例5 **求随机相位正弦波$x(t)=A\sin(\omega_0 t+\theta)$ 的功率谱密度，其中A和ω_0为常数，θ为0 ~ 2π间均匀分布的随机变量**

解： 例2中已计算出：

$$R_x(\tau)=\frac{1}{2}A^2\cos\omega_0\tau$$

$$\therefore \quad S_x(\omega)=\frac{1}{2}\pi A^2[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]$$

例6 **试求二元随机波形的功率谱密度，其中信号取值是二值的(0或A)，每隔时间间隔T取值变1次，但每次的具体取值是随机且互相独立的，取0、A的概率各为1/2。**

解： 例3中已计算出：

$$R_x(\tau)=\begin{cases}\dfrac{A^2}{2}\left(1-\dfrac{|\tau|}{2T}\right) & |\tau|\le T\\[2ex] \dfrac{A^2}{4} & \text{else}\end{cases}$$

自相关函数可以视为：

$$R_x(\tau)=R_{x1}(\tau)+R_{x2}(\tau)=\frac{A^2}{4}+\frac{A^2}{4}\left(1-\frac{|\tau|}{T}\right)rect\left(\frac{\tau}{2T}\right)$$

$$R_{x1}(\tau)=\frac{A^2}{4},\quad R_{x2}(\tau)=\frac{A^2}{4}\left(1-\frac{|\tau|}{T}\right)rect\left(\frac{\tau}{2T}\right)$$

$$R_{x1}(\tau)=\frac{A^2}{4}\quad\Rightarrow\quad S_{x1}(\omega)=\frac{A^2}{4}\cdot 2\pi\delta(\omega)=\frac{\pi A^2}{2}\delta(\omega)$$

$$R_{x2}(\tau) = \frac{A^2}{4}\left(1 - \frac{|\tau|}{T}\right) rect\left(\frac{\tau}{2T}\right) \quad \Rightarrow$$

$$S_{x2}(\omega) = \int_{-\infty}^{\infty} R_{x2}(\tau) e^{-j\omega\tau} d\tau = \frac{A^2}{4}\int_{-T}^{T}\left(1 - \frac{|\tau|}{T}\right) e^{-j\omega\tau} d\tau$$

$$= \frac{A^2}{4}\int_{-T}^{T} e^{-j\omega\tau} d\tau - \frac{A^2}{4}\int_{-T}^{T}\frac{|\tau|}{T} e^{-j\omega\tau} d\tau$$

$$= \frac{A^2}{4}\frac{1}{-j\omega} e^{-j\omega\tau}\Big|_{-T}^{T} + \frac{A^2}{4}\int_{-T}^{0}\frac{\tau}{T} e^{-j\omega\tau} d\tau - \frac{A^2}{4}\int_{0}^{T}\frac{\tau}{T} e^{-j\omega\tau} d\tau$$

$$= \frac{A^2}{4}\frac{1}{-j\omega}(-2j\sin\omega T) + \frac{A^2}{4}\frac{\tau}{T}\frac{1}{-j\omega} e^{-j\omega\tau}\Big|_{-T}^{0} - \frac{A^2}{4}\frac{1}{T}\frac{1}{-j\omega}\int_{-T}^{0} e^{-j\omega\tau} d\tau$$

$$-\left[\frac{A^2}{4}\frac{\tau}{T}\frac{1}{-j\omega} e^{-j\omega\tau}\Big|_{0}^{T} - \frac{A^2}{4}\frac{1}{T}\frac{1}{-j\omega}\int_{0}^{T} e^{-j\omega\tau} d\tau\right]$$

$$S_{x2}(\omega)=\frac{A^2}{2}\frac{\sin\omega T}{\omega}+\frac{A^2}{4}\frac{1}{-j\omega}e^{j\omega T}-\frac{A^2}{4T}\frac{1}{(-j\omega)^2}e^{-j\omega\tau}\Big|_{-T}^{0}$$

$$-\left[\frac{A^2}{4}\frac{1}{-j\omega}e^{-j\omega T}-\frac{A^2}{4T}\frac{1}{(-j\omega)^2}e^{-j\omega\tau}\Big|_{0}^{T}\right]$$

$$=\frac{A^2}{2}\frac{\sin\omega T}{\omega}+\frac{A^2}{4}\frac{1}{-j\omega}e^{j\omega T}-\frac{A^2}{4T}\frac{1}{(-j\omega)^2}+\frac{A^2}{4T}\frac{1}{(-j\omega)^2}e^{j\omega T}$$

$$-\frac{A^2}{4}\frac{1}{-j\omega}e^{-j\omega T}+\frac{A^2}{4T}\frac{1}{(-j\omega)^2}e^{-j\omega T}-\frac{A^2}{4T}\frac{1}{(-j\omega)^2}$$

$$=\frac{A^2}{2}\frac{\sin\omega T}{\omega}+\frac{A^2}{4}\frac{1}{-j\omega}(2j\sin\omega T)+\frac{A^2}{4T}\frac{2}{\omega^2}-\frac{A^2}{4T}\frac{1}{\omega^2}2\cos\omega T$$

$$=\frac{A^2}{2}\frac{\sin\omega T}{\omega}-\frac{A^2}{2}\frac{\sin\omega T}{\omega}+\frac{A^2}{2}\frac{1}{\omega^2 T}(1-\cos\omega T)$$

$$=\frac{A^2}{2}\frac{1}{\omega^2 T}2\sin^2(\omega T/2)=\frac{A^2\sin^2(\omega T/2)}{\omega^2 T}$$

$$\therefore \quad S_x(\omega)=S_{x1}(\omega)+S_{x2}(\omega)=\frac{\pi A^2}{2}\delta(\omega)+\frac{A^2\sin^2(\omega T/2)}{\omega^2 T}$$

- 随机信号独立、不相关和正交的含义：
  - 随机信号**$x(t)$**和**$y(t)$**独立：
    对**$x(t)$**的任一时刻**t_1**和**$y(t)$**的任一时刻**t_2**，均有

$$p(x_1,y_2)=p(x_1)p(y_2)$$

  - 随机信号**$x(t)$**和**$y(t)$**不相关：
    对**$x(t)$**的任一时刻**t_1**和**$y(t)$**的任一时刻**t_2**，均有

$$E(x_1y_2)=E(x_1)E(y_2)$$

推论：$R_{xy}(\tau)=m_x m_y$

$$\text{if } m_x=0 \text{ and/or } m_y=0 \Rightarrow R_{xy}(\tau)=0$$

◆ 随机信号**x(t)**和**y(t)**正交：

对**x(t)**的任一时刻t_1和**y(t)**的任一时刻t_2，均有

$$E(x_1 y_2)=0 \quad \text{即：} R_{xy}(\tau)=0$$

实际中常将正交理解为：对**x(t)**和**y(t)**的任一时刻t_1，有

$$E(x_1 y_1)=0 \quad \text{即：} R_{xy}(0)=0$$

$$p(x_1,y_2)=p(x_1)p(y_2) \quad \Rightarrow \quad E(x_1 y_2)=E(x_1)E(y_2)$$

例7 **随机相位正弦波$x(t)=A\sin(\omega_0 t+\theta)$，其中$A$和$\omega_0$为常数，$\theta$为0～2π间均匀分布的随机变量；二元随机波形$y(t)$的取值是0或$A$，每隔$T$取值变1次，但每次具体取值是随机且互相独立的，取0、A的概率各为1/2。设$x(t)$和$y(t)$是统计独立的，求$z(t)=x(t)y(t)$的自相关函数和功率谱密度。**

解： 例2中已计算出：

$$R_x(\tau)=\frac{1}{2}A^2\cos\omega_0\tau$$

例5中已计算出：

$$S_x(\omega)=\frac{1}{2}\pi A^2[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]$$

例3中已计算出：

$$R_y(\tau)=\begin{cases}\frac{A^2}{2}\left(1-\frac{|\tau|}{2T}\right) & |\tau|\le T\\ \frac{A^2}{4} & \text{else}\end{cases}$$

例6中已计算出：

$$S_y(\omega)=\frac{\pi A^2}{2}\delta(\omega)+\frac{A^2\sin^2(\omega T/2)}{\omega^2 T}$$

$$z(t)=x(t)y(t)\Rightarrow$$

$$R_z(\tau)=E[z(t)z(t-\tau)]=E[x(t)y(t)x(t-\tau)y(t-\tau)]$$

$x(t)$和$y(t)$统计独立，则有：

$$R_z(\tau)=E[x(t)x(t-\tau)]E[y(t)y(t-\tau)]=R_x(\tau)R_y(\tau)$$

$$\therefore \quad R_z(\tau)=\begin{cases}\dfrac{A^4}{4}\left(1-\dfrac{|\tau|}{2T}\right)\cos\omega_0\tau & |\tau|\le T\\ \dfrac{A^4}{8}\cos\omega_0\tau & \text{else}\end{cases}$$

$$R_z(\tau)=R_x(\tau)R_y(\tau)\quad\Rightarrow$$

$$S_z(\omega)=\frac{1}{2\pi}S_x(\omega)*S_y(\omega)$$

$$\therefore \quad S_z(\omega) = \frac{1}{2\pi} S_x(\omega) * S_y(\omega)$$

$$= \frac{1}{2\pi} \frac{\pi A^2}{2} \delta(\omega + \omega_0) * \left[\frac{\pi A^2}{2} \delta(\omega) + \frac{A^2 \sin^2(\omega T / 2)}{\omega^2 T} \right]$$

$$+ \frac{1}{2\pi} \frac{\pi A^2}{2} \delta(\omega - \omega_0) * \left[\frac{\pi A^2}{2} \delta(\omega) + \frac{A^2 \sin^2(\omega T / 2)}{\omega^2 T} \right]$$

$$= \frac{\pi A^4}{8} \delta(\omega + \omega_0) + \frac{A^4}{4T} \frac{\sin^2[(\omega + \omega_0) T / 2]}{(\omega + \omega_0)^2}$$

$$+ \frac{\pi A^4}{8} \delta(\omega - \omega_0) + \frac{A^4}{4T} \frac{\sin^2[(\omega - \omega_0) T / 2]}{(\omega - \omega_0)^2}$$

