Oscillating Rim Hook Tableaux and Colored Matchings

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Abstract

Motivated by the question of finding a type B analogue of the bijection between oscillating tableaux and matchings, we find a correspondence between oscillating m-rim hook tableaux and m-colored matchings, where m is a positive integer. An oscillating m-rim hook tableau is defined as a sequence $(\lambda^0, \lambda^1, \ldots, \lambda^{2n})$ of Young diagrams starting with the empty shape and ending with the empty shape such that λ^i is obtained from λ^{i-1} by adding an m-rim hook or by deleting an m-rim hook. Our bijection relies on the generalized Schensted algorithm due to White. An oscillating 2-rim hook tableau is also called an oscillating domino tableau. When we restrict our attention to two column oscillating domino tableaux of length $2n$, we are led to a bijection between such tableaux and noncrossing 2-colored matchings on $\{1, 2, \ldots, 2n\}$, which are counted by the product C_nC_{n+1} of two consecutive Catalan numbers. A 2-colored matching is noncrossing if there are no two arcs of the same color that are intersecting. We show that oscillating domino tableaux with at most two columns are in one-to-one correspondence with Dyck path packings. A Dyck path packing of length $2n$ is a pair (D, E), where D is a Dyck path of length $2n$, and E is a dispersed Dyck path of length $2n$ that is weakly covered by D. So we deduce that Dyck path packings of length $2n$ are counted by C_nC_{n+1}.

Keywords: Oscillating m-rim hook tableau, m-colored matching, lattice path, bijection, Dyck path packing

AMS Classification: 05A15, 05A18

1 Introduction

The objective of this paper is to provide a rim hook analogue of the correspondence between oscillating tableaux and matchings given by Chen, Deng, Du, Stanley and Yan [3]. We show that there is a one-to-one correspondence between oscillating m-rim hook tableaux and m-(arc) colored matchings. The construction of our bijection relies on the generalized Schensted algorithm for rim hook tableaux introduced by White [9].

We shall pay special attention to the case of oscillating domino (2-rim hook) tableaux with at most two columns. In this case, our main result reduces to a bijection between oscillating domino tableaux with at most two columns and noncrossing 2-colored matchings.

Bear in mind that, a noncrossing 2-colored matching is not meant to be a noncrossing matching with two colors, but a matching that does not contain crossing edges that are of the same color. On the other hand, we find a correspondence between oscillating domino tableaux with at most two columns and Dyck path packings. A Dyck path packing can be viewed as a dispersed Dyck path E weakly covered by a Dyck path D, where a dispersed Dyck path is defined as non-overlapping Dyck paths connected by some horizontal steps on the x-axis. See Figure 1.1 for an illustration.

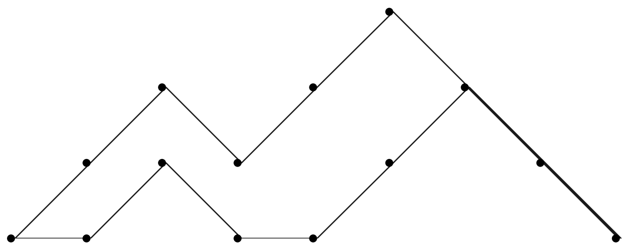


Figure 1.1: A Dyck path packing.

So we are led to a bijection between Dyck path packings and noncrossing 2-colored matchings. It is easy to check that noncrossing 2-colored matchings with $2n$ vertices are counted by the product of two Catalan numbers, that is, C_nC_{n+1}, where

