Labeled Ballot Paths and the Springer Numbers

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Abstract. The Springer numbers are defined in connection with the irreducible root system of type B_n and also arise as the generalized Euler and class numbers introduced by Shanks. Combinatorial interpretations of the Springer numbers have been found by Purtill in terms of André signed permutations, and by Arnol'd in terms of snakes of type B_n. We introduce the inversion code of a snake of type B_n and establish a bijection between labeled ballot paths of length n and snakes of type B_n. Moreover, we obtain the bivariate generating function for the number $B(n,k)$ of labeled ballot paths starting at $(0,0)$ and ending at (n,k). Using our bijection, we find a statistic α such that the number of snakes π of type B_n with $\alpha(\pi)=k$ equals $B(n,k)$. We also show that our bijection specializes to a bijection between labeled Dyck paths of length $2n$ and alternating permutations on $[2n]$.

Keywords: Springer number, snake of type B_n, labeled ballot path, labeled Dyck path, bijection

AMS Subject Classifications: 05A05, 05A19

1 Introduction

The Springer numbers are introduced by Springer [16] in the study of irreducible root system of type B_n. Let S_n denote the n-th Springer number. The sequence $\{S_n\}_{n\geq 0}$ is listed as entry $A001586$ in OEIS [11]. The first few values of S_n are

$$1, 1, 3, 11, 57, 361, 2763, 24611, \ldots.$$

To be more specific, S_n can be defined as follows. Let V be a real vector space, R be a root system of type B_n in V, and W be the Weyl group of R. It is known that for a fixed simple root set S of R, any $\alpha \in R$ is either a positive or a negative linear combination of elements of S, denoted by $\alpha > 0$ or $\alpha < 0$. For a subset $I \subset S$, let $\sigma(I,S)$ denote the number of elements $w \in W$ such that $w\alpha > 0$ for any $\alpha \in I$ and $w\alpha < 0$ for any $\alpha \in S-I$. Then the Springer number S_n can be defined as the maximum value of $\sigma(I,S)$ among $I \subset S$. Springer derived the following generating function,

$$\sum_{n\geq 0} S_n \frac{x^n}{n!} = \frac{1}{\cos x - \sin x}. \tag{1.1}$$

On the other hand, Hoffman [10] pointed out that the Springer numbers also arise as the generalized Euler and class numbers $s_{m,n}$ ($n \geq 0$) for $m = 2$, where the numbers $s_{m,n}$ are introduced by Shanks [15] based on the Dirichlet series

$$L_m(s) = \sum_{k=0}^{\infty} \left(\frac{-m}{2k+1}\right) \frac{1}{(2k+1)^s}.$$

Note that the above notation $(-m/(2k+1))$ is the Jacobi symbol. To be precise, the generalized Euler and class numbers $s_{2,n}$ are defined by

$$s_{2,n} = \begin{cases} c_{2,\frac{n}{2}}, & \text{if } n \text{ is even;} \\ d_{2,\frac{n+1}{2}}, & \text{if } n \text{ is odd,} \end{cases}$$

where the numbers $c_{2,n}$ and $d_{2,n}$ are given by

$$c_{2,n} = \frac{(2n)!}{\sqrt{2}} \left(\frac{\pi}{4}\right)^{-2n-1} L_2(2n+1),$$

$$d_{2,n} = \frac{(2n-1)!}{\sqrt{2}} \left(\frac{\pi}{4}\right)^{-2n} L_{-2}(2n).$$

According to the following recurrence relations for $c_{2,n}$ and $d_{2,n}$ derived by Shanks [15],

$$\sum_{i=0}^{n} (-4)^i \binom{2n}{2i} c_{2,n-i} = (-1)^n,$$

$$\sum_{i=0}^{n-1} (-4)^i \binom{2n-1}{2i} d_{2,n-i} = (-1)^{n-1},$$

one sees that the numbers $s_{2,n}$ are integers. In fact, the above recurrence relations lead to the following formulas

$$\sum_{n\geq 0} c_{2,n} \frac{x^{2n}}{(2n)!} = \sec 2x \cos x,$$

$$\sum_{n\geq 1} d_{2,n} \frac{x^{2n-1}}{(2n-1)!} = \sec 2x \sin x.$$

Shanks raised the question of finding combinatorial interpretations for the Euler and class numbers $s_{m,n}$. For $m = 2$, $s_{2,n}$ is the n-th Springer number. Purtill [12] gave an interpretation of the Springer numbers in terms of the André signed permutations on $[n] = \{1, 2, \ldots, n\}$. Arnol'd [1] found another interpretation of the Springer numbers in terms of snakes of type B_n. Recall that, intuitively, a signed permutation on $[n]$ can be viewed as an ordinary permutation on $[n]$ with some elements associated with minus signs. An element i with a minus sign is often written as $\bar{i}$. A snake of type B_n is an alternating signed permutation $\pi = \pi_1\pi_2\cdots\pi_n$ on $[n]$ such that

$$0 < \pi_1 > \pi_2 < \pi_3 > \pi_4 < \cdots \pi_n. \tag{1.2}$$

The above alternating or up-down condition (1.2) is based on the following order:

$$\bar{n} < \cdots < \bar{1} < 1 < \cdots < n.$$

For example, $1\bar{3}2$ is a snake of type B_3. Arnol'd [1] proved that the Springer number S_n equals the number of snakes of type B_n. Hoffman [10] showed that the exponential generating function for the number of snakes of type B_n also equals the right hand side of (1.1), that is, the generating function of the Springer numbers. Recently, Chen, Fan and Jia [4] obtained a formula for the generating function of $s_{m,n}$ for arbitrary m. When m is square-free, this leads to a combinatorial interpretation of the numbers $s_{m,n}$ in terms of Λ-alternating augmented m-signed permutations. Note that for $m = 2$, Λ-alternating augmented 2-signed permutations are exactly snakes of type B_n.

The objective of this paper is to give a combinatorial interpretation for the Springer numbers in terms of labeled ballot paths. In fact, we shall introduce the inversion code of a snake of type B_n. By using the inversion code, we construct a bijection between the set of snakes of type B_n and the set of labeled ballot paths of length n. Let $B(n, k)$ denote the number of labeled ballot paths starting at $(0, 0)$ and ending at (n, k). Then the numbers $B(n, k)$ can be viewed as a refinement of the Springer numbers. Using the recurrence relation of $B(n, k)$, we obtain the generating function for $B(n, k)$.

Using our bijection, we find a statistic α on snakes of type B_n such that the number of snakes π of type B_n with $\alpha(\pi) = k$ equals $B(n, k)$. A labeled ballot path that eventually returns to the x-axis is called a labeled Dyck path. When $k = 0$, $B(2n, 0)$ is the number of labeled Dyck paths of length $2n$. We find that $B(2n, 0)$ and the number E_{2n} of alternating permutations on $[2n]$ have the same generating function, and we show that our bijection for labeled ballot paths of length n and snakes of type B_n reduces to a bijection between labeled Dyck paths of length $2n$ and alternating permutations on $[2n]$.

The paper is organized as follows. In Section 2, we give descriptions of the map Φ from the set of snakes of type B_n to the set of labeled ballot paths of length n, and the map Ψ from labeled ballot paths of length n to snakes of type B_n. In Section 3, we show that the maps Φ and Ψ are well-defined, and they are inverses of each other. The last section is devoted to the bivariate generating function for the numbers $B(n, k)$ and the classification of snakes of type B_n in accordance with the numbers $B(n, k)$. We also show that the map Ψ restricted to labeled Dyck paths serves as a combinatorial interpretation of the fact that $B(2n, 0)$ equals E_{2n}. To conclude, we point out the connections of the restriction of our bijection to labeled Dyck paths with the restrictions of some known bijections on weighted 2-Motzkin paths.

2 The bijection

In this section, we define a class of labeled ballot paths and establish a bijection between such labeled ballot paths of length n and snakes of type B_n.

