第 29 卷 第 3 期
2014 年 3 月

航空动力学报
Journal of Aerospace Power

Vol. 29 No. 3
Mar. 2014

文章编号:1000-8055(2014)03-0596-08 **doi**:10.13224/j.cnki.jasp.2014.03.016

Statistical properties of fatigue damage of Gaussian random loadings

CHENG Hong-wei, TAO Jun-yong, CHEN Xun, JIANG Yu

(Science and Technology on Integrated Logistics Support Laboratory,
College of Mechatronic Engineering and Automation,
National University of Defense Technology, Changsha 410073, China)

Abstract: Firstly, the fatigue damages associated with the random loadings were always deemed as high-cycle or very-high-cycle fatigue problems, and based on Chebyshev theorem, the number of rainflow cycles in a given time interval could be recognized as a constant by neglecting its randomness. Secondly, the randomness of fatigue damage induced by the distribution of rainflow cycles was analyzed. According to central limit theorem, the fatigue damage could be assumed to follow Gaussian distribution, and the statistical parameters: mean and variance, were derived from Dirlik's solution. Finally, the proposed method was used to a simulate Gaussian random loading and the measured random loading from an aircraft. Comparisons with observed results were carried out extensively. In the first example, the relative errors of the proposed method are 2.29%, 3.52% and 1.16% for the mean, standard deviation and variation coefficient of fatigue damage, respectively. In the second example, these relative errors are 11.70%, 173.32% and 18.20%, and the larger errors are attributable to non-stationary state of the measured loading to some extent.

Key words: random loading; fatigue damage; rainflow counting; probability density function; rainflow cycle range

CLC number: V235; TB123 **Document code**: A

Nomenclature

D_{rf}	Rainflow fatigue damage accumulation	$G(f)$	One-sided spectral density
T	Time duration	β_d	Coefficient of variation of fatigue damage
f	Frequency	m, A	Constants of material
v_0	Expected zero up-crossing rate	γ	Irregular factor
$f(D_{rf})$	Distribution function of fatigue damage	m_n	nth central moment of a random process
v_p	Expected peak rate	σ_d	Standard deviation of fatigue damage
f_m	Mean frequency	M_t	Total number of rainflow cycles
$X(t)$	Gaussian random loading	σ_X	Standard deviation of Gaussian process
$f_{rf}(z)$	Distribution function of rainflow cycles	S	Rainflow cycle range
z	Normalized rainflow cycle range	μ_d	Mean value of fatigue damage

Mechanical components under in-service conditions are often subjected to random loadings, such as the irregular stress processes in some engine components[1-4], the fluctuating pressure on aircraft skin panels[5]. The random loadings are distinct from the regular loadings, and then the fatigue analysis of random loadings is quite complex.

The final purpose of statistical analysis of the random loading fatigue damage is to predict or evaluate the fatigue lives and reliabilities of the mechanical structures. The randomness of the stochastic loadings is in terms of the peak rate[6] and the magnitude of the rainflow cycles[7-10]. They all will influence the computational accuracy of the fatigue damage accumula-

Received:2013-07-24

tion. For the peak rate, namely the expected number of rainflow cycles per unit time, Rice[11] and Bendat[12] have carried out lots of research systematically, and they proposed the expression of peak rate. However, the distribution function of peak rate is not presented in their works due to theoretical complexity[8]. For the statistical properties of rainflow cycles of Gaussian random loadings, Dirlik[7] and Bishop[8] have proposed the empirical expression and theoretical expression respectively. Benasciutti, Rychlik, et al[13-14] have studied the rainflow cycle distribution of non-Gaussian random loadings.

There are lots of investigations and results on the fatigue damage expectation of random loadings. The statistical characteristics and distributions were studied seldom. If one wants to study the statistical properties in time domain, a great deal of sample time histories are needed[14]. And sometimes the Gaussian random loadings are always expressed in frequency domain, say, PSD(power spectral density)[8].