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Notice that there are several combinatorial objects that are enumerated by the number C_nC_{n+1}, such as walks within the first quadrant of $\mathbb{Z}^3$ starting at $(0,0,0)$ and consisting of n steps taken from $\{(-1,0,0),(0,-1,1),(0,1,0),(1,0,-1)\}$, see Bostan and Kauers [1], walks within the first quadrant of $\mathbb{Z}^2$ starting at $(0,0)$, ending on the x-axis and consisting of $2n$ steps taken from $\{(-1,0),(-1,1),(1,-1),(1,0)\}$, see Bousquet-Mélou and Mishna [2], alternating Baxter permutations of length $2n+1$, see Cori, Dulucq and Viennot [4], and walks within the first quadrant of $\mathbb{Z}^2$ starting and ending at $(0,0)$ and consisting of $2n$ steps taken from $\{(-1,0),(0,-1),(0,1),(1,0)\}$, see Guy [6]. We also find a correspondence between noncrossing 2-colored matchings and Guy's walks. It would be interesting to establish further connections between packings of Dyck paths and other combinatorial structures.

Let us give an overview of some definitions. A partition of an integer n is a sequence $(\lambda_1,\lambda_2,\ldots,\lambda_\ell)$ of nonincreasing positive integers such that $\lambda_1+\lambda_2+\cdots+\lambda_\ell=n$. We can also represent a partition by its Young diagram, i.e., a left-justified array of cells (or, squares) with λ_i cells in row i for $1\le i\le \ell$. For example, Figure 1.2 is the Young diagram of the partition $(5,4,2,2)$.

To define the rim hooks of a partition, we note that the outside border of a partition λ is a collection of cells not in λ but immediately bellow or to the right of λ; or in the first

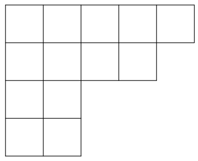


Figure 1.2: The Young diagram of $(5, 4, 2, 2)$

row and to the right of λ; or in the first column and bellow λ. For example, in Figure 1.3 the shaded area illustrates the outside border of $\lambda = (5, 4, 2, 2)$.

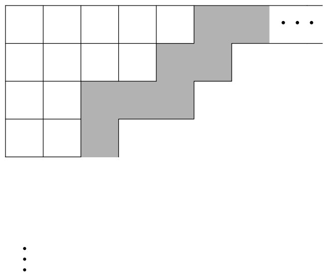
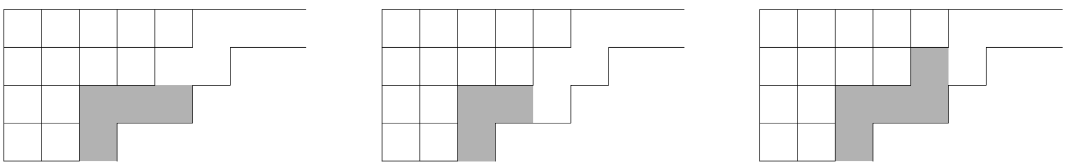


Figure 1.3: The outside border of $(5, 4, 2, 2)$

Let α be a set of contiguous cells in the outside border of λ. We say that α is a rim hook outside λ if the shape $\mu = \lambda \cup \alpha$ is the Young diagram of a partition. For example, in Figure 1.4, it can be seen that among the three sets of contiguous cells in the outside border of λ, there is only one rim hook outside λ, which is the third diagram. If α has



(1) (2) (3)

Figure 1.4: Contiguous cells in the outside border

m cells, then we write $|\alpha| = m$ and call α an m-rim hook outside λ. If α is a rim hook outside λ and $\mu = \lambda \cup \alpha$, then we call α an outer rim hook of μ, and we write $\mu - \alpha$ to mean λ.

We can now introduce the notion of oscillating m-rim hook tableaux. An oscillating m-rim hook tableau of length $2n$ can be defined as a sequence $\lambda = (\lambda^0, \lambda^1, \ldots, \lambda^{2n})$ of

Young diagrams such that $\lambda^0 = \lambda^{2n} = \emptyset$, and for $1 \le i \le 2n$, λ^i is obtained from λ^{i-1} either by adding an m-rim hook outside λ^{i-1} or by deleting an outer m-rim hook of λ^{i-1}. For example, Figure 1.5 is an illustration of an oscillating 3-rim hook tableau.