Recall that a ballot path of length n is a lattice path with n steps from the origin consisting of up steps $u = (1, 1)$ and down steps $d = (1, -1)$ that does not go below the x-axis. As a special case, a Dyck path is a ballot path of length $2n$ that ends at the x-axis. A ballot path is also called a partial Dyck path [3]. The height of a step of a ballot path is defined to be the smaller y-coordinate of its endpoints. By a labeled ballot path we mean a ballot path for which each step is endowed with a nonnegative integer that is less than or equal to its height. If the label of a step equals its height, then we say that this step is saturated. Otherwise, we say this step is unsaturated. A labeled ballot path $P = p_1p_2 \cdots p_n$ for which the step p_i is labeled by w_i is denoted by (P, W), where $W = w_1w_2 \cdots w_n$.

It should be noted that Françon and Viennot [6] defined a class of weighted 2-Motzkin paths. If such a weighted 2-Motzkin path of length $2n$ does not contain any horizontal steps, then it becomes a labeled Dyck path of length $2n$ in our terminology. In Section 4, we shall discuss the connections between the restriction of our bijection to labeled Dyck paths and the restrictions of known bijections for weighted 2-Motzkin paths.

For example, for a ballot path $P = uuudduu$, there are 216 labelings. Figure 1 gives a labeling of the ballot path P.

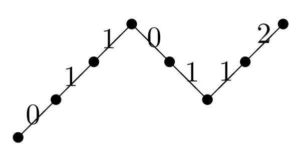

Figure 1: A labeled ballot path $(uuudduu, 0110112)$ of length 7.

For $n = 3$, there are 3 ballot paths $P_1 = uuu, P_2 = uud$ and $P_3 = udu$. There are 6 labelings for P_1, 4 labelings for P_2 and 1 labeling for P_3. On the other hand, there are 11 snakes of type B_3 as listed below:

$$1\bar{2}3, 1\bar{3}2, 1\bar{3}\bar{2}, 213, 2\bar{1}3, 2\bar{3}1, 2\bar{3}\bar{1}, 312, 3\bar{1}2, 3\bar{2}1, 3\bar{2}\bar{1}.$$

In order to establish a bijection between the set of ballot paths of length n and the set of snakes of type B_n, we introduce the inversion code of a snake π of type B_n. Let $\pi = \pi_1 \cdots \pi_n$. We define $c_i(\pi)$ as follows

$$c_i(\pi) = \begin{cases} \#\{(\pi_{2k}, \pi_{2k+1}) | 1 \le k \le (n-1)/2, i < 2k, \pi_{2k} < \pi_i < \pi_{2k+1}\}, & \text{if } n \text{ is odd;} \\ \#\{(\pi_{2k-1}, \pi_{2k}) | 1 \le k \le n/2, i < 2k-1, \pi_{2k} < \pi_i < \pi_{2k-1}\}, & \text{if } n \text{ is even.} \end{cases}$$

The sequence $(c_1(\pi), c_2(\pi), \ldots, c_n(\pi))$, denoted $c(\pi)$, is called the inversion code of π. For example, let $n = 7$ and $\pi = 3\bar{5}214\bar{7}6$. Then the inversion code of π is $(2, 1, 2, 1, 1, 0, 0)$. For $n = 8$ and $\pi = 538\bar{2}\bar{1}\bar{4}76$, the inversion code of π is $(1, 1, 0, 1, 0, 0, 0, 0)$.

As will be seen, we need an alternative way to compute $c_i(\pi)$. For $1 \leq i \leq n$, an element π_i is called a peak of π if π_i is greater than its neighbors, and π_i is called a bottom if it is smaller than its neighbors. Notice that for the first and last element of π, there is only one neighbor. Denote the set of peaks of π by $\mathrm{P}(\pi)$, and the set of bottoms of π by $\mathrm{B}(\pi)$. For a snake π of type B_n, each element π_i is either a peak or a bottom. For a bottom π_i of π, we define

$$a_i(\pi) = \#\{j > i | \pi_j < \pi_i, \pi_j \in \mathrm{B}(\pi)\},$$

$$b_i(\pi) = \#\{j > i | \pi_j < \pi_i, \pi_j \in \mathrm{P}(\pi)\}.$$

For a peak π_i of π, we define

$$a_i(\pi) = \#\{j > i | \pi_j > \pi_i, \pi_j \in \mathrm{B}(\pi)\},$$

$$b_i(\pi) = \#\{j > i | \pi_j > \pi_i, \pi_j \in \mathrm{P}(\pi)\}.$$

Lemma 2.1 *Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a snake of type B_n. For $1 \leq i \leq n$, if π_i is a bottom, then we have*

$$c_i(\pi) = a_i(\pi) - b_i(\pi). \tag{2.3}$$

If π_i is a peak, then we have

$$c_i(\pi) = b_i(\pi) - a_i(\pi). \tag{2.4}$$

Proof. We first consider the case when n is odd and π_i is a bottom. Clearly, for each pair (π_{2k}, π_{2k+1}), π_{2k} is a bottom and π_{2k+1} is a peak and $\pi_{2k} < \pi_{2k+1}$. For a pair (π_{2k}, π_{2k+1}) to the right of π_i, if $\pi_i > \pi_{2k+1}$ or $\pi_i < \pi_{2k}$, then by definition, this pair contributes 0 to $c_i(\pi)$. In the meantime, this pair contributes 0 to the right hand side of (2.3). On the other hand, if $\pi_{2k} < \pi_i < \pi_{2k+1}$, the pair (π_{2k}, π_{2k+1}) contributes 1 to $c_i(\pi)$, and contributes 1 to the right hand side of (2.3) as well. Thus (2.3) is valid when n is odd and π_i is a bottom. The case when n is odd and π_i is a peak and the case when n is even can be justified by using the same argument. Similarly, (2.4) can be verified. This completes the proof. ∎

We are now ready to describe the map Φ from the set of snakes of type B_n to the set of labeled ballot paths of length n. Let $\pi = \pi_1\pi_2\cdots\pi_n$ and $\Phi(\pi) = (P, W) = (p_1p_2\cdots p_n, w_1w_2\cdots w_n)$. The map Φ consists of n steps. Suppose that we are in step k, that is, $p_1, p_2, \ldots, p_{k-1}$ and their labels $w_1, w_2, \ldots, w_{k-1}$ are already determined. We proceed to demonstrate how to determine p_k and its label w_k. Let us look for the element $n-k+1$ or $\overline{n-k+1}$ in π. Assume that $\pi_i = n-k+1$ or $\overline{n-k+1}$. There are two cases.

Case 1: $\pi_i = n-k+1$. If i is odd, then set $p_k = u$; if i is even, then set $p_k = d$. Moreover, set $w_k = c_i(\pi)$.

Case 2: $\pi_i = \overline{n-k+1}$. If i is odd, then set $p_k = d$; if i is even, then set $p_k = u$. Moreover, set $w_k = h_k - c_i(\pi)$, where h_k denotes the height of the k-th step p_k in the ballot path $p_1 p_2 \cdots p_k$.

For example, let $n = 7$ and $\pi = 2\bar{1}547\bar{6}\bar{3}$. The construction of $\Phi(\pi)$ is illustrated in Figure 2.

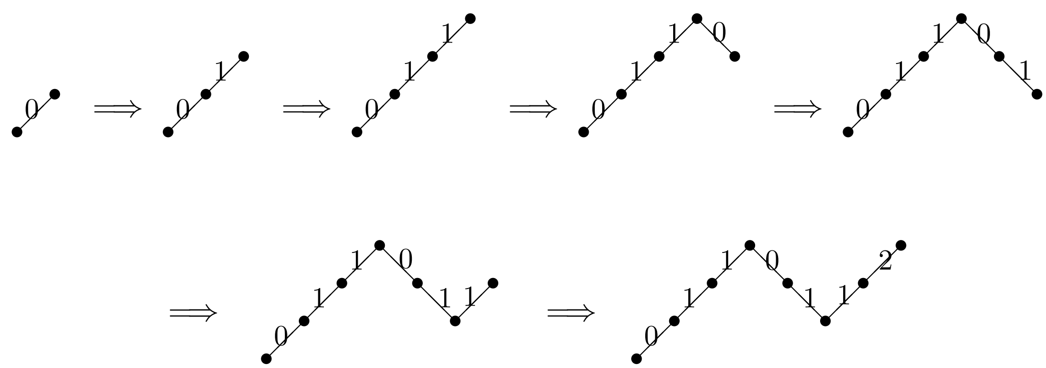

Figure 2: The construction of $\Phi(\pi)$ for $\pi = 2\bar{1}547\bar{6}\bar{3}$.