Based on this fact, we have studied the statistical properties of fatigue damage in the specified time duration from spectral data of Gaussian random loadings. Lastly, the proposed method is used to the simulated loading and the measured loading. Comparisons with the observed results in time domain have verified the correctness and effectiveness of the presented method.

1 Randomness of number of loading cycles

Rice[11] and Bendat[12] have figured out the expression of expected peak rate υ_p, namely number of loading cycles and the expected zero up-crossing rate υ_0 from the PSDs of Gaussian processes. Firstly, we shall introduce the concept of spectral moment which is defined as

$$m_n = \int_0^\infty f^n G(f)\,\mathrm{d}f \qquad n = 0, 1, 2, \cdots \tag{1}$$

where f indicates the frequency, $G(f)$ is the one-sided PSD, n is the order of the spectral moment. Then, the expected zero up-crossing rates is

$$\upsilon_0 = E(N_{0+}) = \left(\frac{m_2}{m_0}\right)^{1/2} \tag{2}$$

where $E(\cdot)$ is the expected value operator, and the expected peak rate is

$$\upsilon_p = E(N_p) = \left(\frac{m_4}{m_2}\right)^{1/2} \tag{3}$$

where N_{0+} is the up-crossing number of zero level in the processes within unit time and N_p is the number of peaks in unit time. They are both random.

In random loadings, the number of peaks is equal to that of rainflow cycles. Then υ_p can also represent the expected rainflow cycle rate. So far, the expected number of cycles is known; however, the distribution function of N_p is not solved[8].

According to the research of Bishop[8], the numbers of cycles in different time intervals can be recognized to be independent identically distributed (IID). Fortunately, most of fatigue problems associated with random loadings is high-cycle fatigue problem. And then, the time duration of the loading, T (or the total number of loading cycles M_t) is always very long (or large). For time duration T, the total number of rainflow cycles is

$$M_t = \sum_{i=1}^{T} N_p^{(i)} \tag{4}$$

and

$$E(M_t) = T \times \upsilon_p = T \times \left(\frac{m_4}{m_2}\right)^{1/2} \tag{5}$$

where $N_p^{(i)}$ is the number of rainflow cycles in the ith unity time interval. When T is long enough, $\hat{\upsilon}_p = \frac{1}{T}\sum_{i=1}^{T} N_p^{(i)}$ will give an accurate estimation of υ_p. According to Chebyshev theorem[15], for any positive number ε, there is

$$\lim_{T\to\infty} P(|\hat{\upsilon}_p - \upsilon_p| < \varepsilon) = 1 \tag{6}$$

For high-cycle fatigue or very-high-cycle fatigue problems, $T\to\infty$, then Eq. (6) will hold for a very small ε, then $\sum_{i=1}^{T} N_p^{(i)} = M_t \approx \upsilon_p T$. Hence, the total number of rainflow cycles, M_t can be referred to as a constant for the specified time duration T.

2 Randomness of rainflow fatigue damage

The randomness of the rainflow fatigue

damage is attributed to that the amplitudes of the rainflow cycles are random. For the Gaussian random loadings, there are several methods to evaluate the rainflow distributions[7,8,16]. Although, the Dirlik solution is an empirical method, lots of researches have verified its accuracy[8,17]. Therefore, in this study, Dirlik's formulation will be used to express the Gaussian rainflow cycle distribution.