§ 2.3 典型的随机信号

一、高斯随机信号

- **通信系统的三类噪声：单频噪声、脉冲噪声、起伏噪声**
- **单频噪声：时间上连续，频谱集中在某个频率附近很窄范围**
- **脉冲噪声：时间上持续很短、间隔较长且无规则，频谱很宽**
- **起伏噪声：时间上连续、无规则，普遍存在**
- **三类噪声以叠加形式干扰信号，称为加性噪声**

- 起伏噪声主要有：热噪声、散粒噪声、宇宙噪声
- 热噪声：导体中大量自由电子热运动产生的
- 散粒噪声：有源电子器件电子发射不均匀所引起的
- 宇宙噪声：天体的电磁辐射所引起的
- 起伏噪声是高斯随机过程，又称为高斯噪声
- 高斯噪声$n(t)$的概率密度函数表示为：

$$p(n)=\frac{1}{\sqrt{2\pi}\sigma_n}\exp\left[-\frac{(n-a)^2}{2\sigma_n^2}\right]$$

其中a为信号的均值， σ_n^2 为信号的方差

- **起伏噪声的均值一般为0 (a=0)，此时的概率密度函数为：**

$$p(n) = \frac{1}{\sqrt{2\pi}\sigma_n} \exp\left(-\frac{n^2}{2\sigma_n^2} \right)$$

- **此时噪声的方差为：**

$$\sigma_n^2 = E[|n(t)|^2] = R_n(0) = P_n$$ **方差等于平均功率**

- **高斯信号经线性运算(加、减、积分、微分) 后，其结果仍是随机信号**

二、白噪声

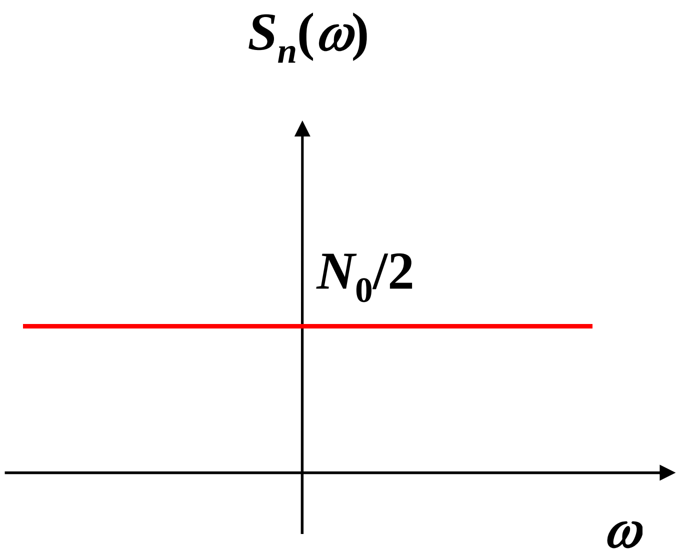
- **白噪声(white noise)：功率谱密度函数为常数的噪声**

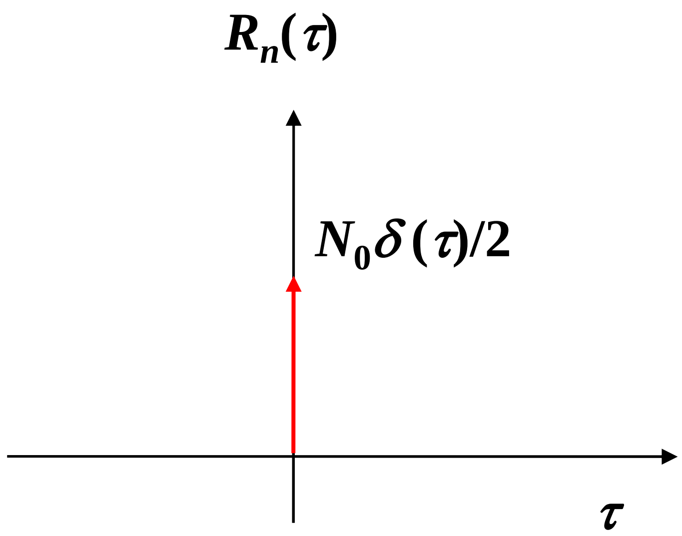
$$S_n(\omega) = \frac{N_0}{2}$$

- **白噪声的自相关函数为：**

$$R_n(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_n(\omega)e^{j\omega\tau}d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{N_0}{2}e^{j\omega\tau}d\omega=\frac{N_0}{2}\delta(\tau)$$

不同时刻的白噪声取值总不相关





- **白噪声的平均功率为：**

$$P_n = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_n(\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{N_0}{2}d\omega \to \infty$$

白噪声实际不存在，常又称为理想白噪声

- **理想白噪声经过实际系统时，其频带受到系统带宽的限制⇒一定频带内功率谱密度为常数、此频带外功率谱密度为0的随机噪声(带限白噪声)**

- **带限白噪声主要有两类：理想低通白噪声、理想带通白噪声**

1、理想低通白噪声

- **理想低通白噪声的功率谱密度函数为：**

$$S_n(\omega) = \frac{N_0}{2} rect\left(\frac{\omega}{2\omega_c}\right)$$

其中ω_c为理想低通白噪声的带宽

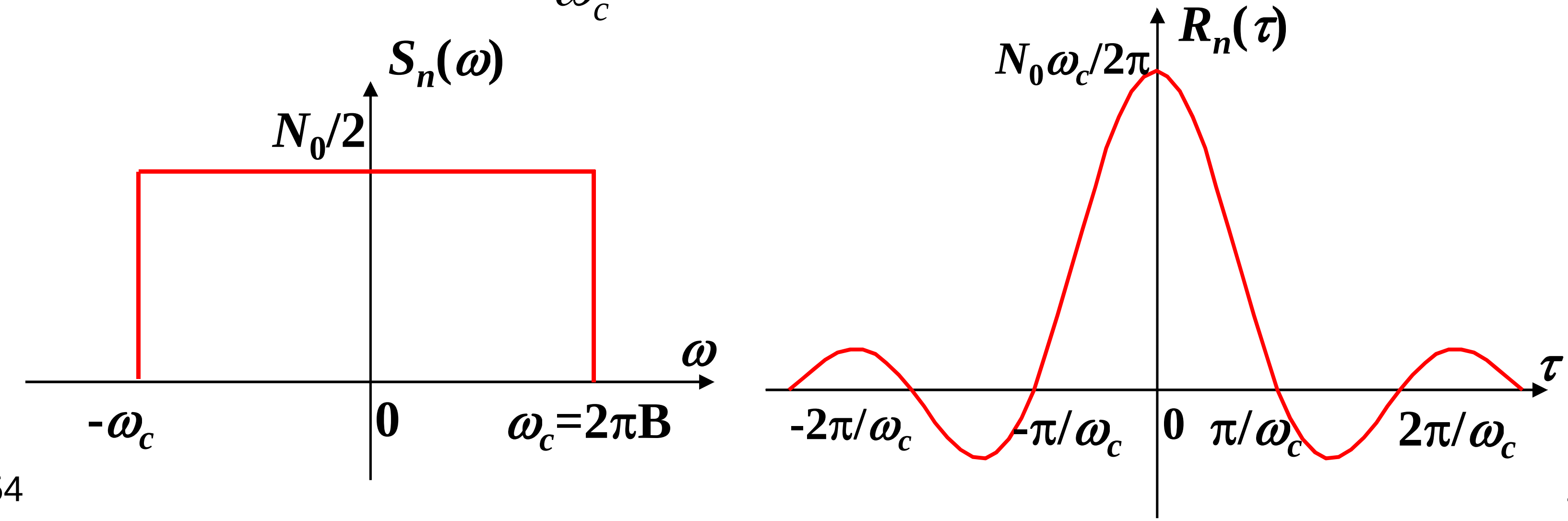
- **理想低通白噪声的自相关函数为：**

$$R_n(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_n(\omega)e^{j\omega\tau}d\omega=\frac{1}{2\pi}\int_{-\omega_c}^{\omega_c}\frac{N_0}{2}e^{j\omega\tau}d\omega$$

$$=\frac{1}{2\pi}\cdot\frac{N_0}{2}\cdot\frac{1}{j\tau}e^{j\omega\tau}\Big|_{-\omega_c}^{\omega_c}=\frac{1}{2\pi}\cdot\frac{N_0}{2}\cdot\frac{1}{j\tau}(2j\sin\omega_c\tau)$$

$$=\frac{1}{2\pi}N_0\omega_c\cdot\frac{\sin\omega_c\tau}{\omega_c\tau}=\frac{1}{2\pi}N_0\omega_c Sa(\omega_c\tau)$$

相隔时间 $\tau=n\Delta\tau=\dfrac{n\pi}{\omega_c}$ **的理想低通白噪声的取值不相关**



- **理想低通白噪声的平均功率为：**

$$P_n = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_n(\omega)d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} \frac{N_0}{2}d\omega = \frac{\omega_c}{2\pi}N_0 = BN_0$$