We now give the inverse map Ψ from the set of labeled ballot paths to the set of snakes of type B_n. By a partial signed permutation of $[n]$, we mean a permutation on some subset of $[n]$ with some elements having minus signs. For example, $\gamma = \bar{4}2\bar{5}$ is a partial signed permutation of $\{1, 2, \ldots, 6\}$.

Given a labeled ballot path $(P, W) = (p_1 p_2 \cdots p_n, w_1 w_2 \cdots w_n)$, we shall construct a sequence of partial signed permutations $\Gamma_0, \Gamma_1, \Gamma_2, \ldots, \Gamma_n$, such that $\Gamma_0 = \emptyset$ and $\Gamma_n = \pi = \Psi(P, W)$ is the desired snake of type B_n. To reach this goal, we generate a sequence of labeled ballot paths $(P_1, W_1), (P_2, W_2), \ldots, (P_n, W_n)$, where $(P_1, W_1) = (P, W)$, P_{i+1} is obtained from P_i by contracting a certain step p_{r_i} of P_i into a single point, and W_{i+1} is obtained from W_i by deleting the label of the step p_{r_i} and updating the labels of other steps. Below is a procedure to determine (P_{i+1}, W_{i+1}) and Γ_i from (P_i, W_i) and Γ_{i-1}. There are two cases.

Case 1: P_i has an odd number of steps. If there exists a saturated down step in P_i, namely, a down step whose label equals equals its height, then we assume that p_{r_i} is the leftmost saturated down step. Contract p_{r_i} into a single point to form a ballot path P_{i+1} and add 1 to the labels of all down steps of P_{i+1}. Let (P_{i+1}, W_{i+1}) denote the resulting labeled ballot path and set $\Gamma_i = \overline{n - r_i + 1}\Gamma_{i-1}$.

For the case when all the down steps of P_i are unsaturated, that is, the label of each down step is less than its height, the first step is an up step labeled by 0. We assume that p_{r_i} is the rightmost up step labeled by 0. Contract p_{r_i} into a single point to form a ballot path P_{i+1}. Then subtract 1 from the labels of up steps of P_{i+1} that are originally to the right of p_{r_i} and add 1 to the labels of down steps of P_{i+1} that are originally to the left of p_{r_i}. Denote the resulting labeled ballot path by (P_{i+1}, W_{i+1}) and set $\Gamma_i = (n - r_i + 1)\Gamma_{i-1}$.

Case 2: P_i has an even number of steps. If there exists a down step of P_i labeled by 0, we assume that p_{r_i} is the leftmost down step labeled by 0. Contract p_{r_i} into a single point to form a ballot path P_{i+1}. Then add 1 to the labels of up steps of P_{i+1} which are originally to the right of p_{r_i} and subtract 1 from the labels of down steps of P_{i+1} which are originally to the left of p_{r_i}. Denote the resulting labeled ballot path by (P_{i+1}, W_{i+1}) and set $\Gamma_i = (n - r_i + 1)\Gamma_{i-1}$.

For the case when there are no down steps of P_i labeled by 0, the first step is an up step labeled by 0. We assume that p_{r_i} is the rightmost saturated up step. Contract p_{r_i} into a single point to form a ballot path P_{i+1}. Then subtract 1 from the labels of all down steps of P_{i+1}. Denote the resulting path by (P_{i+1}, W_{i+1}) and set $\Gamma_i = \overline{n - r_i + 1}\Gamma_{i-1}$.

For example, for the labeled ballot path $(P, W) = (uuudduu, 0110112)$ in Figure 1, the construction of $\Psi(P, W)$ is illustrated in Figure 3. The order of the contracted steps is $p_5, p_2, p_1, p_4, p_3, p_7, p_6$. The labeled ballot paths (P_i, W_i) are given in Figure 3, and the partial signed permutations Γ_i are given below:

$$\Gamma_0 = \emptyset, \Gamma_1 = \bar{3}, \Gamma_2 = \bar{6}\bar{3}, \Gamma_3 = 7\bar{6}\bar{3}, \Gamma_4 = 47\bar{6}\bar{3}, \Gamma_5 = 547\bar{6}\bar{3}, \Gamma_6 = \bar{1}547\bar{6}\bar{3}, \Gamma_7 = 2\bar{1}547\bar{6}\bar{3}.$$

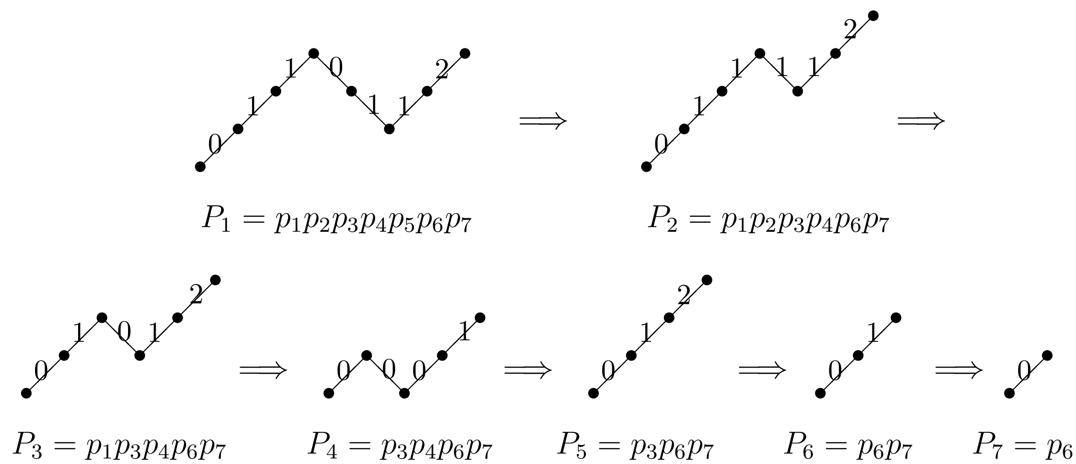

Figure 3: The construction of $\Psi(P, W)$ for the labeled ballot path in Figure 1.

3 The proof

In this section, we shall show that the maps Φ and Ψ described in the previous section are well-defined and are inverses of each other. Thus the map Φ and the map Ψ induce a bijection between the set of labeled ballot paths of length n and the set of snakes of type B_n.

Proposition 3.1 *The map Φ is well-defined, that is, for any snake $\pi = \pi_1\pi_2\cdots\pi_n$ of type B_n, $\Phi(\pi)$ is a labeled ballot path.*

Proof. Before we show that $\Phi(\pi) = (P, W)$ is a labeled ballot path, it is necessary to prove that $P = p_1p_2\cdots p_n$ is a ballot path, that is, for any $1 \leq k \leq n$, the number of up steps is greater than or equal to the number of down steps among the first k steps of P. By the definition of Φ, we have $p_1 = u$. Assume that in step k of the construction of $\Phi(\pi)$, we have already constructed a ballot path $p_1p_2\cdots p_{k-1}$. The task of this step is to locate $n-k+1$ or $\overline{n-k+1}$ in π in order to obtain p_k. We consider two cases.

If $\pi_i = n-k+1$ and i is odd or $\pi_i = \overline{n-k+1}$ and i is even, then we set $p_k = u$. Clearly, $p_1p_2\cdots p_k$ is a ballot path. If $\pi_i = n-k+1$ and i is even or $\pi_i = \overline{n-k+1}$ and i is odd, then we set $p_k = d$ and we wish to show that the height of p_k is nonnegative. In other words, the step p_k does not go below the x-axis while assuming that the first step p_1 starts at the origin. Consider the case $\pi_i = \overline{n-k+1}$ and i is odd. Note that the height of p_k equals the number of up steps among $p_1, p_2, \ldots, p_{k-1}$ minus the number of down steps among $p_1, p_2, \ldots, p_k$. By the definition of Φ, we find

$$\begin{aligned} h_k = \#\{&1 \leq j \leq n \,|\, \pi_j > 0, n-k+1 < \pi_j \text{ and } j \text{ is odd}\} \\ &+ \#\{1 \leq j \leq n \,|\, \pi_j < 0, n-k+1 < |\pi_j| \text{ and } j \text{ is even}\} \\ &- \#\{1 \leq j \leq n \,|\, \pi_j < 0, n-k+1 \leq |\pi_j| \text{ and } j \text{ is odd}\} \\ &- \#\{1 \leq j \leq n \,|\, \pi_j > 0, n-k+1 < \pi_j \text{ and } j \text{ is even}\}. \end{aligned} \tag{3.5}$$