2.1 Dirlik's formulation

Dirlik's formulation[7] is derived based on extensive simulations and theoretical analysis. For Gaussian random process $X(t)$, there are important relationships between the spectral moments and its statistics. The standard deviation is

$$\sigma_X = \sqrt{m_0} \tag{7}$$

The irregular factor is

$$\gamma = \frac{v_0}{v_p} = \sqrt{\frac{m_2^2}{m_0 m_4}} \tag{8}$$

The mean frequency is

$$f_m = \frac{m_1}{m_0}\sqrt{\frac{m_2}{m_4}} \tag{9}$$

The rainflow distribution function $f_{rf}(\cdot)$ is always in the form of normalized variable, $z=S/(2\sigma_X)$, where σ_X is the standard deviation of the Gaussian random loading, and S is the range of the rainflow cycle (two times of amplitude). The normalized rainflow distribution is

$$f_{rf}(z)= c_1\frac{1}{\tau}e^{-z/\tau}+c_2\frac{z}{\alpha^2}e^{-z^2/(2\alpha^2)}+c_3 z e^{-z^2/2} \tag{10}$$

where

$$\begin{cases} c_1 = \dfrac{2(f_m-\gamma^2)}{1+\gamma^2} \\ \alpha = \dfrac{\gamma-f_m-c_1^2}{1-\gamma-c_1+c_1^2} \\ c_2 = \dfrac{1-\gamma-c_1+c_1^2}{1-\alpha} \\ c_3 = 1-c_1-c_2, \\ \tau = 1.25(\gamma-c_3-\alpha c_2)/c_1 \end{cases} \tag{11}$$

substituting $z=S/(2\sigma_X)$ into Eq. (10), the rainflow cycle distribution of Gaussian random loading with specified PSD can be derived.

2.2 Randomness of fatigue damage

The focus of this study is on the randomness of fatigue damage due to the random loading itself. And the randomness of the fatigue characteristics of the materials and structures is not included. For those theories, one can refer to Ref. [18-19]. So the mediam life S-N curve will be used, namely, 50%-S-N, in this study. The expression of S-N curve is normally as

$$NS^m = A$$

$$N = \frac{A}{S^m} \tag{12}$$

where S is the stress level; N is the number of cycles to failure at stress level S; m is the stress-life exponent, and A is the fatigue strength coefficient.

According to Palmgren-Miner's rule, the fatigue damage with time duration T is

$$D_{rf} = \sum_{i=1}^{M_t}\frac{S_i^m}{A} \tag{13}$$

where S_i is the range of the ith rainflow cycle which is random; resultingly, D_{rf} will be random too. In addition, the critical fatigue damage to failure is always assumed to be $D_{cr}=1$[6-8]. According to Bishop[8], we can assume that the ranges of different rainflow cycles are independent identically distributed. Therefore, D_{rf} is the sum of many random variables. A step further, based on the central limit theorem, D_{rf} should follow Gaussian distribution.

The expectation of S_i^m is

$$E(S_i^m)=E(S^m)=\int_0^\infty S^m f_{rf}(S)\mathrm{d}S \tag{14}$$

The variance is

$$\mathrm{Var}(S^m)=\int_0^\infty[S^m-E(S^m)]^2 f_{rf}(S)\mathrm{d}S = E(S^{2m})-E^2(S^m) \tag{15}$$

Based on the hypothesis that different rainflow cycles in the random loading are independent identically distributed, the mean value and variance of the cumulated fatigue damage are

$$\mu_d = E(D_{rf}) = M_t\frac{E(S^m)}{A} \tag{16}$$

$$\sigma_d^2 = \mathrm{Var}(D_{rf}) = \frac{M_t}{A^2}[E(S^{2m})-E^2(S^m)] \tag{17}$$

where the subscript "d" denotes "damage". The coefficient of variation of the fatigue damage is

$$\beta(D_{rf})=\frac{\sigma_d}{\mu_d}=\frac{\sigma(S^m)}{\sqrt{M_t}\,\mu(S^m)}=\frac{\beta(S^m)}{\sqrt{M_t}} \quad (18)$$

Because D_{rf} follows Gaussian distribution, the mean value and variance can define its distribution function totally

$$f(D_{rf})=\frac{1}{\sqrt{2\pi}\,\sigma_d}\exp\left[-\frac{(D_{rf}-\mu_d)}{2\sigma_d^2}\right] \quad (19)$$

Overall, the fatigue damage from the random loading aspect can be characterized by a Gaussian distribution. And the algorithm followed by the proposed method is sketched in Fig. 1.