其中*B*为单位为**Hz的带宽**

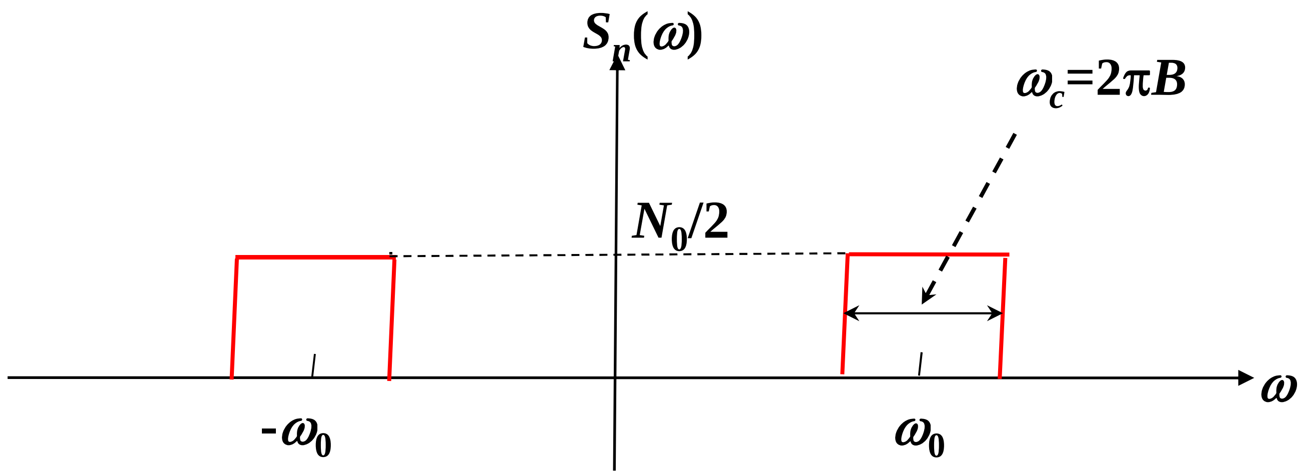
2、理想带通白噪声

- **理想带通白噪声的功率谱密度函数为：**

$$S_n(\omega) = \frac{N_0}{2} rect\left(\frac{\omega-\omega_0}{\omega_c}\right) + \frac{N_0}{2} rect\left(\frac{\omega+\omega_0}{\omega_c}\right)$$

其中ω_0为理想带通白噪声的中心频率

ω_c为理想带通白噪声的带宽



$$\omega_0 > \frac{\omega_c}{2}$$

- **理想带通白噪声的自相关函数为：**

$$R_n(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_n(\omega) e^{j\omega\tau} d\omega$$

$$= \frac{1}{2\pi}\int_{\omega_0-\omega_c/2}^{\omega_0+\omega_c/2} \frac{N_0}{2} e^{j\omega\tau} d\omega + \frac{1}{2\pi}\int_{-\omega_0-\omega_c/2}^{-\omega_0+\omega_c/2} \frac{N_0}{2} e^{j\omega\tau} d\omega$$

$$= \frac{1}{2\pi}\cdot\frac{N_0}{2}\cdot\frac{1}{j\tau} e^{j\omega\tau}\Big|_{\omega_0-\omega_c/2}^{\omega_0+\omega_c/2} + \frac{1}{2\pi}\cdot\frac{N_0}{2}\cdot\frac{1}{j\tau} e^{j\omega\tau}\Big|_{-\omega_0-\omega_c/2}^{-\omega_0+\omega_c/2}$$

$$\Rightarrow \quad R_n(\tau) = \frac{1}{2\pi} \cdot \frac{N_0}{2} \cdot \frac{1}{j\tau} [e^{j(\omega_0+\omega_c/2)\tau} - e^{j(\omega_0-\omega_c/2)\tau}$$

$$+ e^{j(-\omega_0+\omega_c/2)\tau} - e^{j(-\omega_0-\omega_c/2)\tau}]$$

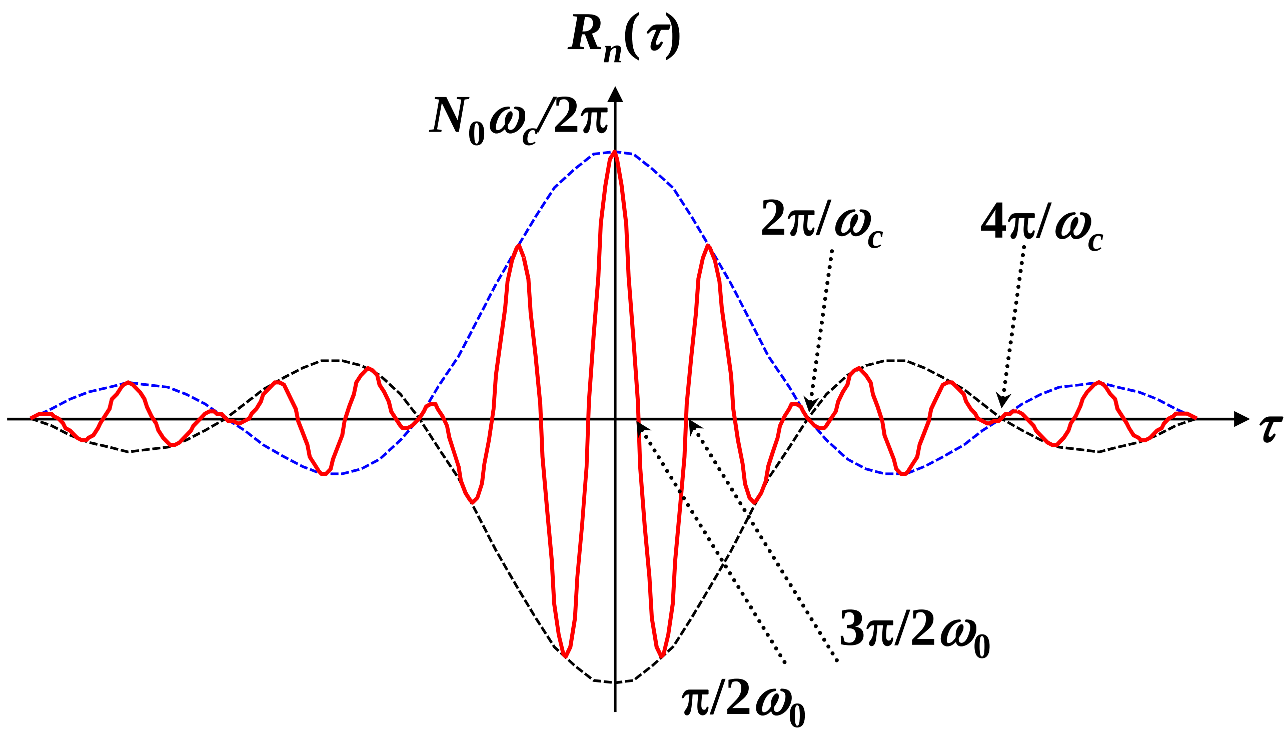
$$= \frac{1}{2\pi} \cdot \frac{N_0}{2} \cdot \frac{1}{j\tau} \{2j \sin[(\omega_0 + \omega_c / 2)\tau]$$

$$- 2j \sin[(\omega_0 - \omega_c / 2)\tau]\}$$

$$= \frac{1}{2\pi} \cdot \frac{N_0}{\tau} [2\cos\omega_0\tau \sin(\omega_c\tau / 2)]$$

$$= \frac{1}{2\pi} N_0\omega_c \cdot \frac{\sin(\omega_c\tau / 2)}{\omega_c\tau / 2} \cos\omega_0\tau$$

$$= \frac{1}{2\pi} N_0\omega_c Sa\left(\frac{\omega_c\tau}{2}\right) \cos\omega_0\tau$$



- **理想带通白噪声的自相关函数以Sa($\omega_c\tau/2$)为包络**
- 理想带通白噪声的自相关函数与**cos$\omega_0\tau$**相同间隔出现零点
- 相隔时间 $\tau = \dfrac{(2n+1)\pi}{2\omega_0}$ 或 $\tau = n\Delta\tau = \dfrac{2n\pi}{\omega_c}$ 理想带通白噪声的取值不相关

