_{\rho_0}\phi H\left(\frac{\psi_{f_0}}{\psi_{\rho_0}}\right)$, is the product of a stochastically passive scalar, ψ_{ρ_0} and ϕ. By the previous theorem we have that

$$\mathcal{E}(t) = \int \psi_{\rho_0}(x,t) H\left(\frac{\psi_{f_0}(x,t)}{\psi_{\rho_0}(x,t)}\right)\phi(x,t)dx \qquad (38)$$

is a martingale. The expected value is then constant in time:

$$\frac{d}{dt}\mathbb{E}\left\{\int \psi_{\rho_0} H\left(\frac{\psi_{f_0}}{\psi_{\rho_0}}\right)\phi dx\right\} = 0. \qquad (39)$$

If we denote

$$f(x,t) = \mathbb{E}\psi_{f_0}(x,t) \qquad (40)$$

and

$$\rho(x,t) = \mathbb{E}\psi_{\rho_0}(x,t) \qquad (41)$$

we have from (27) that f solves (10), $\rho > 0$ solves (11). We prove that we have (13).

The starting point is (39). In view of (40) and (41), the statement that needs to be proved is

$$\int \mathbb{E}\left(\psi_{\rho_0}\right) H\left(\frac{\mathbb{E}(\psi_{f_0})}{\mathbb{E}(\psi_{\rho_0})}\right)\phi dx \leq \mathbb{E}\left\{\int \psi_{\rho_0} H\left(\frac{\psi_{f_0}}{\psi_{\rho_0}}\right)\phi dx\right\} \qquad (42)$$

The conservation (39) works for any H, but we expect (42) to hold only for convex H. Indeed, (42) can be reduced to a Jensen inequality. We claim more, that for all x, t we have

$$\mathbb{E}\left(\psi_{\rho_0}\right) H\left(\frac{\mathbb{E}(\psi_{f_0})}{\mathbb{E}(\psi_{\rho_0})}\right) \leq \mathbb{E}\left\{\psi_{\rho_0} H\left(\frac{\psi_{f_0}}{\psi_{\rho_0}}\right)\right\} \qquad (43)$$

Considering the functions

$$g = \frac{\psi_{\rho_0}}{\mathbb{E}(\psi_{\rho_0})} \qquad (44)$$

and

$$v = \frac{\psi_{f_0}}{\mathbb{E}(\psi_{\rho_0})} \qquad (45)$$

we see that (43) becomes

$$H\left(\mathbb{E}(v)\right) \leq \mathbb{E}\left\{gH\left(\frac{v}{g}\right)\right\}. \qquad (46)$$

9

This, however, is nothing but Jensen's inequality for the probability measure

$$Ph = \mathbb{E}(gh),$$

$$H\left(P\left(\frac{v}{g}\right)\right) \leq PH\left(\frac{v}{g}\right).$$

A   Proof of Theorem 2

We devote this appendix to proving Theorem 2. The original proof can be found in [3] for constant coefficients, and in [4] for variable coefficients.

Lemma 1 *Let X be the stochastic flow defined by (15), (16). Then the map X is spatially smooth (almost surely), and the determinant $D = \det(\nabla X)$ satisfies the equation*

$$dD = D\left[(\nabla \cdot v + 2\nu E)\,dt + \sqrt{2\nu}\partial_k \sigma_{kp}\,dW_p\right]$$

where

$$E = \tfrac{1}{2}\left[\partial_i \sigma_{ip}\partial_j \sigma_{jp} - \partial_j \sigma_{ip}\partial_i \sigma_{jp}\right].$$

Proof. Differentiating (15) we have

$$d(\partial_a X_j) = \partial_k v_j \partial_a X_k\,dt + \sqrt{2\nu}\partial_k \sigma_{jp}\partial_a X_k\,dW_p. \tag{47}$$