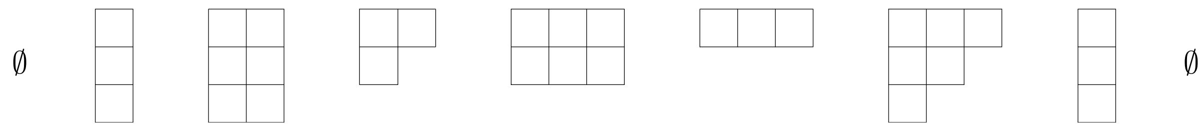


Figure 1.5: An oscillating 3-rim hook tableau

When $m = 1$, an oscillating m-rim hook tableau is an ordinary oscillating tableau. An oscillating 2-rim hook tableau will be also called an oscillating domino tableau. We shall use $r(\lambda)$ (resp., $c(\lambda)$) to denote the maximum number of rows (resp., columns) of the shape λ^i appearing in λ for $0 \le i \le 2n$.

The main objective of this paper is to show that there is a one-to-one correspondence between oscillating m-rim hook tableaux and m-colored matchings. An m-colored matching M on $[2n]$ is a matching on $[2n]$ with each arc assigned one of m colors, say, $c_1, c_2, \ldots, c_m$. For example, Figure 1.6 gives a 2-colored matching, where we use solid lines to represent arcs assigned the color c_1, and dotted lines to represent arcs assigned the color c_2. A k-crossing of M is a k-subset $\{(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)\}$ of arcs of the

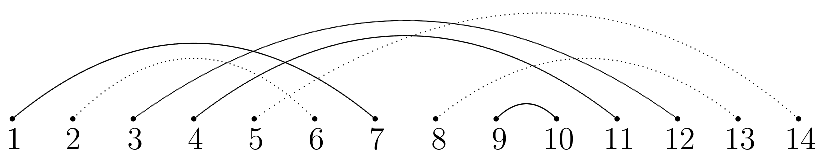


Figure 1.6: A 2-colored matching

same color such that $i_1 < i_2 < \cdots < i_k < j_1 < j_2 < \cdots < j_k$. Similarly, we define a k-nesting of M as a k-subset $\{(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)\}$ of arcs of the same color such that $i_1 < i_2 < \cdots < i_k < j_k < \cdots < j_2 < j_1$. Denote by $\mathrm{cr}(M)$ (resp., $\mathrm{ne}(M)$) the maximal number k such that M has a k-crossing (resp., k-nesting). We say that M is k-noncrossing (resp., k-nonnesting) if M has no k-crossing (resp., k-nesting). A 2-noncrossing (or, 2-nonnesting) m-colored matching is called a noncrossing (or, nonnesting) m-colored matching.

Our bijection can be used to characterize the crossing number and the nesting number of an m-colored matching in terms of the maximum number of columns and the maximum number of rows of shapes in the corresponding oscillating m-rim hook tableau. The construction of our bijection is based on the generalized Schensted algorithm for rim hook tableaux due to White [9].

This paper is organized as follows. We shall give a brief review of White's algorithm in Section 2. Based on this algorithm, we give a description of the bijection between oscillating m-rim hook tableaux and m-colored matchings in Section 3. Section 4 is concerned with oscillating domino tableaux with at most two columns and noncrossing 2-colored matchings. We show that such tableaux are in one-to-one correspondence with Dyck path packings. We also give a bijection between noncrossing 2-colored matchings and Guy's walks.

2 The generalized Schensted algorithm

To give a combinatorial proof of the orthogonality of the characters of the symmetric group S_n, White [9] extended the ordinary Schensted algorithm [7] to rim hook tableaux. For our purpose, we shall be concerned with only a special case of White's construction when all rim hooks and hooks are restricted to m-rim hooks and m-hooks. This version of White's algorithm has been further studied by Stanton and White [8]. As remarked by White [9], when $m = 2$, White's algorithm reduces to a one-to-one correspondence between elements of the hyperoctahedral group and pairs of domino tableaux of the same shape, which was first obtained by Lusztig. We shall adopt the notation and terminology in [8].