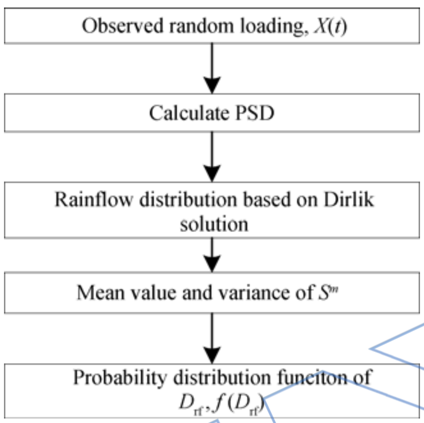

Fig. 1 Algorithm flowchart of probability density functions of rainflow fatigue damage for Gaussian loadings

3 Validation examples

For validating the correctness, the proposed method is used to the simulated Gaussian process with flat PSD and the observed Gaussian process on an aircraft. One should note that the true distribution function of fatigue damage is unknown. So the results of Monte-Carlo method in time domain are used to derive the empirical distribution of fatigue damage. Then the empirical distribution is compared with the result based on the presented method to verify its accuracy and capability.

3.1 Simulated signals

The simulated Gaussian loading is shown in Fig. 2. The statistical properties of fatigue damage has be calculated in time domain and frequency domain respectively. The time duration $T=8$ s.

3.1.1 Empirical results from time domain

200 random loadings are simulated with the identical PSD shown in Fig. 2(b). Then rainflow counting method is used to the 200 samples, and the rainflow fatigue damages are calculated for each sample. In this example, assumed the S-N curve is

$$NS^{4.38}=1.23\times10^{15} \quad (20)$$

Then, the rainflow fatigue damage for each sample time history is

$$D_i=\sum_{i=1}^{M_j}\frac{1}{N_i}=\sum_{i=1}^{M_j}\frac{S_i^{4.38}}{1.23\times10^{15}} \quad (21)$$

where M_j is the number of cycles in the ith simulated sample time history and N_i is the number of cycles to failure at stress lever S_i. Then the fatigue damage sequence $\{D_i\}$, $i=1, 2, \cdots, 200$ can be obtained. The values of $\{D_i\}$ are shown in Fig. 2(c). Based on Kolmogorov-Smirnov test[20], $\{D_i\}$ follows Gaussian distribution in the confidence of 90%. The mean of $\{D_i\}$ is

$$\hat{\mu}_d=\frac{1}{200}\sum_{i=1}^{200}D_i=0.0218 \quad (22)$$

The standard deviation is

$$\hat{\sigma}_d=1.3143\times10^{-3} \quad (23)$$

The coefficient of variation is

$$\hat{\beta}_d=\frac{\hat{\sigma}_d}{\hat{\mu}_d}=0.0603 \quad (24)$$

3.1.2 Results of proposed method

By substituting the PSD data shown in Fig. 2(b) into Eq. (10), the normalized rainflow distribution can be obtained. Through a simple variable change, we can get the rainflow distribution. Then by substituting the distribution function into Eq. (14)—Eq. (18), the mean of the fatigue damage is

$$\mu_d=0.0213 \quad (25)$$

The standard deviation is

$$\sigma_d=1.2680\times10^{-3} \quad (26)$$

The coefficient of variation is

$$\beta_d=\frac{\sigma_d}{\mu_d}=0.0596 \quad (27)$$

By comparing the results in Eq. (25)—Eq. (27) with that in Eq. (22)—Eq. (23), the relative errors of the proposed method are $e(\mu_d)=2.29\%$,

$e(\sigma_d)=3.52\%$, $e(\beta_d)=1.16\%$ can be seen, where e(•) indicates the relative error.