Let S^n be the permutation group on n symbols, and ϵ_τ denote the signature of the permutation $\tau \in S^n$. By Itô's formula,

$$dD = \sum_{\substack{\tau \in S^n \\ b=1\ldots n}} \epsilon_\tau \left[\prod_{c\neq b}\partial_c X_{\tau_c}\,d\left(\partial_b X_{\tau_b}\right) + \sum_{c<b}\prod_{d\neq b,c}\partial_d X_{\tau_d}\,d\left\langle \partial_b X_{\tau_b}, \partial_c X_{\tau_c}\right\rangle\right]$$

$$= \sum_{\substack{\tau \in S^n \\ b=1\ldots n}} \epsilon_\tau \left[\partial_k v_{\tau_b}\partial_b X_k \prod_{c\neq b}\partial_c X_{\tau_c}\,dt + \sqrt{2\nu}\partial_k \sigma_{\tau_b,p}\partial_b X_k \prod_{c\neq b}\partial_c X_{\tau_c}\,dW_p + \right.$$

$$\left. + \nu \sum_{c\neq b}\partial_b X_l \partial_l \sigma_{\tau_b,p}\,\partial_c X_m \partial_m \sigma_{\tau_c,p}\prod_{d\neq b,c}\partial_d X_{\tau_d}\,dt\right] \tag{48}$$

10

We compute each of the terms above individually:

$$\sum_{\substack{\tau \in S^n \\ b=1...n}} \epsilon_\tau \partial_k v_{\tau_b} \partial_b X_k \prod_{c \neq b} \partial_c X_{\tau_c} = \sum_{\substack{\tau \in S^n \\ b=1...n}} \epsilon_\tau \partial_{\tau_b} v_{\tau_b} \partial_b X_{\tau_b} \prod_{c \neq b} \partial_c X_{\tau_c} +$$

$$+ \sum_{\substack{\tau \in S^n \\ b=1...n}} \sum_{k \neq \tau_b} \epsilon_\tau \partial_k v_{\tau_b} \partial_b X_k \partial_{\tau_k^{-1}} X_k \prod_{c \neq b, \tau_k^{-1}} \partial_c X_{\tau_c}$$

$$= (\nabla \cdot v) \det(\nabla X) +$$

$$+ \sum_{\substack{b=1...n \\ k \neq b}} \sum_{\tau \in S^n} \epsilon_\tau \partial_{\tau_k} v_{\tau_b} \partial_b X_{\tau_k} \partial_k X_{\tau_k} \prod_{c \neq b, k} \partial_c X_{\tau_c}$$

$$= (\nabla \cdot v) \det(\nabla X) + 0$$

The second term above is zero because replacing replacing τ with $\tau \circ (b\ k)$ in the inner sum produces a negative sign.

Similarly we have

$$\sum_{\substack{\tau \in S^n \\ b=1...n}} \epsilon_\tau \partial_k \sigma_{\tau_b, p} \partial_b X_k \prod_{c \neq b} \partial_c X_{\tau_c} \, dW_p = \partial_k \sigma_{k,p} \det(\nabla X) \, dW_p.$$

For the last term in (48), the only difference is that we have a few extra cases to consider: When $l = \tau(b)$, $m = \tau(c)$, we will get $\det(\nabla X)\partial_i \sigma_{ip} \partial_j \sigma_{jp}$. When $l = \tau(c)$ and $m = \tau(b)$, we will get $-\det(\nabla X)\partial_j \sigma_{ip} \partial_i \sigma_{jp}$. In all other cases we get 0. This concludes proof of Lemma 1.

Lemma 2 *For any time t, the map X_t has a (spatially) smooth inverse.*

Proof. Define λ by

$$\lambda = \exp\left[\int_0^t \left(\nabla \cdot v + 2\nu E - \nu(\partial_k \sigma_{kp})^2\right) \, dt + \sqrt{2\nu} \int_0^t \partial_k \sigma_{kp} \, dW_s^{(p)}\right]$$

The Itô's formula immediately shows that λ satisfies equation (19). Since (19) is a linear SDE with smooth coefficients, uniqueness of the solution guarantees $D = \exp(\lambda)$ almost surely, and hence $D > 0$ almost surely.

The spatial invertibility of X now follows as X_t is locally orientation preserving and has degree 1 (because X_t is properly homotopic to X_0, the identity map). The (spatial) smoothness of the inverse is guaranteed by the inverse function theorem.

11

The above lemma shows existence of a spatial inverse of X. As before, we let A denote the spatial inverse of X. We now derive a stochastic evolution equation of A [equation (21)].