三、高斯一马尔可夫(Markov)信号

- **高斯－马尔可夫信号：自相关函数为指数型的平稳高斯信号**

$$R_n(\tau) = \sigma^2 e^{-\beta|\tau|}$$

其中σ²为信号的均方值 $\sigma^2 = R_n(0) = E[|n(t)|^2]$

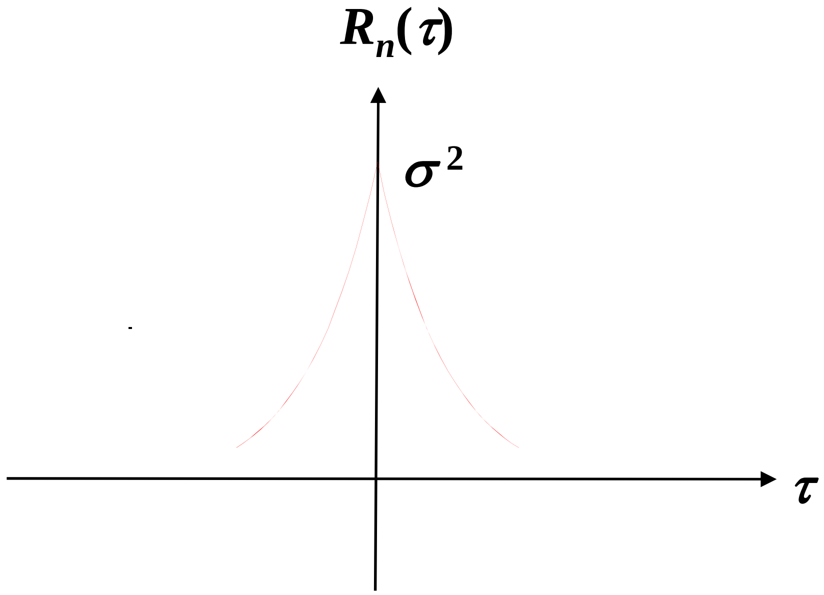
- **高斯－马尔可夫信号的功率谱密度函数为：**

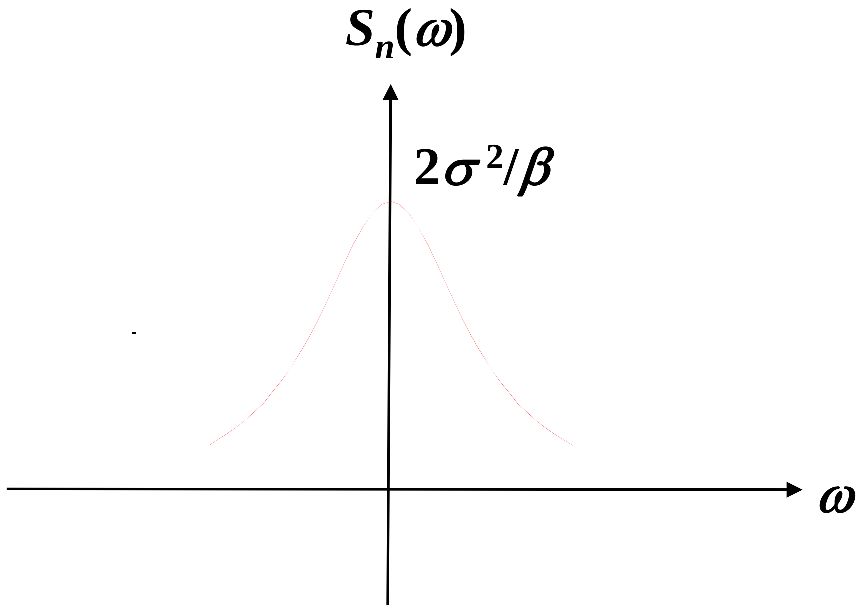
$$S_n(\omega) = \int_{-\infty}^{\infty} R_n(\tau) e^{-j\omega\tau} d\tau = \int_{-\infty}^{\infty} \sigma^2 e^{-\beta|\tau|} e^{-j\omega\tau} d\tau$$

$$= \sigma^2 \left[\int_0^{\infty} e^{-\beta\tau} e^{-j\omega\tau} d\tau + \int_{-\infty}^{0} e^{\beta\tau} e^{-j\omega\tau} d\tau \right]$$

$$= \sigma^2 \left[\frac{1}{-\beta - j\omega} e^{-\beta\tau} e^{-j\omega\tau} \Big|_0^{\infty} + \frac{1}{\beta - j\omega} e^{\beta\tau} e^{-j\omega\tau} \Big|_{-\infty}^{0} \right]$$

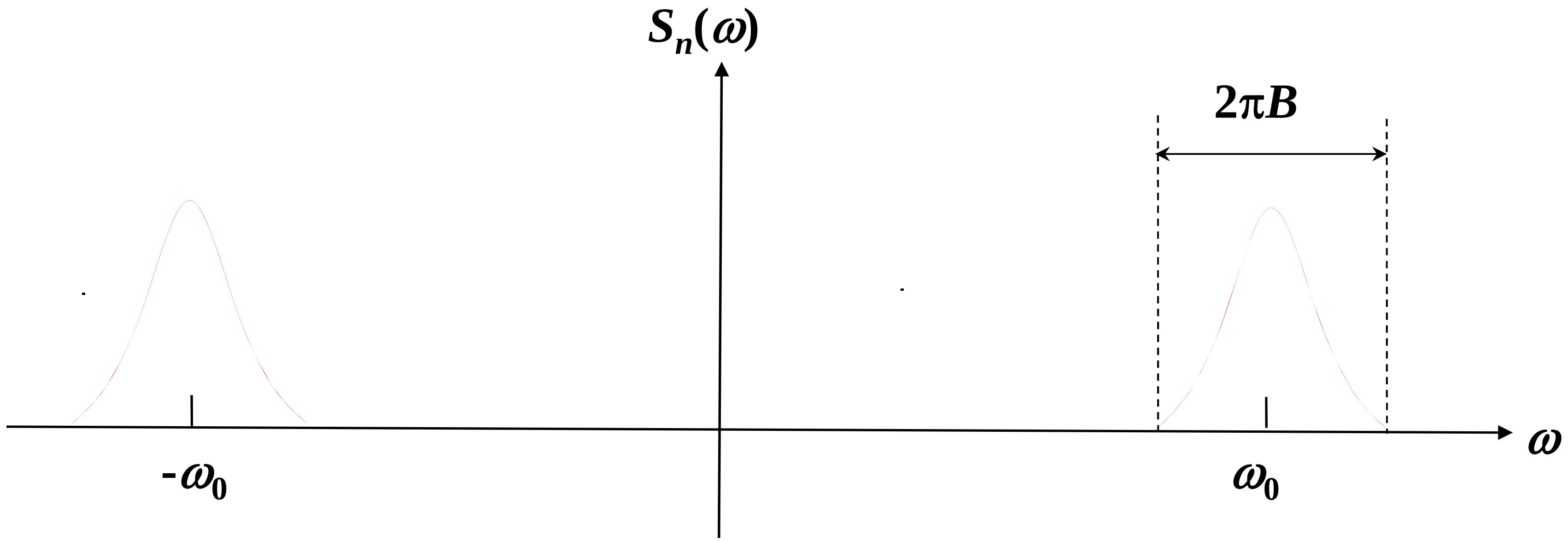
$$\therefore \quad S_n(\omega) = \sigma^2\left[0 - \frac{1}{-\beta - j\omega} + \frac{1}{\beta - j\omega} - 0\right] = \frac{2\sigma^2\beta}{\beta^2 + \omega^2}$$



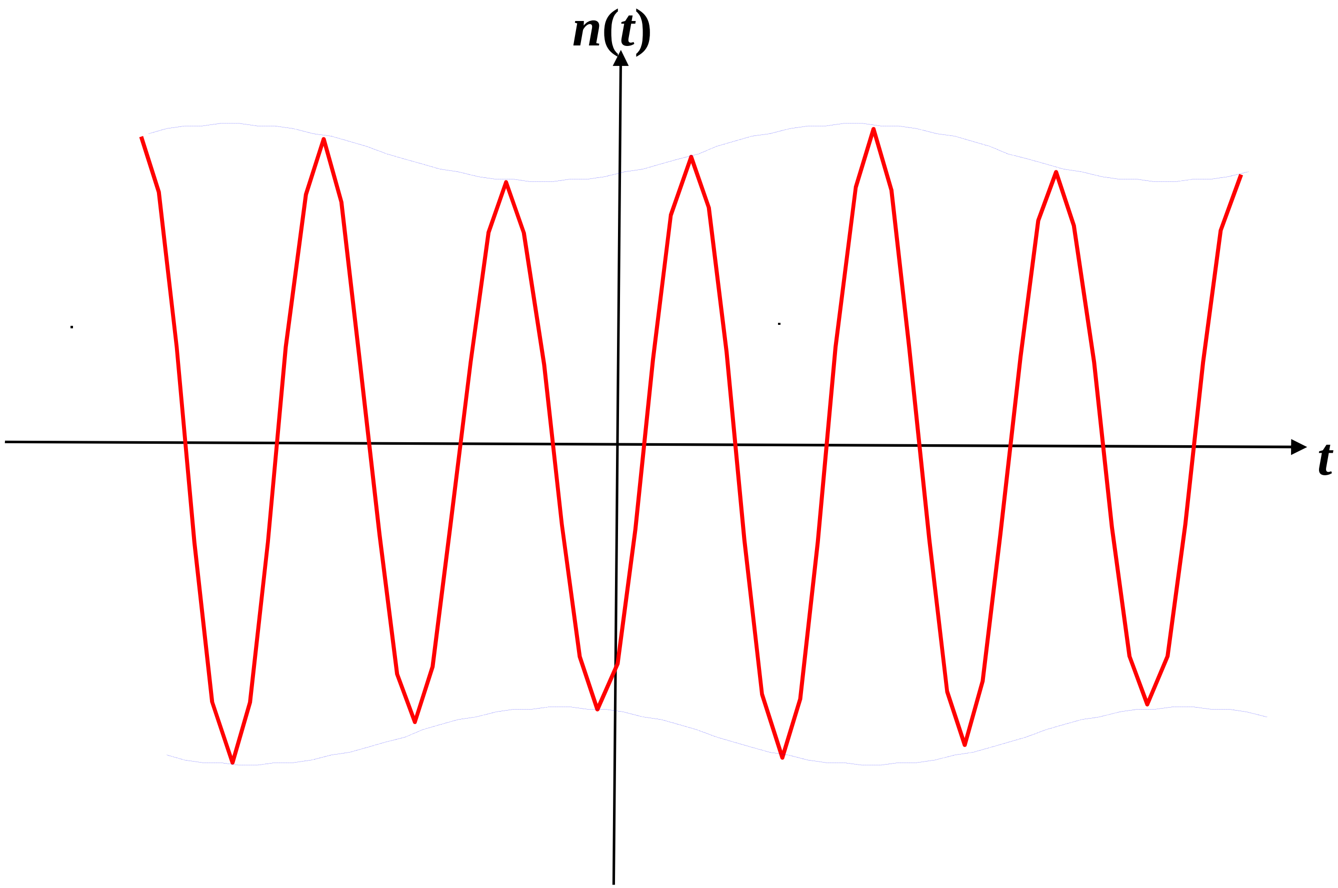


四、窄带高斯噪声

- **窄带系统**：带宽B远小于中心频率f_0的带通系统
- **窄带高斯噪声**：高斯白噪声经过窄带系统的输出



- **窄带高斯噪声的频谱分量集中在频率f_0附近，其样本函数很像一个包络和相位随时间缓慢变换的正弦波**



- **窄带高斯噪声可以表示为：** $n(t) = \rho_n(t)\cos[\omega_0 t + \varphi_n(t)]$

  其中$p_n(t)$为包络，$\varphi(t)$为相位

  它们均为随机信号，变化比$\cos\omega_0 t$缓慢得多

$$\because \quad n(t) = \rho_n(t)\cos\varphi_n(t)\cos\omega_0 t - \rho_n(t)\sin\varphi_n(t)\sin\omega_0 t$$