In view of the alternating property of π, for any negative element π_{2i+1} at an odd position, we have $\pi_{2i} < \pi_{2i+1}$ and hence π_{2i} is negative. Consequently,

$$\begin{aligned} &\#\{1 \leq j \leq n \,|\, \pi_j < 0, n-k+1 \leq |\pi_j| \text{ and } j \text{ is odd}\} \\ &\quad \leq \#\{1 \leq j \leq n \,|\, \pi_j < 0, n-k+1 < |\pi_j| \text{ and } j \text{ is even}\}. \end{aligned}$$

On the other hand, for any positive element π_{2i} at an even position, we see that $\pi_{2i-1} > \pi_{2i}$ and hence π_{2i-1} is also positive. This implies that

$$\begin{aligned} &\#\{1 \leq j \leq n \,|\, \pi_j > 0, n-k+1 < \pi_j \text{ and } j \text{ is even}\} \\ &\quad \leq \#\{1 \leq j \leq n \,|\, \pi_j > 0, n-k+1 < \pi_j \text{ and } j \text{ is odd}\}. \end{aligned}$$

Thus we deduce that whenever there is a negative term contributing to h_k, there is at least one positive term contributing to h_k as well. So we conclude that $h_k \geq 0$. A similar argument applies to the case when $\pi_i = n-k+1$ and i is even. Hence we have shown that P is a ballot path.

We next prove that for any step in $\Phi(\pi)$, its label does not exceed its height. Assume we are in step k in the construction of $\Phi(\pi)$ and we have determined a labeled ballot $(p_1\cdots p_{k-1}, w_1\cdots w_{k-1})$ of length $k-1$. We proceed to locate $n-k+1$ or $\overline{n-k+1}$ in π in order to determine whether p_k is an up step or a down step and the label w_k of p_k. Suppose that $\pi_i = \overline{n-k+1}$ and i is odd. In this case, by the definition of Φ, we see that $p_k = d$ and $w_k = h_k - c_i(\pi)$. We claim that $c_i(\pi) \leq h_k$. To compute h_k by using formula (3.5), we shall consider two cases with respect to the range of j. The first case is $1 \leq j \leq i$ and the second case is $i+1 \leq j \leq n$. In other words, we shall consider the contributions of $\pi_1\pi_2\cdots\pi_i$ and $\pi_{i+1}\cdots\pi_n$ to the value of h_k.

We claim that $c_i(\pi)$ does not exceed the contribution of $\pi_{i+1}\cdots\pi_n$ to h_k. Suppose that n is odd. By the definition of $c_i(\pi)$, a pair (π_{2j}, π_{2j+1}) of consecutive elements of π with $i < 2j \leq n-1$ contributes 1 to the value of $c_i(\pi)$ if $\pi_{2j} < \pi_i < \pi_{2j+1} < 0$ or $\pi_{2j} < \pi_i < 0$ and $\pi_{2j+1} > 0$. If there is a pair (π_{2j}, π_{2j+1}) with $\pi_{2j} < \pi_i < \pi_{2j+1} < 0$, then this pair contributes 1 to both h_k and $c_i(\pi)$. If there is a pair (π_{2j}, π_{2j+1}) with $\pi_{2j} < \pi_i < 0$ and $\pi_{2j+1} > 0$, then this pair contributes 1 or 2 to h_k (depends on whether π_{2j+1} is greater than $n-k+1$), while it contributes 1 to $c_i(\pi)$. It is straightforward to check that if a pair (π_{2j}, π_{2j+1}) contributes 0 to $c_i(\pi)$, then it contributes 0 or 1 to h_k. On the other hand, because $\pi_1\cdots\pi_i$ contributes 0 to $c_i(\pi)$, it remains to show that the contribution of $\pi_1\cdots\pi_i$ to h_k is nonnegative. Let

$$\begin{aligned} g_i(\pi) = {} & \#\{1 \leq j \leq i \,|\, \pi_j > 0, n-k+1 < \pi_j \text{ and } j \text{ is odd}\} \\ & + \#\{1 \leq j \leq i \,|\, \pi_j < 0, n-k+1 < |\pi_j| \text{ and } j \text{ is even}\} \\ & - \#\{1 \leq j \leq i \,|\, \pi_j < 0, n-k+1 \leq |\pi_j| \text{ and } j \text{ is odd}\} \\ & - \#\{1 \leq j \leq i \,|\, \pi_j > 0, n-k+1 < \pi_j \text{ and } j \text{ is even}\}. \end{aligned}$$

By the same reasoning as in the proof of the fact that $h_k \geq 0$, we can verify that $g_i(\pi) \geq 0$. Thus we have completed the proof for the case when $\pi_i = \overline{n-k+1}$ and both n and i are odd. All the other cases with respect to the sign of π_i and the parities of n and i can be treated in the same manner. The details are omitted. This completes the proof. ∎

Proposition 3.2 *The map Ψ is well-defined, that is, for any labeled ballot path (P, W) of length n, the signed permutation $\pi = \Gamma_n = \pi_1\pi_2\cdots\pi_n$ is a snake of type B_n.*

Proof. Suppose that at the i-th step in the construction of $\Psi(P, W)$, we have already constructed a labeled ballot path (P_i, W_i). We first consider the case when P_i has an odd number of steps. In this case we aim to show that after contracting a certain step of P_i, we get a ballot path P_{i+1}.

By the construction of Ψ, if there exists a saturated down step in P_i, then we contract the leftmost saturated down step. In this case, we get a ballot path P_{i+1}. We now assume that all the down steps are unsaturated. This means that there are no down steps with height 0, that is, there are no down steps touching the x-axis. By the construction of Ψ, we shall contract the rightmost up step labeled by 0. After this step is contracted, it is easily seen that the height of every step in P_{i+1} is nonnegative, since by assumption there are no down steps that touch the x-axis. Hence we also get a ballot path P_{i+1} in this case. Moreover, one can check that after updating the labels of the steps in P_{i+1}, each step has a nonnegative label that does not exceed its height.

The case when P_i has an even number of steps can be dealt with by the same argument. Thus we conclude that once we have accomplished the task in step i, we are led to a labeled ballot path (P_{i+1}, W_{i+1}).

We now turn to the proof of the alternating property of π. It is apparent from the construction of Ψ that $\pi_1 > 0$. Now we prove that $\pi_1 > \pi_2 < \pi_3 > \cdots \pi_n$. Suppose that in step i we have already constructed a partial signed permutation Γ_{i-1} and a labeled ballot path (P_i, W_i). To determine Γ_i, by our construction, we are supposed to contract a certain step p_{r_i} in P_i to form a ballot path P_{i+1} and to set $\Gamma_i = (n - r_i + 1)\Gamma_{i-1}$ or $\Gamma_i = \overline{n - r_i + 1}\Gamma_{i-1}$ depending on whether p_{r_i} is an up step or a down step and the parity of the number of steps of P_i.

Similarly, in order to determine Γ_{i+1}, we contract a certain step $p_{r_{i+1}}$ of P_{i+1} to form P_{i+2} and set $\Gamma_{i+1} = (n-r_{i+1}+1)\Gamma_i$ or $\Gamma_{i+1} = \overline{n - r_{i+1} + 1}\Gamma_i$. For notational convenience, set $t_i = n-r_i+1$ and $t_{i+1} = n-r_{i+1}+1$. There are four possibilities for the construction of Γ_{i+1}, namely, $\bar{t}_{i+1}\bar{t}_i\Gamma_{i-1}, t_{i+1}t_i\Gamma_{i-1}$, $\bar{t}_{i+1}t_i\Gamma_{i-1}$ and $t_{i+1}\bar{t}_i\Gamma_{i-1}$.

We only consider the case that P_i has an odd number of steps and so t_i is at an odd position of π. To prove the alternating property of Γ_n, we aim to verify that

$$\begin{aligned}
\bar{t}_{i+1} < \bar{t}_i \quad &\text{if} \quad \Gamma_{i+1} = \bar{t}_{i+1}\bar{t}_i\Gamma_{i-1},\\
t_{i+1} < \bar{t}_i \quad &\text{if} \quad \Gamma_{i+1} = t_{i+1}\bar{t}_i\Gamma_{i-1},\\
t_{i+1} < t_i \quad &\text{if} \quad \Gamma_{i+1} = t_{i+1}t_i\Gamma_{i-1},
\end{aligned}$$

and that the situation $\Gamma_{i+1} = t_{i+1}\bar{t}_i\Gamma_{i-1}$ can never happen.