Lastly, the empirical(EM) distribution derived from $\{D_i\}$ and the theoretical(TH) expression based on the proposed method are shown in Fig. 2(d), where the proposed method can give a perfect accuracy to estimate the distribution of the Gaussian fatigue damage in given time duration.

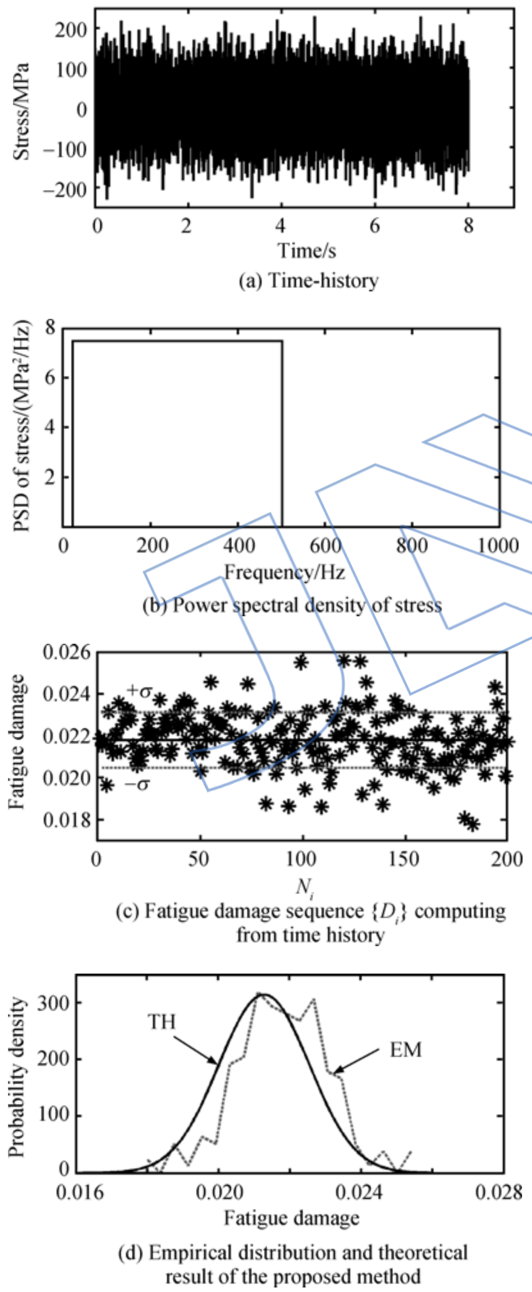

Fig. 2 Statistical properties of simulated Gaussian loading

3.2 Observed signals

For validating the proposed method a step further, the measured signal on an aircraft have be analyzed in Fig. 3. The originally measured signal is acceleration sequence, shown in Fig. 3(a). Through the frequency domain two-fold integral[21], the displacement signal can be obtained from the acceleration one. The displacement signal is shown in Fig. 3(c). Then, according to deflection formulation, the stress time history can be derived. For the given structure, the stress response signal is proportional to the displacement signal, namely, $s(t)\propto x(t)$.