For two partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$, we write $\mu \subseteq \lambda$ if $\mu_i \leq \lambda_i$ for all i. If $\mu \subseteq \lambda$, then the skew diagram of shape λ/μ is defined as the set of cells obtained from λ by deleting the cells in μ. For example, the shaded area in Figure 2.1 represents a skew diagram.

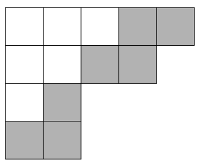


Figure 2.1: The skew diagram of shape $(5, 4, 2, 2)/(3, 2, 1)$

Let $i_1, i_2, \ldots, i_n$ be n positive integers with $i_1 < i_2 < \cdots < i_n$. An m-rim hook tableau P of shape λ on $\{i_1, i_2, \ldots, i_n\}$ is an assignment of $i_1, i_2, \ldots, i_n$ to the squares of λ with each integer appearing exactly m times such that for $r = 1, 2, \ldots, n$, the cells occupied by i_r is an outer m-rim hook of the shape obtained from λ by deleting the cells occupied by $i_{r+1}, \ldots, i_n$. We say that $i_1, i_2, \ldots, i_n$ are the contents of P, and write $\mathrm{content}(P) = \{i_1, i_2, \ldots, i_n\}$. Similarly, we can define skew m-rim hook tableaux of shape λ/μ. Figure 2.2 is an illustration of a 4-rim hook tableau.

Recall that an m-hook is the Young diagram corresponding to a partition $(t, 1, 1, \ldots, 1)$ of m, where $1 \leq t \leq m$. An m-hook tableau is an m-rim hook tableau whose shape is an m-hook. Figure 2.3 gives a 4-hook tableau of shape $(2, 1, 1)$.

1	1	1	1	4	4
2	3	3	4	4	
2	3	5			
2	3	5			
2	5	5			

Figure 2.2: A 4-rim hook tableau

3	3
3	
3	

Figure 2.3: A 4-hook tableau

Let P be an m-rim hook tableau, and H be an m-hook tableau. The generalized Schensted algorithm is an algorithm to generate an m-rim hook tableau by inserting H to P. When $m = 1$, it reduces to the usual Schensted algorithm. To describe the generalized Schensted algorithm, we need recall more definitions.

Let α be a set of contiguous cells contained in the outside border of λ. The head (resp., tail) of α is the upper rightmost (resp., lower leftmost) square in α. The head (resp., tail) of α is said to be illegal with respect to λ if the cell above (resp., to the left) the head (resp., tail) is in the outside border of λ. For example, in Figure 1.4 the contiguous cells in (1) have an illegal head, whereas the contiguous cells in (2) have an illegal tail. Clearly, if α has neither an illegal head nor an illegal tail with respect to λ, then α is a rim hook outside λ. Even though α is not a rim hook outside λ, it may be a rim hook outside another partition. We say that α is a rim hook if it is a rim hook outside a certain partition.

Let σ be an m-rim hook contained in the outside border of λ. Let $\mathit{slitherup}(\lambda, \sigma)$ denote the m-rim hook contained in the outside border of λ whose tail is adjacent to the head of σ. Similarly, we can define $\mathit{slitherdown}(\lambda, \sigma)$. Figure 2.4 gives illustrations of these two operations.