We shall give the proof only for the case $\Gamma_{i+1} = \bar{t}_{i+1}\bar{t}_i\Gamma_{i-1}$. To this end, we assume that in the i-th step in the construction of $\Psi(P, W)$, we contract a down step p_{r_i} of P_i, and in the $(i+1)$-st step, we contract an up step $p_{r_{i+1}}$ of P_{i+1}. By the implementation of Ψ, we see that $\Gamma_{i+1} = \bar{t}_{i+1}\bar{t}_i\Gamma_{i-1}$. We claim that $t_i < t_{i+1}$, that is, $r_i > r_{i+1}$. Assume to the contrary that $r_i < r_{i+1}$. Once the down step p_{r_i} is contracted, the height of all steps to the right of the step p_{r_i} will increase by 1. However, by the construction of Ψ, the labels of up steps remain unchanged. Since the height of each up step has been increased by 1, this implies that the up steps to the right of p_{r_i} are unsaturated. Therefore, the up step $p_{r_{i+1}}$ cannot be chosen in the $(i + 1)$-st step, which is a contradiction. So we deduce that $\bar{t}_{i+1} < \bar{t}_i$. The discussions for the other three cases are similar, and hence are omitted.

It remains to consider the case when the number of steps of P_i is even. But the argument for this case is analogous to the case when the number of steps of P_i is odd. Hence we have reached the conclusion that $\Gamma_n = \pi_1\pi_2 \cdots \pi_n$ is a snake of type B_n. This completes the proof. ∎

Proposition 3.3 *The maps Φ and Ψ are inverses of each other.*

Proof. It suffices to show that Ψ is the inverse of Φ. In fact, we need to prove a stronger property. To this end, let $\pi = \pi_1\pi_2 \cdots \pi_n$ be a snake of type B_n. For $1 \le i \le n$, let $\Pi_i = \pi_1\pi_2\cdots\pi_i$. For convenience, we use $\Phi(\Pi_i)$ to denote the labeled ballot path obtained by applying the map Φ to the partial signed permutation Π_i. Set $\Phi(\Pi_0)$ to be the empty path. Roughly speaking, by the construction of Φ, it can be seen that $\Phi(\Pi_i)$ is obtained from $\Phi(\Pi_{i-1})$ by adding a step to $\Phi(\Pi_{i-1})$ and updating the labels of other steps. A precise description of the process to obtain $\Phi(\Pi_i)$ from $\Phi(\Pi_{i-1})$ enables us to show that

$$\Psi(\Phi(\Pi_i)) = \Pi_i. \tag{3.6}$$

Of course, this implies that Ψ is the inverse of Φ.

Before we examine how to obtain $\Phi(\Pi_i)$ from $\Phi(\Pi_{i-1})$, we need to recall the construction of $\Phi(\Pi_i)$. The labeled ballot path $\Phi(\Pi_i)$ consists of i steps. Let $p_1p_2\cdots p_i$ denote this path. For any k with $1 \leq k \leq i$, we can find the element π_j among $\pi_1, \pi_2, \ldots, \pi_i$ such that $|\pi_j|$ is the k-th largest element of the set $\{|\pi_1|, \ldots, |\pi_i|\}$. Then by the construction of $\Phi(\Pi_i)$, we have $p_k = u$ if $\pi_j > 0$ and j is odd, or $\pi_j < 0$ and j is even; otherwise we have $p_k = d$. If $\pi_j > 0$, the label of the step p_k in $\Phi(\Pi_i)$ is equal to $c_j(\Pi_i)$. If $\pi_j < 0$, then the label of the step p_k in $\Phi(\Pi_i)$ is equal to $h_k - c_j(\Pi_i)$, where h_k is the height of the step p_k in $\Phi(\Pi_i)$. For example, for a snake $\pi = 2\bar{1}547\bar{6}\bar{3}$, we have $\Pi_3 = 2\bar{1}5$ and $\Phi(\Pi_3)$ consists of 3 steps $p_1p_2p_3$, where $p_1 = u$, $p_2 = u$ and $p_3 = u$, and the labels are $0, 1$ and 2.

Notice that for $i < n$, Π_i is a partial signed permutation on $[n]$, which is not necessarily a permutation of the set $\{1, 2, \ldots, i\}$. For example, for a snake $\pi = 2\bar{1}547\bar{6}\bar{3}$, we have $\Pi_3 = 2\bar{1}5$. For this reason, we should note that for $1 \leq k \leq i$, the k-th largest element of the set $\{|\pi_1|, \ldots, |\pi_i|\}$ is not necessarily the element $i + 1 - k$ in the construction of the k-th step p_k of the labeled ballot path $\Phi(\Pi_i)$. This explains why we should keep track of the k-th largest element of the set $\{|\pi_1|, \ldots, |\pi_i|\}$ in the construction of the k-th step of the labeled ballot path $\Phi(\Pi_i)$.

Given a labeled ballot path (P, W) of length n, let us look at the construction of $\Psi(P, W)$, which consists of n steps. At each step, we contract a certain step of P, delete the label of the contracted step and update the labels of other steps. We also obtain a partial signed permutation at every step.

For $i < n$, at the first step of applying Ψ to $\Phi(\Pi_i)$, let p_k denote the contracted step of the labeled ballot path $\Phi(\Pi_i) = p_1p_2\cdots p_i$. Assume the k-th largest element of the set $\{|\pi_1|, \ldots, |\pi_i|\}$ is $|\pi_j|$. Set $\Gamma_0 = \emptyset$. The partial signed permutation obtained at the first step of applying Ψ to $\Phi(\Pi_i)$ is $\Gamma_1 = (n - (n - |\pi_j| + 1) + 1)\Gamma_0 = |\pi_j|\Gamma_0 = |\pi_j|$ or $\Gamma_1 = \overline{n - (n - |\pi_j| + 1) + 1}\Gamma_0 = \overline{|\pi_j|}\Gamma_0 = \overline{|\pi_j|}$.

To prove the claim (3.6), we proceed by induction on i. When $i = 1$, $\Phi(\Pi_1) = \Phi(\pi_1)$ is a ballot path consisting of a single up step labeled by 0. As stated before, we should keep track of π_1. When applying Ψ to $\Phi(\Pi_1)$, we contract p_1 and obtain the partial signed permutation $n - (n - \pi_1 + 1) + 1 = \pi_1$. Assume that $\Psi(\Phi(\Pi_{i-1})) = \Pi_{i-1}$. We continue to show that $\Psi(\Phi(\Pi_i)) = \Pi_i$. In fact, we need to prove the following properties:

(1) At the first step of applying Ψ to $\Phi(\Pi_i)$, the contracted step is exactly the step

that is added to $\Phi(\Pi_{i-1})$ in order to get $\Phi(\Pi_i)$.

(2) After contracting the added step in $\Phi(\Pi_i)$ and updating the labels of other steps, we get the labeled ballot path $\Phi(\Pi_{i-1})$.

(3) The partial signed permutation obtained at the first step is π_i.

To describe the process to construct $\Phi(\Pi_i)$ from $\Phi(\Pi_{i-1})$, we need to consider four cases according to the parity of i and the sign of π_i. We shall restrict our attention only to the case when i is even and $\pi_i > 0$, because the same argument applies to other cases. In this case, assume that $|\pi_i|$ is the r-th largest element of the set $\{|\pi_1|, \ldots, |\pi_i|\}$. So we deduce that $\Phi(\Pi_i)$ can be obtained from $\Phi(\Pi_{i-1})$ by adding a down step p_r labeled by 0.

We now prove property (1), that is, at the first step of applying Ψ to $\Phi(\Pi_i)$, we must contract the down step p_r. Since p_r is labeled by 0, it is sufficient to show that there are no down steps of $\Phi(\Pi_i)$ labeled by 0 that appear to the left of p_r. Again, by the construction of $\Phi(\Pi_i)$, we see that there is a down step p_t of $\Phi(\Pi_i)$ appearing to the left of p_r if there exists an element of Π_i that is either

(a) a positive element at an even position which is greater than π_i, or

(b) a negative element at an odd position whose absolute value is greater than π_i.