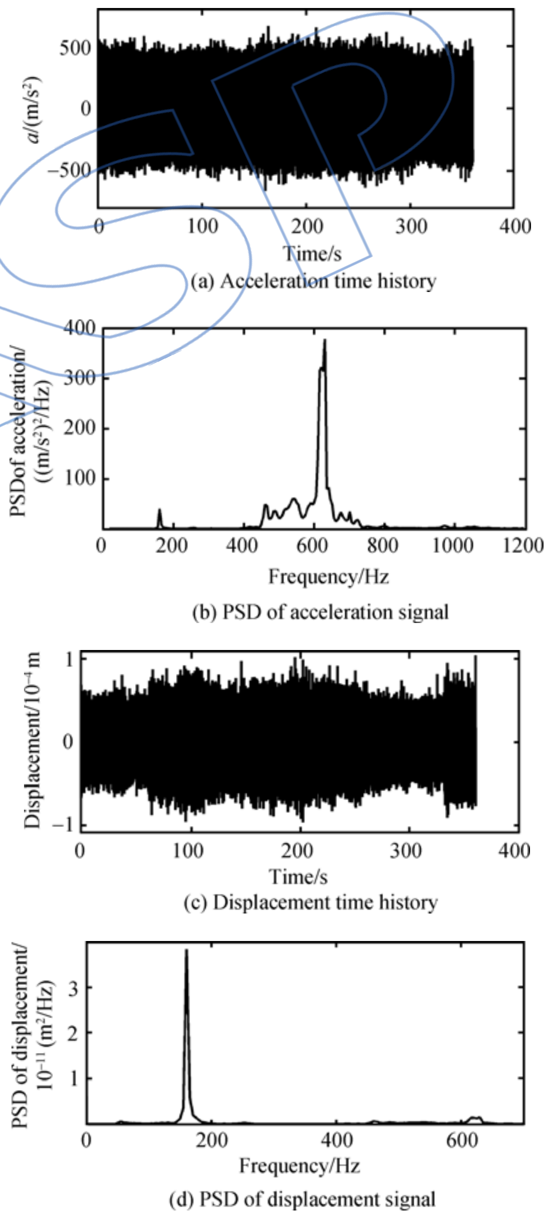

Fig. 3 Observed Gaussian vibration signal on an aircraft

Then we can obtain the loading sequence.

For the observed acceleration signal, the mean $\mu_a = 0\ \mathrm{m/s^2}$, where the subscript "a" denote "acceleration", the standard deviation $\sigma_a = 141.5379\ \mathrm{m/s^2}$, and the process is Gaussian in the confidence of 90%. The mean of the displacement signal, $\mu_x = 0$ m; the standard deviation $\sigma_x = 1.9980 \times 10^{-5}$ m, and it is Gaussian in the confidence of 90%. Assumed that the linear relationship between the stress process and the displacement process shown in Fig. 3 (c) is

$$S(t) = L[x(t)] = 5 \times 10^6 x(t) \tag{28}$$

Where $L(\cdot)$ indicates the linear function relationship. Then the statistics of the stress process is: the mean $\mu_s = 0$ MPa, the standard deviation $\sigma_s = 99.9020$ MPa. And it is normal distributed in the confidence of 90%.

For validating the capability of the proposed method, the measured signal, whose time duration $T = 360.0939$ s, is divided to 40 segments averagely. Fig. 4 shows the statistical properties of the measured random loading on an aircraft. The time duration of each segment is 9 s, which is shown in Fig. 4(a).

The mediam S-N of the supporting material is supposed to be

$$NS^4 = 1.5 \times 10^{15} \tag{29}$$

The fatigue damage will be calculated in both time domain and frequency domain.

3.2.1 Empirical results from time domain

The computational procedure is the same with that in subsection 3.1. The fatigue damage sequence $\{D_i\}$ of the 40 segments have be gotten, shown in Fig. 4(c). It is obviously, $\{D_i\}$ is random. The mean value of $\{D_i\}$ is

$$\hat{\mu}_d = \frac{1}{40}\sum_{i=1}^{40} D_i = 0.0171 \tag{30}$$

The standard deviation is

$$\hat{\sigma}_d = 3.4841 \times 10^{-4} \tag{31}$$

The coefficient of variation is

$$\hat{\beta}_d = \frac{\hat{\sigma}_d}{\hat{\mu}_d} = 0.0610 \tag{32}$$

3.2.2 Results of proposed method

By substituting the PSD data shown in Fig. 4(b) into Eq. (10), and through a simple variable change, the rainflow distribution function can be got. Then substituting the PDF of rainflow cycles into Eq. (14)—Eq. (18), one can get the statistics of the fatigue damage. The mean of the fatigue damage is