$$= n_c(t)\cos\omega_0 t - n_s(t)\sin\omega_0 t$$

$$n_c(t) = \rho_n(t)\cos\varphi_n(t)$$

$$n_s(t) = \rho_n(t)\sin\varphi_n(t)$$

- **窄带高斯噪声*n*(*t*)的同相分量：$n_c(t)$**
- **窄带高斯噪声*n*(*t*)的正交分量：$n_s(t)$**

它们均为随机信号，变化比$\cos\omega_0 t$缓慢得多

$$\rho_n(t) = \sqrt{n_c^2(t) + n_s^2(t)}$$

$$\varphi_n(t) = tg^{-1}\frac{n_s(t)}{n_c(t)}$$

- **设窄带高斯噪声$n(t)$是平稳的随机信号，其均值为0，方差为σ^2，则振幅分布为：**

$$p(n)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{n^2}{2\sigma^2}\right)$$

- **下面讨论同相分量$n_c(t)$与正交分量$n_s(t)$的统计特性**

$$\begin{aligned}E[n(t)]&=E[n_c(t)\cos\omega_0 t-n_s(t)\sin\omega_0 t]\\&=E[n_c(t)]\cos\omega_0 t-E[n_s(t)]\sin\omega_0 t\end{aligned}$$

$n(t)$是平稳的，均值为0 $\Rightarrow$ 对任何时刻t，$E[n(t)]=0$

$$\therefore\quad E[n_c(t)]=E[n_s(t)]=0$$

$$R_n(\tau)=E[n(t)n(t-\tau)]=E\{[n_c(t)\cos\omega_0 t-n_s(t)\sin\omega_0 t]$$
$$\cdot[n_c(t-\tau)\cos\omega_0(t-\tau)-n_s(t-\tau)\sin\omega_0(t-\tau)]\}$$
$$=E[n_c(t)n_c(t-\tau)]\cos\omega_0 t\cos\omega_0(t-\tau)$$
$$-E[n_s(t)n_c(t-\tau)]\sin\omega_0 t\cos\omega_0(t-\tau)$$
$$-E[n_c(t)n_s(t-\tau)]\cos\omega_0 t\sin\omega_0(t-\tau)$$
$$+E[n_s(t)n_s(t-\tau)]\sin\omega_0 t\sin\omega_0(t-\tau)$$
$$=R_{n_c}(t,t-\tau)\cos\omega_0 t\cos\omega_0(t-\tau)$$
$$-R_{n_s n_c}(t,t-\tau)\sin\omega_0 t\cos\omega_0(t-\tau)$$
$$-R_{n_c n_s}(t,t-\tau)\cos\omega_0 t\sin\omega_0(t-\tau)$$
$$+R_{n_s}(t,t-\tau)\sin\omega_0 t\sin\omega_0(t-\tau)$$

$n(t)$是平稳的 $\Rightarrow$ **$R_n(\tau)$与t 无关** $\Rightarrow$ 取**$t=0$**，则有：

$$R_n(\tau) = R_{n_c}(t, t-\tau)\cos\omega_0\tau + R_{n_c n_s}(t, t-\tau)\sin\omega_0\tau$$

显然要求 $R_{n_c}(\tau)$**和**$R_{n_s}(\tau)$**均与**t **无关**，则有：

$$R_{n_c}(\tau) = R_{n_c}(t, t-\tau)$$

$$R_{n_c n_s}(\tau) = R_{n_c n_s}(t, t-\tau)$$

$$\therefore \quad R_n(\tau) = R_{n_c}(\tau)\cos\omega_0\tau + R_{n_c n_s}(\tau)\sin\omega_0\tau$$

同理取$t=\pi/2\omega_0$，则有：

$$R_n(\tau) = R_{n_s}(t, t-\tau)\cos\omega_0\tau - R_{n_s n_c}(t, t-\tau)\sin\omega_0\tau$$

$$\therefore \quad R_n(\tau) = R_{n_s}(\tau)\cos\omega_0\tau - R_{n_s n_c}(\tau)\sin\omega_0\tau$$

$$R_{n_s}(\tau) = R_{n_s}(t, t-\tau) \qquad R_{n_s n_c}(t, t-\tau) = R_{n_s n_c}(\tau)$$

$$R_n(\tau)=R_{n_c}(\tau)\cos\omega_0\tau+R_{n_cn_s}(\tau)\sin\omega_0\tau$$

and $$R_n(\tau)=R_{n_s}(\tau)\cos\omega_0\tau-R_{n_sn_c}(\tau)\sin\omega_0\tau$$

它们对所有τ均成立，则有：

$$R_{n_c}(\tau)=R_{n_s}(\tau) \qquad R_{n_cn_s}(\tau)=-R_{n_sn_c}(\tau)$$

$$R_{n_cn_s}(\tau)=E[n_c(t)n_s(t-\tau)]=E[n_s(t-\tau)n_c(t)]$$

$$=E[n_s(t')n_c(t'+\tau)]=R_{n_sn_c}(-\tau)$$

$$\therefore\quad R_{n_sn_c}(-\tau)=-R_{n_sn_c}(\tau)$$

$R_{n_sn_c}(\tau)$是τ的奇函数 $\qquad \therefore\quad R_{n_sn_c}(0)=0$

同理得： $\qquad R_{n_cn_s}(0)=0$

$n_c(t)$和$n_s(t)$在同一时刻的取值互不相关

当τ=0

$$R_n(\tau)=R_{n_c}(\tau)\cos\omega_0\tau+R_{n_cn_s}(\tau)\sin\omega_0\tau \quad\Rightarrow\quad R_n(0)=R_{n_c}(0)$$

$$R_n(\tau)=R_{n_s}(\tau)\cos\omega_0\tau-R_{n_sn_c}(\tau)\sin\omega_0\tau \quad\Rightarrow\quad R_n(0)=R_{n_s}(0)$$

$$\because\quad E[n_c(t)]=E[n_s(t)]=E[n(t)]=0$$

$$\therefore\quad E[n_c^2(t)]=E[n_s^2(t)]=E[n^2(t)]=\sigma^2$$

$n_c(t)$和$n_s(t)$的方差均为σ^2

$n_c(t)$和$n_s(t)$的自相关函数与t无关

$n_c(t)$和$n_s(t)$是平稳的

$$n(t) = n_c(t)\cos\omega_0 t - n_s(t)\sin\omega_0 t$$

$$\xrightarrow{t_1=0} \quad n(t_1) = n_c(t_1)$$

$$\xrightarrow{t_2=\frac{3\pi}{2\omega_0}} \quad n(t_2) = n_s(t_2)$$

$n(t)$是平稳的 $\Rightarrow$ $n(t_1)$和$n(t_2)$高斯分布 $\Rightarrow$ $n_c(t_1)$和$n_s(t_2)$高斯分布

$n_c(t)$和$n_s(t)$也是平稳的

$$\therefore \quad p(n_c) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{n_c^2}{2\sigma^2}\right)$$

$$p(n_s) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{n_s^2}{2\sigma^2}\right)$$

- **$n_c(t)$和$n_s(t)$均值均为0，方差均为σ^2，同一时刻取值不相关**

$n_c(t)$和$n_s(t)$互相独立

$$\therefore \quad p(n_c, n_s) = p(n_c)p(n_s)$$

$$= \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{n_c^2}{2\sigma^2}\right)\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{n_s^2}{2\sigma^2}\right)$$

$$= \frac{1}{2\pi\sigma^2}\exp\left(-\frac{n_c^2+n_s^2}{2\sigma^2}\right)$$

- 根据概率论联合概率密度的关系：

$$p(\rho_n, \varphi_n) = \begin{vmatrix} \dfrac{\partial n_c}{\partial \rho_n} & \dfrac{\partial n_s}{\partial \rho_n} \\ \dfrac{\partial n_c}{\partial \varphi_n} & \dfrac{\partial n_s}{\partial \varphi_n} \end{vmatrix} p(n_c, n_s)$$

$$\because \quad n_c(t) = \rho_n(t)\cos\varphi_n(t), \quad n_s(t) = \rho_n(t)\sin\varphi_n(t)$$

$$\therefore \quad \begin{vmatrix} \dfrac{\partial n_c}{\partial \rho_n} & \dfrac{\partial n_s}{\partial \rho_n} \\ \dfrac{\partial n_c}{\partial \varphi_n} & \dfrac{\partial n_s}{\partial \varphi_n} \end{vmatrix} = \begin{vmatrix} \cos\varphi_n & \sin\varphi_n \\ -\rho_n \sin\varphi_n & \rho_n \cos\varphi_n \end{vmatrix}$$

$$= \rho_n \cos^2\varphi_n + \rho_n \sin^2\varphi_n = \rho_n$$

$$\therefore \quad p(\rho_n, \varphi_n) = \begin{vmatrix} \dfrac{\partial n_c}{\partial \rho_n} & \dfrac{\partial n_s}{\partial \rho_n} \\ \dfrac{\partial n_c}{\partial \varphi_n} & \dfrac{\partial n_s}{\partial \varphi_n} \end{vmatrix} p(n_c, n_s) = \frac{\rho_n}{2\pi\sigma^2} \exp\left(-\frac{n_c^2 + n_s^2}{2\sigma^2} \right)$$