It can be shown that if the step p_t is labeled by 0, then neither cases can happen. If (a) happens, that is, there exists an element of Π_i, say π_{2k}, such that $\pi_{2k} > \pi_i > 0$ and π_{2k} is the t-th largest element of the set $\{|\pi_1|, \ldots, |\pi_i|\}$. Since the step p_t is labeled by 0, we have $c_{2k}(\Pi_i) = 0$. By the definition of $c_{2k}(\Pi_i) = 0$, there are no pairs (π_{2j-1}, π_{2j}) of Π_i such that $\pi_{2j-1} > \pi_{2k} > \pi_{2j}$. In particular, by the assumption $\pi_{2k} > \pi_i$, we have $\pi_{2k} > \pi_{i-1}$. Since i is even, by the alternating property of Π_i, we have $\pi_{i-1} > \pi_{i-2}$. Thus we obtain $\pi_{2k} > \pi_{i-2}$. Moreover, since $c_{2k}(\Pi_i) = 0$, we get $\pi_{2k} > \pi_{i-3}$. Continuing this process, we reach the conclusion that $\pi_{2k} > \pi_{2k+1}$, contradicting the alternating property of Π_i.

If (b) happens, that is, there is an element of Π_i, say π_{2k+1}, such that $\pi_{2k+1} < 0$, $|\pi_{2k+1}| > \pi_i$ and $|\pi_{2k+1}|$ is the t-th largest element of the set $\{|\pi_1|, \ldots, |\pi_i|\}$. Since the step p_t is labeled by 0, we find that $c_{2k+1}(\Pi_i)$ equals the height of the step p_t. However, to the contrary, we can show that the height of p_t is greater than $c_{2k+1}(\Pi_i)$. Consider any pair (π_{2j-1}, π_{2j}) such that $\pi_{2j-1} > \pi_{2k+1} > \pi_{2j}$. Suppose that $|\pi_{2j}|$ is the j_1-th largest element of the set $\{|\pi_1|, \ldots, |\pi_i|\}$. Then the step p_{j_1} is an up step and it appears to the left of p_t. On the other hand, suppose that $|\pi_{2j-1}|$ is the j_2-th largest element of the set $\{|\pi_1|, \ldots, |\pi_i|\}$. Then one can check that the step p_{j_2} cannot be a down step that appears to the left of p_t. In other words, the pair (π_{2j-1}, π_{2j}) contributes 1 to $c_{2k+1}(\Pi_i)$, and it contributes at least 1 to the height of p_t. Moreover, suppose that $|\pi_{2k+2}|$ is the j_3-th largest element of the set $\{|\pi_1|, \ldots, |\pi_i|\}$. Since $\pi_{2k+2} < \pi_{2k+1} < 0$, we see that the step p_{j_3} is an up step that appears to the left of p_t. This means that the step p_{j_3} increases the height of the step p_t by 1. So we deduce that the height of p_t is at least $c_{2k+1}(\Pi_i)+1$.

Next we verify property (2), that is, after contracting the added step p_r in $\Phi(\Pi_i)$ and updating the labels of other steps, we get the labeled ballot path $\Phi(\Pi_{i-1})$. By the construction of Ψ, after contracting the down step p_r, we add 1 to the labels of up steps to the right of p_r, and subtract 1 from the labels of down steps to the left of p_r, all the labels of other steps remain unchanged.

We proceed to show that in the process of constructing $\Phi(\Pi_i)$ from $\Phi(\Pi_{i-1})$, after we add the down step p_r to $\Phi(\Pi_{i-1})$, the labels of all the up steps to the right of p_r decrease by 1, the labels of all the down steps to the left of p_r increase by 1, the labels of other steps remain unchanged.

Since the labels of the steps of $\Phi(\Pi_{i-1})$ involve the inversion code $c(\Pi_{i-1})$, let us examine the change of $c(\Pi_{i-1})$ after adding the element π_i to Π_{i-1}. According to Lemma 2.1, we see that after adding the element π_i to Π_{i-1}, for any element π_{2j} of Π_{i-1} with $\pi_{2j} > \pi_i$, $c_{2j}(\Pi_{i-1})$ increases by 1. For any element π_{2j+1} of Π_{i-1} with $\pi_{2j+1} < \pi_i$, $c_{2j+1}(\Pi_{i-1})$ decreases by 1. Let us consider the changes of the labels of the steps in $\Phi(\Pi_{i-1})$ in the process of constructing $\Phi(\Pi_i)$ from $\Phi(\Pi_{i-1})$.

Suppose that $|\pi_{2j}|$ ($|\pi_{2j+1}|$ resp.) is the j_4-th (j_5-th resp.) largest element of the set $\{|\pi_1|, \ldots, |\pi_i|\}$. By the construction of $\Phi(\Pi_{i-1})$, p_{j_4} is a down step whose label equals $c_{2j}(\Pi_{i-1})$. Thus the label of the step p_{j_4} increases by 1 in $\Phi(\Pi_i)$.

Next, we examine the change of the label of the step p_{j_5}. There are three cases: (i) If $\pi_{2j+1} > 0$, then p_{j_5} is an up step appearing to the right of p_r. Since the label of p_{j_5} equals $c_{2j+1}(\Pi_i)$, we see that its label decreases by 1 in $\Phi(\Pi_i)$. (ii) If $\pi_{2j+1} < 0$ and $|\pi_{2j+1}| < \pi_i$, then p_{j_5} is a down step appearing to the right of p_r. Since the label of the step p_{j_5} equals its height minus $c_{2j+1}(\Pi_i)$, and the height of p_{j_5} decreases by 1, we deduce that label of p_{j_5} remains unchanged. (iii) If $\pi_{2j+1} < 0$ and $|\pi_{2j+1}| > \pi_i$, then p_{j_5} is a down step appearing to the left of p_r. Since the height of p_{j_5} remains unchanged and $c_{2j+1}(\Pi_{i-1})$ decreases by 1, we deduce that the label of p_{j_5} increases by 1 in $\Phi(\Pi_i)$.

The proof of property (3) is obvious according to the construction of Ψ. Thus we have completed the proof for the case when i is even and $\pi_i > 0$. The other cases are omitted as mentioned before. ∎

4 A refinement

In this section, we obtain a formula for the bivariate generating function for the number $B(n,k)$ of labeled ballot paths of length n that end at the point (n,k), where $0 \le k \le n$. The numbers $B(n,k)$ can be considered as a refinement of the Springer numbers. By a restriction of our bijection, we also obtain a correspondence between labeled Dyck paths of length $2n$ and alternating permutations on $[2n]$. By considering the last step of a labeled ballot path, it is easy to derive the following recurrence relation.

Theorem 4.1 *For* $1 \leq k \leq n$, *we have*

$$B(n,k) = (k+1)B(n-1,k+1) + kB(n-1,k-1). \tag{4.7}$$

Since a ballot path can never end at a point (n,k) when $n+k$ is odd, we have $B(n,k) = 0$ if $n+k$ is odd.