$$\mu_d = 0.0191 \tag{33}$$

The standard deviation is

$$\sigma_d = 9.5227 \times 10^{-4} \tag{34}$$

The coefficient of variation is

$$\beta_d = 0.0499 \tag{35}$$

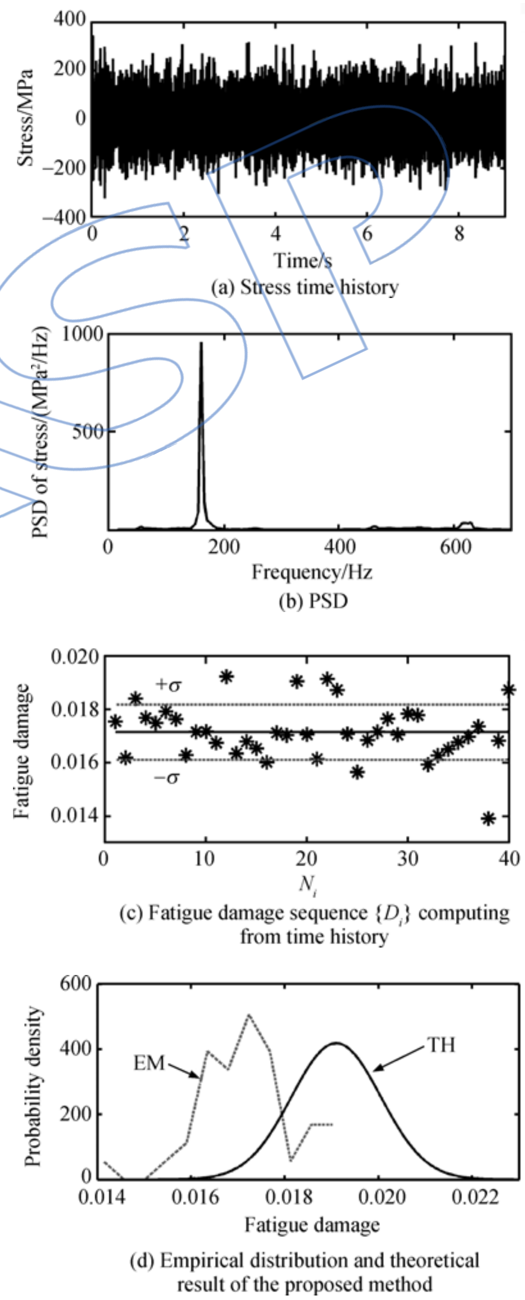

Fig. 4 Statistical properties of measured random loading on an aircraft

The results in Eq. (33)—Eq. (35) are compared with that in Eq. (30)—Eq. (32). Compared with the empirical results the relative errors of the proposed method are $e(\mu_d)=11.70\%$, $e(\sigma_d)=173.32\%$, $e(\beta_d)=18.20\%$.

In contrast to the simulated signal in subsection 3.1, the relative errors for the real data are larger, but in the rational range, for this measured signal, the reasons of larger errors are summarized as following:

1) The sample size for estimating the empirical distribution is 40, which is a bit smaller for getting a stable result.

2) It's obvious that the measured signal is not stationary seriously, and this is the main reason of computational errors.

4 Conclusions

In this paper, the randomness of the fatigue damage has be studied induced by Gaussian random loadings in a given time duration and the following conclusions are obtained:

1) The fatigue damage associated with the random loadings is high-cycle fatigue or very high-cycle fatigue problem where the time duration of the loading process is very long. Hence, during the computation of the fatigue damage, the randomness of the number of rainflow cycles is negligible.

2) The fatigue damage induced by Gaussian random loading is normal distributed based on the central limit theorem. And the distribution parameters are obtained based on Dirlik's formulation.

3) The proposed method has been used to simulated signals and the measured signal on an aircraft. And the results have verified its correctness and applicability.

4) This study is very meaningful for the analysis and estimation of the fatigue lives and reliabilities of structures subjected to random vibration loadings.