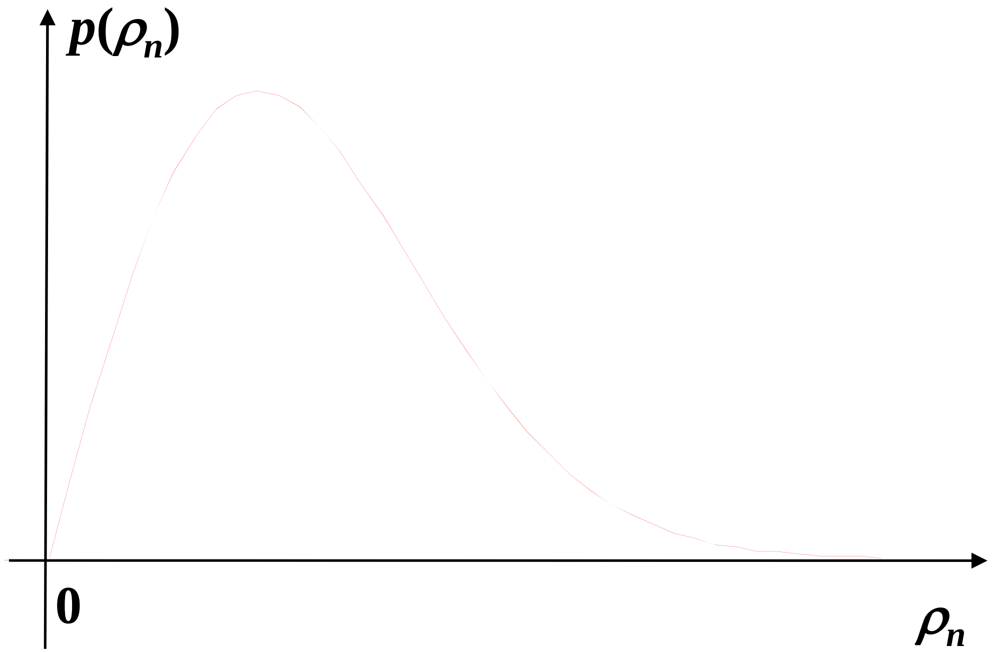
$$= \frac{\rho_n}{2\pi\sigma^2} \exp\left(-\frac{\rho_n^2}{2\sigma^2} \right), \quad \rho_n \geq 0, \quad 0 \leq \varphi_n \leq 2\pi$$

- **包络$\rho_n(t)$的一维概率密度(边缘分布)为：**

$$p(\rho_n)=\int_0^{2\pi}p(\rho_n,\varphi_n)d\varphi_n=\int_0^{2\pi}\frac{\rho_n}{2\pi\sigma^2}\exp\left(-\frac{\rho_n^2}{2\sigma^2}\right)d\varphi_n$$

$$=\frac{\rho_n}{\sigma^2}\exp\left(-\frac{\rho_n^2}{2\sigma^2}\right)$$

包络是瑞利分布



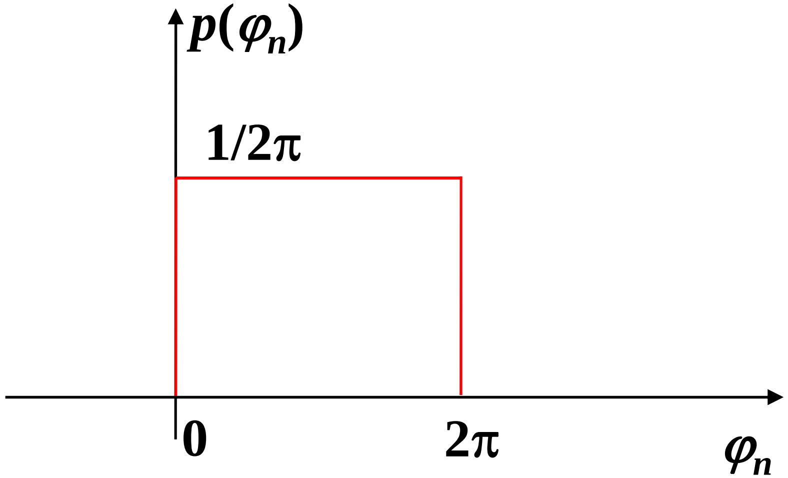
- **相位$\varphi_n(t)$的一维概率密度(边缘分布)为：**

$$p(\varphi_n)=\int_0^{\infty}p(\rho_n,\varphi_n)d\rho_n=\int_0^{\infty}\frac{\rho_n}{2\pi\sigma^2}\exp\left(-\frac{\rho_n^2}{2\sigma^2}\right)d\rho_n$$

$$=-\int_0^{\infty}\frac{1}{2\pi}\exp\left(-\frac{\rho_n^2}{2\sigma^2}\right)d\left(-\frac{\rho_n^2}{2\sigma^2}\right)$$

$$=-\frac{1}{2\pi}\exp\left(-\frac{\rho_n^2}{2\sigma^2}\right)\Bigg|_0^{\infty}=\frac{1}{2\pi}$$

相位是均匀分布



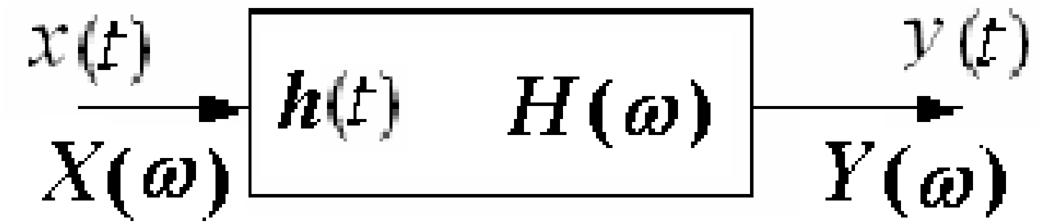
$$\therefore \quad p(\rho_n, \varphi_n) = p(\rho_n) p(\varphi_n)$$

$\rho_n(t)$和$\varphi_n(t)$互相独立

§ 2.4 随机信号通过线性时不变系统

一、输出信号的数字特征

- **确定性信号通过线性时不变系统**



$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau, \quad Y(\omega) = X(\omega) H(\omega)$$

- **随机信号不能作傅里叶变换 → 不能直接套用上述公式**
- **对随机信号$X(t)$的一个样本函数$x(t)$而言，其通过线性时不变系统的输出是$Y(t)$的一个样本函数$y(t)$，则有：**

$$y(t)=h(t)*x(t)=\int_{-\infty}^{\infty}h(\tau)x(t-\tau)d\tau$$

$$\therefore \quad Y(t)=h(t)*X(t)=\int_{-\infty}^{\infty}h(\tau)X(t-\tau)d\tau$$

- 当随机信号$X(t)$平稳时，输出信号的集均值为：

$$E[Y(t)]=E\left[\int_{-\infty}^{\infty}h(\tau)X(t-\tau)d\tau\right]=\int_{-\infty}^{\infty}h(\tau)E[X(t-\tau)]d\tau$$

$$=\int_{-\infty}^{\infty}h(\tau)E[X(t)]d\tau=E[X(t)]\int_{-\infty}^{\infty}h(\tau)d\tau$$

$$H(\omega)=\int_{-\infty}^{\infty}h(\tau)e^{-j\omega\tau}d\tau \quad \Rightarrow \quad H(\omega)\Big|_{\omega=0}=\int_{-\infty}^{\infty}h(\tau)d\tau$$

$$\therefore \quad E[Y(t)] = E[X(t)]H(\omega)\Big|_{\omega=0}$$

输出信号的集均值与时间*t* 无关

- 当随机信号***X*(*t*)**平稳时，输出信号的自相关函数为：

$$R_Y(t, t-\tau) = E[Y(t)Y^*(t-\tau)]$$

$$= E\left\{\int_{-\infty}^{\infty} h(u)X(t-u)du \left[\int_{-\infty}^{\infty} h(v)X(t-\tau-v)dv\right]^*\right\}$$

$$= E\left[\int_{-\infty}^{\infty} h(u)X(t-u)du \cdot \int_{-\infty}^{\infty} h^*(v)X^*(t-\tau-v)dv\right]$$

$$= E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u)h^*(v)X(t-u)X^*(t-\tau-v)dudv\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u)h^*(v)E[X(t-u)X^*(t-\tau-v)]dudv$$

$$\therefore \quad R_Y(t,t-\tau)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h(u)h^*(v)R_X(\tau-u+v)\,dudv$$

- 输出随机信号***Y*(*t*)**的自相关函数与时间***t***无关，则有：

$$R_Y(\tau)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h(u)h^*(v)R_X(\tau-u+v)\,dudv$$

- 令***v*=*u*-*l***，则有：

$$R_Y(\tau)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h(u)h^*(u-l)R_X(\tau-l)\,dudl$$

$$=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}h(u)h^*(u-l)du\right]R_X(\tau-l)dl$$

$$\because \quad R_h(l)=\int_{-\infty}^{\infty}h(u)h^*(u-l)du$$ **冲激响应的自相关函数**

$$\therefore \quad R_Y(\tau)=\int_{-\infty}^{\infty}R_h(l)R_X(\tau-l)dl=R_h(\tau)*R_X(\tau)$$