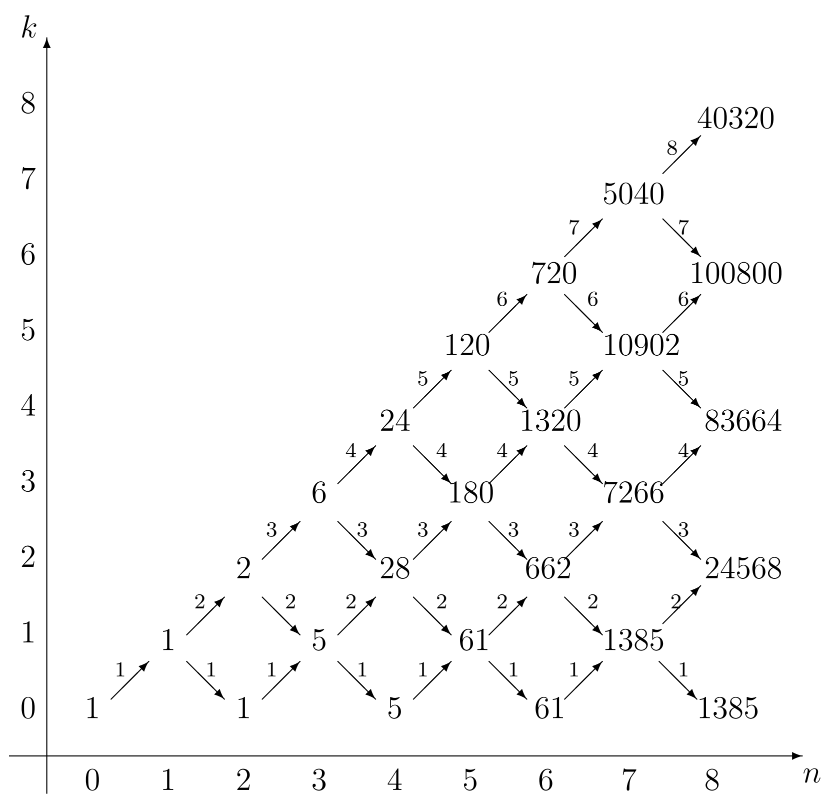

Figure 4: The table for $B(n,k)$ when $0 \leq k, n \leq 8$.

When $k = 0$, $B(2n,0)$ is the number of labeled Dyck paths of length $2n$, where a labeled Dyck path of length $2n$ is a labeled ballot path of length $2n$ that ends at a point on the x-axis. It should be noticed that the numbers $B(2n,0)$ are the secant numbers and they are closely related to alternate level codes of ballots, see Strehl [18]. Recall that an alternate level code of ballots of length n is an integer sequence $\lambda = \lambda_1\lambda_2\cdots\lambda_n$ such that $\lambda_1 = 1$, and for $2 \leq j \leq n$,

$$\lambda_{j-1} + 1 \geq \lambda_j \geq 1.$$

Denote by Λ_n the set of alternate level codes of ballots of length n. For example,

$$\Lambda_3 = \{111, 112, 121, 122, 123\}.$$

Rosen [14] derived the formula

$$\sum_{n\geq 0}\left(\sum_{\lambda\in\Lambda_n}\prod_{i=1}^{n}\lambda_i(\lambda_i+1)\right)\frac{x^n}{n!}=\tan x. \tag{4.8}$$

Strehl [18] deduced the secant companion equation of (4.8):

$$\sum_{n\geq 0}\left(\sum_{\lambda\in\Lambda_n}\prod_{i=1}^{n}\lambda_i^2\right)\frac{x^n}{n!}=\sec x. \tag{4.9}$$

To make a connection between labeled Dyck paths and alternate level codes of ballots, we need the following bijection, see Stanley [17, Ex. 6.19].

Theorem 4.2 *There is a bijection between the set of Dyck paths of length $2n$ and the set of alternate level codes of ballots of length n.*

Proof. Let $\lambda=\lambda_1\lambda_2\cdots\lambda_n\in\Lambda_n$ be an alternate level code of ballots of length n. We shall construct a Dyck path P of length $2n$ from λ. For convenience, we set $\lambda_{n+1}=1$. Let $P=P_1P_2\cdots P_n$, where $P_i=ud^{\lambda_i-\lambda_{i+1}+1}$. We proceed to prove that P is a Dyck path of length $2n$. First, we show that for $1\leq i\leq n$, the number of down steps is less than or equal to the number of up steps in $P_1\cdots P_i$, that is,

$$\sum_{j=1}^{i}(\lambda_j-\lambda_{j+1}+1)\leq i,$$

which is evident since $\lambda_1=1$ and $\lambda_{i+1}\geq 1$. Furthermore, one can check that there are exactly n down steps in P, namely,

$$\sum_{j=1}^{n}(\lambda_j-\lambda_{j+1}+1)=n.$$

Thus P is indeed a Dyck path of length $2n$.

Conversely, given a Dyck path P of length $2n$, we can construct an alternate level code of ballots $\lambda=\lambda_1\lambda_2\cdots\lambda_n$ of length n. Let λ_i be the larger y-coordinate of the endpoints of the i-th up step of P. It is straightforward to verify that λ is an alternate level code of ballots of length n. This completes the proof. ∎

For instance, let $\lambda=122\in\Lambda_3$. Then the Dyck path corresponding to λ is *uududd*. Using the above bijection we are led to a connection between the number $B(2n,0)$ and alternate level codes of ballots.

Corollary 4.3 *We have*

$$B(2n,0)=\sum_{\lambda\in\Lambda_n}\prod_{i=1}^{n}\lambda_i^2. \tag{4.10}$$

Proof. Relation (4.10) follows from the observation that for a given Dyck path, the number of labelings equals the product of squares of the elements of the corresponding alternate level code of ballots. ∎

In passing, we mention that Getu, Shapiro and Woen [8] considered a generalization of the formula of Rosen [14] on tangent numbers, namely, equation (4.8). More precisely, for a given ballot path, they defined the weight of the path to be the product of the y-coordinates of all the endpoints, except for the last point. Let $T(n,k)$ denote the sum of weights of ballot paths from $(1,1)$ to (n,k). It is easily checked that

$$T(n,k) = (k-1)T(n-1,k-1) + (k+1)T(n-1,k+1).$$

When $k=1$, $T(n,1)$ is the tangent number, that is,

$$\sum_{n\geq 1} T(n,1)\frac{x^n}{n!} = \tan x.$$

They gave a table for $T(n,k)$ similar to the table in Figure 4, where the first row consists of the tangent numbers. For $k \geq 1$, they obtained the generating function

$$\sum_{n\geq 1} T(n,k)\frac{x^n}{n!} = \frac{\tan^k x}{k}.$$

By replacing the first row of their table with the secant numbers, they introduced another number $E(n,k)$ and considered the following recurrence relation

$$E(n,k) = (k-1)E(n-1,k-1) + kE(n-1,k+1),$$

where $E(0,1)=1, E(1,2)=E(2,1)=1$ and $E(n,k)=0$ for $n<k-1$ or $k<1$. When $k=1$, $E(n,1)$ is the secant number. However, no combinatorial interpretation was given for the numbers $E(n,k)$. Using the recurrence relation of $E(n,k)$, Getu, Shapiro and Woen [8] derived the exponential generating function

$$\sum_{n\geq k} E(n,k)\frac{x^n}{n!} = \tan^{k-1} x \sec x. \tag{4.11}$$

Comparing the recurrence relations and the initial values of $B(n,k)$ and $E(n,k)$, we find that

$$B(n,k) = E(n,k+1).$$

Therefore $B(n,k)$ can be viewed as a combinatorial explanation for $E(n,k)$. Moreover, we obtain the generating function $G_n(y)$ for the n-th row of the table for $B(n,k)$. Let

$$G_n(y) = \sum_{k=0}^{n} B(n,k)y^k. \tag{4.12}$$

Note that

$$G_n(1) = \sum_{k=0}^{n} B(n,k)$$

equals the n-th Springer number. Let $B(x,y)$ be the generating function for $G_n(y)$, that is,

$$B(x,y) = \sum_{n\geq 0} G_n(y) \frac{x^n}{n!}.$$

Then we have the following formula.

Theorem 4.4 *We have*

$$B(x,y) = \frac{1}{\cos x - y \sin x}.$$

Proof. Let

$$F_k(x) = \sum_{n\geq k} B(n,k)\frac{x^n}{n!} = \sum_{n\geq k} E(n,k+1)\frac{x^n}{n!} = \tan^k x \sec x.$$

Hence

$$B(x,y) = \sum_{n\geq 0}\sum_{0\leq k\leq n} B(n,k) y^k \frac{x^n}{n!} = \sum_{k\geq 0} F_k(x) y^k = \frac{1}{\cos x - y\sin x},$$

as required. ∎

As applications of our bijection, we shall give a classification of snakes of type B_n and establish a connection between labeled Dyck paths and alternating permutations.

Define the statistic

$$\begin{aligned}\alpha(\pi) = \#\{1 \leq j \leq n \,|\, \pi_j > 0 \text{ and } j \text{ is odd}\} \\ + \#\{1 \leq j \leq n \,|\, \pi_j < 0 \text{ and } j \text{ is even}\} \\ - \#\{1 \leq j \leq n \,|\, \pi_j < 0 \text{ and } j \text{ is odd}\} \\ - \#\{1 \leq j \leq n \,|\, \pi_j > 0 \text{ and } j \text{ is even}\}.\end{aligned}$$

Then we have the following classification of snakes of type B_n.