References:

[1] CAO Maoguo, LI Lin. Experimental investigation on thin shell random vibration excited by noise[J]. Journal of Aerospace Power, 2001, 16(3): 291-294. (in Chinese)

[2] JIN Yishan, LI Lin. Sonic fatigue life prediction of aero-engine structure[J]. Journal of Aerospace Power, 2003, 18(3): 373-377. (in Chinese)

[3] YANG Zichun, SHEN Wencai, PENG Maolin. Research onresidual life prediction of turbine blade-disc under random environment[J]. Journal of Aerospace Power, 2009, 24(8): 1677-1683. (in Chinese)

[4] CHENG Li, ZHANG Haiwei, WANG Limin. Fatigue damage of aero-engine in intensive formation flying[J]. Journal of Aerospace Power, 2010, 25(3): 587-591. (in Chinese)

[5] Steinwolf A. Non-Gaussian analysis of turbulent boundary layer fluctuating pressure on aircraft skin panels[J]. Journal of Aircraft, 2006, 43(6): 1662-1675.

[6] Benasciutti D, Tovo R. Fatigue life assessment in non-Gaussian random loadings[J]. International Journal of Fatigue, 2006, 28(7): 733-746.

[7] Dirlik T. Application of computers in fatigue analysis[D]. Warwick, UK: University of Warwick, 1985.

[8] Bishop N W M. The use of frequency domain parameters to predict structural fatigue[D]. Warwick, UK: University of Warwick, 1988.

[9] Bengtsson A, Bogsjö K, Rychlik I. Uncertainty of estimated rainflow damage for random loads[J]. Marine Structures, 2009, 22(2): 261-274.

[10] QU Kai, XING Yaoguo, ZHANG Xudong. Fatigue damage of shipborne solid rocket motor propellant under swing loading[J]. Journal of Aerospace Power, 2011, 26(11): 2637-2640. (in Chinese)

[11] Rice S O. Mathematical analysis of random noise[J]. Bell System Technical Journal, 1944, 23(3): 282-332.

[12] Bendat J S. Probability functions for random responses [R]. NASA Report, NAS-5-4590, 1964.

[13] Benasciutti D, Tovo R. Cycle distribution and fatigue damage assessment in broad-band non-Gaussian random processes[J]. Probabilistic Engineering Mechanics, 2005, 20(2): 115-127.

[14] Rychlik I, Gupta S. Rain-flow fatigue damage for transformed Gaussian loads[J]. International Journal of Fatigue, 2007, 29(3): 406-420.

[15] Gatti P L. Probability theory and mathematical statistics for engineers[M]. New York: Spon Press, 2005: 142-149.

[16] Madhavan P T M, Meher P A. Fatigue reliability analysis in time domain for inspection strategy of fixed offshore structures[J]. Ocean Engineering, 2000, 27(2): 167-186.

[17] Benasciutti D, Tovo R. Comparison of spectral methods for fatigue analysis of broad-band Gaussian random processes [J]. Probabilistic Engineering Mechanics, 2006, 21(4): 287-299.

[18] Shimizu S, Tosha K, Tsuchiya K. New data analysis of probabilistic stress-life (P-S-N) curve and its application

for structural materials[J]. International Journal of Fatigue, 2010, 32(3): 565-575.

[19] Ling J, Pan J. A maximum likelihood method for estimating P-S-N curves[J]. International Journal of Fatigue, 1997, 19(5): 415-419.

[20] Marsaglia G, Tsang W, Wang J. Evaluating Kolmogorov's distribution[J]. Journal of Statistical Software, 2003, 8 (18): 1-4.

[21] Han S, Chung J W. Retrieving displacement signal from measured acceleration signal[C]// Proceedings of 2002 IMAC-XX: Conference and Exposition on Structural Dynamics. Los Angeles, US: International Modal Analysis Conference, 2002: 1178-1184.