- 输出随机信号**Y(t)**的均方为：

$$E[|Y(t)|^2]=E[Y(t)Y^*(t)]=R_Y(0)=\int_{-\infty}^{\infty}R_h(l)R_X(-l)dl$$

输出信号的均方与时间*t*无关

- 输出随机信号**Y(t)**的方差也与时间*t*无关，且为：

$$D[Y(t)]=E[|Y(t)|^2]-\{E[Y(t)]\}^2$$

- 结论：广义平稳随机信号通过线性时不变系统，其输出也是广义平稳的

$$\because \quad R_Y(\tau)=R_h(\tau)*R_X(\tau)$$

$$R_Y(\tau)\overset{F}{\longleftrightarrow}S_Y(\omega)\quad R_X(\tau)\overset{F}{\longleftrightarrow}S_X(\omega)$$

$$\text{if}\quad R_h(\tau)\overset{F}{\longleftrightarrow}S_h(\omega)\quad\Rightarrow\quad S_Y(\omega)=S_X(\omega)S_h(\omega)$$

$$\because \quad R_h(l)=\int_{-\infty}^{\infty} h(u)h^*(u-l)du=\int_{-\infty}^{\infty} h(u)h^*[-(l-u)]du$$

$$=h(l)*h^*(-l)$$

$$\int_{-\infty}^{\infty} h^*(-l)e^{-j\omega l}dl=\left[\int_{-\infty}^{\infty} h(-l)e^{j\omega l}dl\right]^*=\left[\int_{-\infty}^{\infty} h(-l)e^{-j\omega(-l)}dl\right]^*$$

$$=\left[\int_{-\infty}^{\infty} h(u)e^{-j\omega u}du\right]^*=H^*(\omega)$$

$$\therefore \quad S_h(\omega)=H(\omega)H^*(\omega)=|H(\omega)|^2$$

$$\Rightarrow \quad S_Y(\omega)=S_X(\omega)|H(\omega)|^2$$

输出信号的功率谱等于输入信号的功率谱和系统幅频响应平方的乘积

二、输出与输入随机信号间的互相关

- **输出随机信号和输入随机信号的互相关函数为：**

$$R_{YX}(\tau)=E[Y(t)X^*(t-\tau)]=E\left[\int_{-\infty}^{\infty}h(u)X(t-u)du\cdot X^*(t-\tau)\right]$$

$$=E\left[\int_{-\infty}^{\infty}h(u)X(t-u)X^*(t-\tau)du\right]$$

$$=\int_{-\infty}^{\infty}h(u)E[X(t-u)X^*(t-\tau)]du$$

$$=\int_{-\infty}^{\infty}h(u)R_X(\tau-u)du=R_X(\tau)*h(\tau)$$

- **输出随机信号和输入随机信号的互功率谱密度函数为：**

$$S_{YX}(\omega)=S_X(\omega)H(\omega)$$

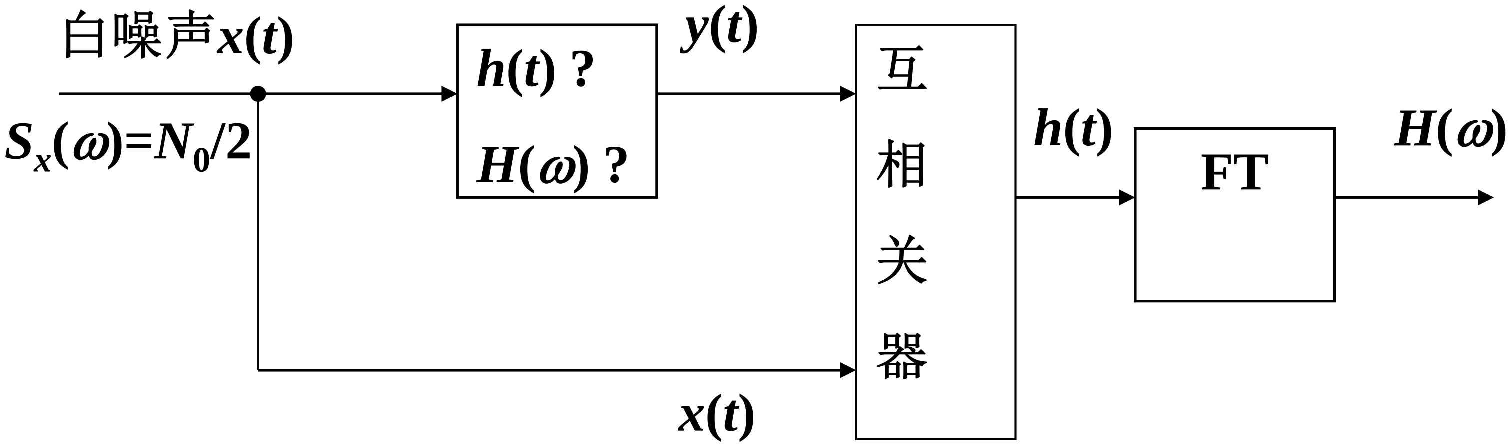
- **当输入随机信号为白噪声时：**

$$S_X(\omega)=\frac{N_0}{2},\quad R_X(\tau)=\frac{N_0}{2}\delta(\tau)$$

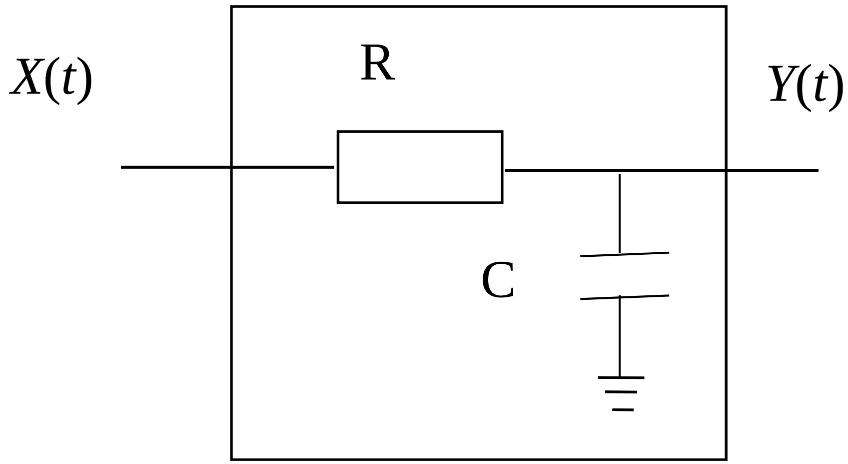
$$\therefore\quad S_{YX}(\omega)=S_X(\omega)H(\omega)=\frac{N_0}{2}H(\omega)$$

$$R_{YX}(\tau)=R_X(\tau)*h(\tau)=\frac{N_0}{2}\delta(\tau)*h(\tau)=\frac{N_0}{2}h(\tau)$$

- **测定未知系统冲激响应和频率函数的方法框图：**



例1 **求白噪声通过RC低通电路的输出信号的自相关函数和功率谱。**



解： RC电路的冲激响应和频率函数为：

$$h(t) = ae^{-at}u(t)$$

$$H(\omega) = \frac{a}{a + j\omega}$$ 其中 $a = \frac{1}{\text{RC}}$

白噪声的功率谱密度和自相关函数为：

$$S_X(\omega) = \frac{N_0}{2}, \quad R_X(\tau) = \frac{N_0}{2}\delta(\tau)$$

$$|H(\omega)|^2 = H(\omega)H^*(\omega) = \frac{a}{a+j\omega}\cdot\frac{a}{a-j\omega} = \frac{a^2}{a^2+\omega^2}$$

$$\therefore\quad S_Y(\omega) = S_X(\omega)|H(\omega)|^2 = \frac{N_0}{2}\cdot\frac{a^2}{a^2+\omega^2}$$

$$\because\quad e^{-a|t|} \overset{F}{\longleftrightarrow} \frac{2a}{a^2+\omega^2}$$

$$S_Y(\omega) = \frac{N_0}{2}\cdot\frac{a^2}{a^2+\omega^2} \quad\Rightarrow\quad R_Y(\tau) = \frac{aN_0}{4}e^{-a|\tau|}$$

自相关函数还可以从卷积计算得到

$$\because\quad h(t) = ae^{-at}u(t)$$

$$R_h(\tau) = \int_{-\infty}^{\infty} h(v)h^*(v-\tau)dv = \int_{-\infty}^{\infty} ae^{-av}u(v)ae^{-a(v-\tau)}u(v-\tau)dv$$

$$\text{when } \tau \ge 0,\quad R_h(\tau) = a^2e^{a\tau}\int_{\tau}^{\infty} e^{-2av}dv = a^2e^{a\tau}\left(-\frac{1}{2a}e^{-2av}\right)\Bigg|_{\tau}^{\infty}$$

$$R_h(\tau) = a^2 e^{a\tau} \cdot \left(\frac{1}{2a} e^{-2a\tau} \right) = \frac{1}{2} a e^{-a\tau}$$

$$\text{when } \tau < 0, \quad R_h(\tau) = a^2 e^{a\tau} \int_0^{\infty} e^{-2av} dv = a^2 e^{a\tau} \left(-\frac{1}{2a} e^{-2av} \right) \Bigg|_0^{\infty}$$

$$= a^2 e^{a\tau} \cdot \frac{1}{2a} = \frac{1}{2} a e^{a\tau}$$

$$\therefore \quad R_h(\tau) = \frac{1}{2} a e^{-a|\tau|}$$

$$\Rightarrow \quad R_Y(\tau) = R_h(\tau) * R_X(\tau) = \frac{N_0}{2} \delta(\tau) * \frac{1}{2} a e^{-a|\tau|} = \frac{aN_0}{4} e^{-a|\tau|}$$