Theorem 4.5 *For $0 \leq k \leq n$, $B(n,k)$ equals the number of snakes $\pi = \pi_1\pi_2\cdots\pi_n$ with $\alpha(\pi) = k$.*

Proof. From the construction of Φ from the set of snakes of type B_n to the set of labeled ballot paths of length n, it can be seen that $\alpha(\pi)$ equals the number of up steps minus the number of down steps of $\Phi(\pi)$. Consequently, if $\alpha(\pi) = k$, then the ballot path

$\Phi(\pi)$ ends at a point with y-coordinate k. So the theorem follows from the definition of $B(n,k)$. This completes the proof. ∎

We remark that Theorem 4.4 and Theorem 4.5 lead to a combinatorial interpretation for a sequence of derivative polynomials for the secant function $\sec x$, as introduced by Hoffman [9]. Let $\{Q_n(y)\}_{n\geq 0}$ be a sequence of polynomials defined by

$$\frac{d^n}{dx^n}\sec x = Q_n(\tan x)\sec x.$$

Hoffman [9] obtained the following exponential generating function for $Q_n(y)$

$$\sum_{n=0}^{\infty} Q_n(y)\frac{x^n}{n!} = \frac{1}{\cos x - y\sin x}.$$

Hence we have

$$Q_n(y) = G_n(y) = \sum_{\pi} y^{\alpha(\pi)},$$

where the sum ranges over snakes of type B_n.

We now consider a restriction of our bijection to labeled Dyck paths and alternating permutations. Substituting (4.10) into(4.9), we obtain

$$\sum_{n\geq 0} B(2n,0)\frac{x^n}{n!} = \sec x.$$

Since $\sec x$ is the generating function for the number E_{2n} of alternating permutations on $[2n]$, we see that $B(2n,0)$ equals E_{2n}. Recall that $B(2n,0)$ equals the number of labeled Dyck paths of length $2n$. The following theorem asserts that the restriction of the map Ψ to labeled Dyck paths serves as a combinatorial interpretation of the fact that $B(2n,0)=E_{2n}$. When restricted to labeled Dyck paths, the map Ψ does not involve any negative elements. On the other hand, when restricted to alternating permutations, the map Φ generates labeled Dyck paths.

Theorem 4.6 *The maps Ψ and Φ induce a bijection between labeled Dyck paths of length $2n$ and alternating permutations on $[2n]$.*

Proof. Let $(P,W)=(p_1\cdots p_{2n}, w_1\cdots w_{2n})$ be a labeled Dyck path of length $2n$. We wish to show that $\pi=\Psi(P,W)=\pi_1\cdots\pi_{2n}$ contains no negative elements. Since (P,W) is a labeled Dyck path, we see that in the first step of Ψ there exists a down step labeled by 0. Assume that p_{r_1} is the leftmost among such down steps. Applying the map Ψ, we are supposed to contract p_{r_1} into a single point to form a ballot path P_2. Then we add 1 to the labels of up steps of P_2 which are originally to the right of p_{r_1} and subtract 1 from the labels of down steps of P_2 which are originally to the left of p_{r_1}. Then we get a labeled ballot path (P_2,W_2) and a partial signed permutation $\Gamma_1=(n-r_1+1)\Gamma_0=(n-r_1+1)$, which contains no negative elements.

Similarly, in step 2, in the labeled ballot path (P_2, W_2), there does not exist any down step of P_2 whose label equals its height. So we can find an up step of P_2 labeled by 0. Suppose that p_{r_2} is the rightmost up step of P_2 with label 0. Contracting p_{r_2} gives a ballot path P_3. Subtract 1 from the labels of up steps of P_3 that are originally to the right of p_{r_2} and add 1 to the labels of down steps of P_3 that are originally to the left of p_{r_2}. Then we obtain a labeled ballot path (P_3, W_3) and a partial signed permutation $\Gamma_2 = (n - r_2 + 1)\Gamma_1 = (n - r_2 + 1)(n - r_1 + 1)$ without negative elements.

Note that P_1 is a Dyck path of length $2n$, and an up step in P_1 and a down step in P_2 are contracted. Hence there are $n-1$ up steps and $n-1$ down steps in P_3. It follows that (P_3, W_3) is a labeled Dyck path. Iterating the above process, we eventually obtain an alternating permutation.

Conversely, let $\pi = \pi_1\pi_2\cdots\pi_{2n}$ be an alternating permutation of length $2n$. We wish to show that $\Phi(\pi) = p_1 \cdots p_{2n}$ is a labeled Dyck path. Since $\Phi(\pi)$ is a labeled ballot path already, it suffices to show that $\Phi(\pi)$ has the same number of up steps as down steps. In step k of the map Φ, we are supposed to find the position of the element $n - k + 1$ in π so that we can determine whether p_k is an up step or a down step. Assume that $\pi_i = n - k + 1$. If i is odd, then $p_k = u$, and if i is even, then $p_k = d$. Since π has $2n$ elements, there are n odd positions as well as n even positions in π. So $\Phi(\pi)$ has n up steps and n down steps. Thus $\Phi(\pi)$ is a labeled Dyck path. This completes the proof. ∎

To conclude, we point out a connection between a special case of our bijection for labeled Dyck paths and the special cases of some known bijections. Françon and Viennot [6] found a bijection, denoted Φ_{FV}, between weighted 2-Motzkin paths of length $n-1$ and permutations on $[n]$ with last elements being n. As a variant of Φ_{FV}, Clarke, Steingrímsson and Zeng [5, p.255–257] obtained a bijection, denoted Φ_{CSZ}, between weighted 2-Motzkin paths of length n and permutations on $[n]$.

If the bijection Φ_{FV} is restricted to weighted 2-Motzkin paths of length $2n$ without horizontal steps, then the corresponding permutations become alternating permutations on $[2n+1]$ with last elements being $2n+1$, or equivalently, alternating permutations on $[2n]$. In fact, a weighted 2-Motzkin path of length $2n$ without horizontal steps is exactly a labeled Dyck path of length $2n$ in our terminology. This means that the above restriction of Φ_{FV} is a bijection between labeled Dyck path of length $2n$ and alternating permutations on $[2n]$.

Similarly, if we restrict the bijection Φ_{CSZ} to weighted 2-Motzkin paths of length $2n$ without horizontal steps, then the corresponding permutations become alternating permutations on $[2n]$. Thus the restriction of Φ_{CSZ} can be also regarded as a bijection between labeled Dyck paths of length $2n$ and alternating permutations on $[2n]$.

It should be noted that the restriction of our bijection to labeled Dyck paths is closely related to the restriction of the bijection Φ_{CSZ}. More precisely, we have the following assertions. Given an alternating permutation $\pi = \pi_1 \cdots \pi_{2n}$, let $\Phi(\pi) = (p_1 \cdots p_{2n}, w_1 \cdots w_{2n})$. Then we have $\Phi_{CSZ}(\pi) = (p'_{2n} \cdots p'_1, w_{2n} \cdots w_1)$, where $u' = d$

and $d' = u$. Conversely, given a labeled ballot path $(P, W) = (p_1 \cdots p_{2n}, w_1 \cdots w_{2n})$, let $(P', W') = (p'_{2n} \cdots p'_1, w_{2n} \cdots w_1)$, where $u' = d$ and $d' = u$. Then we have $\Psi(P, W) = \Phi_{CSZ}^{-1}(P', W')$. On the other hand, it can be seen that the restriction of our bijection to labeled Dyck paths is different from the restriction of Φ_{FV}.

It is also worth mentioning that Foata and Zeilberger [7] found a bijection, denoted Φ_{FZ}, between weighted 2-Motzkin paths of length n and permutations on $[n]$. This bijection can be reduced to a correspondence between certain weighted 2-Motzkin paths of length $2n$ and alternating permutations on $[2n]$. Biane [2] gave a bijection with a different weight assignment for 2-Motzkin paths, denoted Φ_B, between weighted 2-Motzkin paths of length n and permutations on $[n]$. The relations among the bijections $\Phi_{FV}, \Phi_{FZ}, \Phi_B$ and Φ_{CSZ} are discussed in [5, 13].

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