Lemma 3 *Let Y be a C^1 stochastic flow of semi-martingales adapted to $\mathcal{F}_t$, the filtration of W_t. If for all $a \in \mathbb{R}^n$, $t > 0$ we have*

$$\int_0^t Y(X_s(a), ds) = \int_0^t b(X_s(a), s)\, ds + \int_0^t \sigma'(X_s(a), s) dW_s$$

then

$$Y_t(a) = Y_0(a) + \int_0^t b(a, t)\, dt + \int_0^t \sigma'(a, t)\, dW_t.$$

Proof. Let Y' be the process defined by

$$Y_t'(a) = Y_0(a) + \int_0^t b(a, t)\, dt + \int_0^t \sigma'(a, t)\, dW_t,$$

and set $\delta = Y - Y'$. Since δ is adapted to $\mathcal{F}_t$, there exists a non-negative predictable function a such that

$$\int_0^t a(x, y, s)\, ds = \langle \delta(x), \delta(y) \rangle_t.$$

Now, by definition of the generalized Itô integral we have $\int_0^t \delta(X_s, s) \equiv 0$ almost surely, and hence $\int_0^t a(X_s, X_s, s)\, ds \equiv 0$ almost surely. Since X is a flow of homeomorphisms (diffeomorphisms actually), we must have $\forall t$, $a(x, x, t) \equiv 0$ almost surely. Thus $\delta = Y - Y'$ is of bounded variation.

Since we have shown above that δ has bounded variation,

$$\int_0^t \delta(X_s, ds) = \int_0^t \partial_t \delta\big|_{X_s, s}\, ds$$

and hence $\forall t$, $\partial_t \delta_t \equiv 0$. At time 0, $\delta_0 \equiv 0$ by definition, and hence $\delta_t \equiv 0$ almost surely for all t, concluding the proof.

Lemma 4 *There exists a process B of bounded variation such that*

$$A_t = B_t - \sqrt{2\nu} \int_0^t (\nabla A_s) \sigma\, dW_s \tag{49}$$

12

Proof. Applying the generalized Itô formula to $A \circ X$ we have

$$
\begin{aligned}
0 &= \int_{t'}^{t} A(X_s, ds) + \int_{t'}^{t} \nabla A\big|_{X_s,s} \, dX_s + \tfrac{1}{2} \int_{t'}^{t} \partial_{ij}^2 A\big|_{X_s,s} d\left\langle X^{(i)}, X^{(j)} \right\rangle_s + \\
&\quad + \left\langle \int_{t'}^{t} \partial_i A(X_s, ds), X_t^{(i)} - X_{t'}^{(i)} \right\rangle \\
&= \int_{t'}^{t} A(X_s, ds) + \int_{t'}^{t} \left[\nabla A\big|_{X_s,s} v + \nu a_{ij} \partial_{ij}^2 A\big|_{X_s,s} \right] ds + \qquad (50) \\
&\quad + \sqrt{2\nu} \int_{t'}^{t} \nabla A\big|_{X_s,s} \sigma \, dW_s + \left\langle \int_{t'}^{t} \partial_i A(X_s, ds), X_t^{(i)} - X_{t'}^{(i)} \right\rangle.
\end{aligned}
$$

Notice that the second and fourth terms on the right are of bounded variation. Applying Lemma 3 we conclude the proof.

Lemma 5 *The process A satisfies the equation*

$$
dA_t + (v \cdot \nabla) A_t \, dt - \nu a_{ij} \partial_{ij}^2 A_t \, dt - \sqrt{2\nu} \partial_j A_t (\partial_i \sigma_{jk}) \sigma_{ik} \, dt + (\nabla A_t) \sigma \, dW_t = 0 \quad (51)
$$

Proof. Since the joint quadratic variation term in (50) depends only on the martingale part of $\partial_i A$, we can compute it explicitly by

$$
\begin{aligned}
\left\langle \int_{t'}^{t} \partial_i A(X_s, ds), X^{(i)} \right\rangle &= -2\nu \int_{t'}^{t} \left(\partial_{ij}^2 A_s \sigma_{jk} \sigma_{ik} + \partial_j A_s (\partial_i \sigma_{jk}) \sigma_{ik} \right) \circ X_s \, ds \\
&= -2\nu \int_{t'}^{t} \left(a_{ij} \partial_{ij}^2 A_s + \partial_j A_s (\partial_i \sigma_{jk}) \sigma_{ik} \right) \circ X_s \, ds.
\end{aligned}
$$
$$(52)$$

Substituting (52) in (50) and applying Lemma 3 we conclude the proof.

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