输出信号的平均功率为：

$$E[|Y(t)|^2] = R_Y(0) = \frac{aN_0}{4}$$

- **例1中输出随机信号的功率谱为：**

$$S_Y(\omega)=\frac{N_0}{2}\cdot\frac{a^2}{a^2+\omega^2}$$

- **例1引申出：功率谱一般可表示为两个有理多项式的分式**

$$S_Y(\omega)=\frac{P(\omega)}{Q(\omega)}$$

- **实信号的功率谱是实偶函数 $\Rightarrow$ 多项式只含ω的偶次项**

$$S_Y(\omega)=\frac{P(\omega^2)}{Q(\omega^2)}=\frac{a_{2N}\omega^{2N}+a_{2N-2}\omega^{2N-2}+\cdots+a_2\omega^2+a_0}{b_{2M}\omega^{2M}+b_{2M-2}\omega^{2M-2}+\cdots+b_2\omega^2+b_0}$$

- **代入s =jω $\Rightarrow$ 多项式只含s 的偶次项**

$$S_Y(s)=\frac{P(s^2)}{Q(s^2)}$$

- 功率谱的零极点分布有如下特点：
  - 对于P和Q的实根：

$$s^2 = K \quad \Rightarrow \quad s_1 = \sqrt{K},\ s_2 = -\sqrt{K}$$

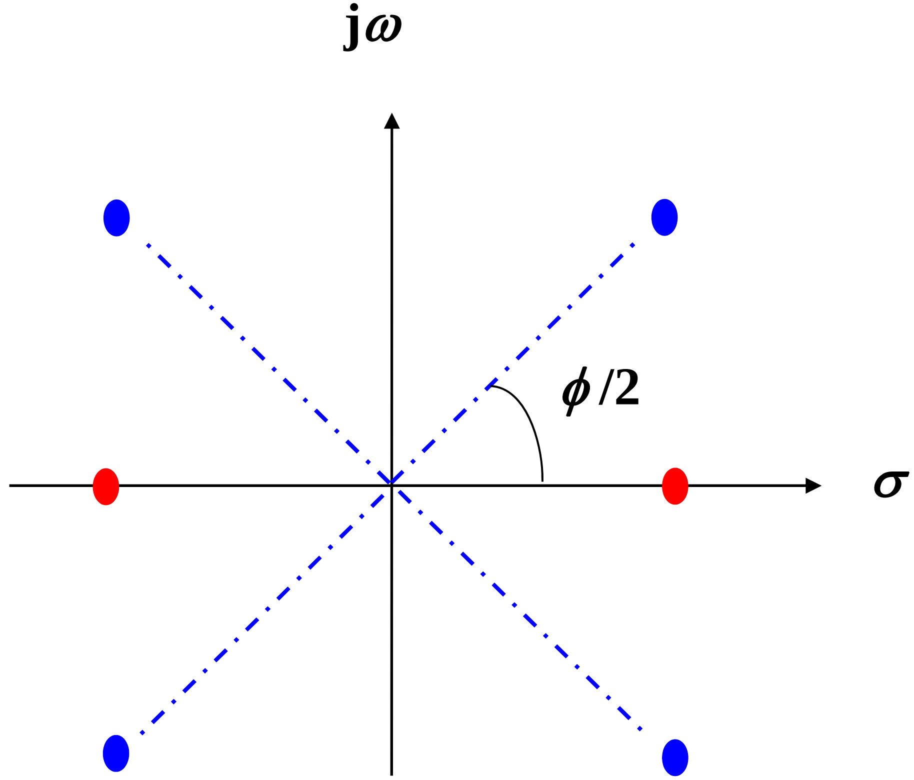
在s平面成对出现在实轴上，且关于虚轴对称

  - 对于P和Q的复根：

$$s^2 = Ae^{\pm j\phi} \quad \Rightarrow \quad s_1 = \sqrt{Ae^{j\frac{\phi}{2}}},\ s_2 = \sqrt{Ae^{j\left(\frac{\phi}{2}+\pi\right)}},$$

$$s_3 = \sqrt{Ae^{-j\frac{\phi}{2}}},\ s_4 = \sqrt{Ae^{-j\left(\frac{\phi}{2}+\pi\right)}}$$

在s平面四个构成一组，相互共轭对称



例2 **自相关函数** $R_X(\tau)=\sigma^2 e^{-\beta|\tau|}$，$\beta>0$ **的随机信号通过RC低通电路，求输出信号的自相关函数和功率谱。**

解： **RC电路的冲激响应和频率函数为：**

$$h(t)=ae^{-at}u(t) \qquad H(\omega)=\frac{a}{a+j\omega} \qquad \text{其中} \quad a=\frac{1}{\mathrm{RC}}$$

$$\therefore \quad |H(\omega)|^2=\frac{a^2}{a^2+\omega^2}$$

$$\because \quad e^{-a|t|} \overset{F}{\longleftrightarrow} \frac{2a}{a^2+\omega^2}$$

输入信号的功率谱密度为：

$$R_X(\tau)=\sigma^2 e^{-\beta|\tau|}, \quad \beta>0 \quad \Rightarrow \quad S_X(\omega)=\frac{2\sigma^2\beta}{\beta^2+\omega^2}$$

$$\therefore \quad S_Y(\omega)=S_X(\omega)|H(\omega)|^2=\frac{2\sigma^2\beta}{\beta^2+\omega^2}\cdot\frac{a^2}{a^2+\omega^2}$$

$$S_Y(\omega)=\frac{2\sigma^2\beta}{\beta^2+\omega^2}\cdot\frac{a^2}{a^2+\omega^2}=\frac{2\sigma^2\beta a^2}{a^2-\beta^2}\left(\frac{1}{\beta^2+\omega^2}-\frac{1}{a^2+\omega^2}\right)$$

$$=\frac{\sigma^2 a}{a^2-\beta^2}\left(\frac{2\beta a}{\beta^2+\omega^2}-\frac{2\beta a}{a^2+\omega^2}\right)$$

$$=\frac{\sigma^2 a}{a^2-\beta^2}\left(a\cdot\frac{2\beta}{\beta^2+\omega^2}-\beta\cdot\frac{2a}{a^2+\omega^2}\right)$$

$$\because \quad e^{-a|t|}\overset{F}{\longleftrightarrow}\frac{2a}{a^2+\omega^2}$$

$$\therefore \quad R_Y(\tau)=\frac{\sigma^2 a}{a^2-\beta^2}\left(ae^{-\beta|\tau|}-\beta e^{-a|\tau|}\right)$$

本例若用时域卷积法计算自相关函数，推导过程比较复杂

例3 **输入到线性时不变系统的平稳随机信号的功率谱和相应的输出信号的功率谱为：**

$$S_X(\omega)=\frac{\omega^2+4}{\omega^2+9},\quad S_Y(\omega)=1$$

系统稳定且为最小相位，求该系统的频率函数。

解： 根据：$S_Y(\omega)=S_X(\omega)|H(\omega)|^2$

$$\therefore\quad |H(\omega)|^2=\frac{S_Y(\omega)}{S_X(\omega)}=\frac{1}{\dfrac{\omega^2+4}{\omega^2+9}}=\frac{\omega^2+9}{\omega^2+4}$$

令$s=j\omega$：则有：

$$|H(\omega)|^2=\frac{-\omega^2-9}{-\omega^2-4}=\frac{(j\omega)^2-9}{(j\omega)^2-4}$$

$$\therefore \quad |H(\omega)|^2 \quad \leftrightarrow \quad G(s)=\frac{s^2-9}{s^2-4}=\frac{(s-3)(s+3)}{(s-2)(s+2)}$$

系统稳定 → 传递函数的极点在左半平面

系统为最小相位 → 传递函数的零点在左半平面

$$\therefore \quad H(s)=\frac{s+3}{s+2}$$

令$s=j\omega$：则有：

$$\Rightarrow \quad H(\omega)=\frac{j\omega+3}{j\omega+2}$$

例4 线性时不变系统是一微分器： $y(t)=\dfrac{dx(t)}{dt}$

平稳随机信号***X*(*t*)**经过该系统，求输出信号的自相关函数、输出信号和输入信号的互相关函数。

解：

$$y(t)=\frac{dx(t)}{dt} \quad \Rightarrow \quad Y(\omega)=j\omega X(\omega)$$

$$\therefore \quad H(\omega)=\frac{Y(\omega)}{X(\omega)}=j\omega$$

$$\Rightarrow \quad S_Y(\omega)=S_X(\omega)|H(\omega)|^2=S_X(\omega)|j\omega|^2$$

$$=\omega^2 S_X(\omega)=-(j\omega)^2 S_X(\omega)$$

傅里叶变换性质：

$$\frac{d^n x(t)}{dt^n} \overset{F}{\longleftrightarrow} (j\omega)^n X(\omega)$$

$$S_Y(\omega) = -(j\omega)^2 S_X(\omega) \quad \Rightarrow \quad R_Y(\tau) = -\frac{d^2 R_X(\tau)}{d\tau^2}$$

类似地推导：

$$S_{YX}(\omega) = S_X(\omega) H(\omega) = j\omega S_X(\omega)$$

$$\therefore \quad R_{YX}(\tau) = \frac{dR_X(\tau)}{d\tau}$$

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