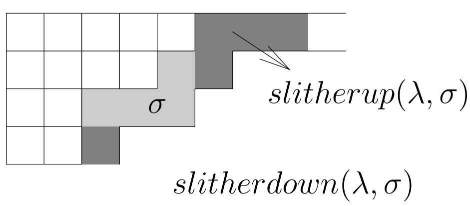


Figure 2.4: $\mathit{slitherup}(\lambda, \sigma)$ and $\mathit{slitherdown}(\lambda, \sigma)$

If α is a set of cells which are not necessarily contiguous, then *bumpout*(α) is the set of cells obtained by moving each cell in α to the position in the next row and in the next column. In other words, a cell in row i and column j is bumped out to the position in row $i+1$ and column $j+1$. If σ and τ are two distinct rim hooks outside λ such that $\sigma \cap \tau \neq \emptyset$, then define $\sigma[\tau]$ as a rim hook outside $\lambda \cup \tau$, that is,

$$\sigma[\tau] = (\sigma - \sigma \cap \tau) \cup bumpout(\sigma \cap \tau).$$

See Figure 2.5 for an illustration.

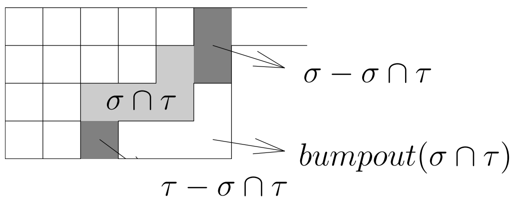


Figure 2.5: $\sigma[\tau]$

Let U be an m-rim hook tableau of shape λ, and V a skew m-rim hook tableau of shape μ/ω. The pair (U, V) is called an overlapping pair if $\sigma = \lambda/\omega$ is a rim hook outside ω, and any content in U is smaller than any content in V, see Figure 2.6.

$U =$

1	1	1	1	4	4
2	3	3	4	4	
2	3	5			
2	3	5			
2	5	5			

$V =$

					8	8
				8	8	
		7	7	9	9	
		7	9	9		
		7				

Figure 2.6: An overlapping pair

Given an overlapping pair (U, V), Stanton and White [8] defined an operator A acting on the pair (U, V), which generates an overlapping pair $(U^1, V^1) = A(U, V)$. To be precise, the operator A can be described as follows. Suppose that r is the minimum content in V, and τ is the m-rim hook containing r. For a rim hook α and an integer s, let $\alpha(s)$ denote the rim kook α with each cell of α filled with s. The skew m-rim hook tableau V^1 is obtained from V by removing $\tau(r)$. Then U^1 can be constructed depending on how the shapes σ and τ overlap. Note that $\sigma = \lambda/\omega$, where λ is the shape of U and μ/ω is the shape of V. There are three cases.

Case 1: $\sigma \cap \tau = \emptyset$. Set $U^1 = U \cup \tau(r)$.

Case 2: $\sigma \cap \tau \neq \emptyset$ and $\sigma \neq \tau$. Set $U^1 = U \cup \tau[\sigma](r)$.

Case 3: $\sigma = \tau$. In this case, we construct a sequence $(\tau = \tau_0, \tau_1, \tau_2, \ldots)$ of m-rim hooks contained in the outside border of λ, where $\tau_i = \mathit{slitherup}(\lambda, \tau_{i-1})$ for $i \geq 1$. Assume that m_0 is the smallest integer such that the head of τ_{m_0} is legal with respect to λ. Set $U^1 = U \cup \tau_{m_0}(r)$.

It is not difficult to check that $A(U, V)$ is an overlapping pair, see [9]. Let

$$A^n(U, V) = A^{n-1}(A(U, V)),$$

and write

$$(U^n, V^n) = A^n(U, V).$$

Assume that n_0 is the smallest integer such that V^{n_0} is empty. Then, define

$$\mathit{Combine}(U, V) = U^{n_0}.$$

We can now state the rim hook insertion algorithm. Suppose that P is an m-rim hook tableau of shape λ with contents not containing r, and H is an m-hook tableau with content r. The rim hook insertion algorithm gives an m-rim hook tableau by inserting H to P, denoted $P \leftarrow H$. To obtain $P \leftarrow H$, we need to define an overlapping pair (U, V). Let $P = P_1 \cup P_2$, where P_1 (resp., P_2) is the m-rim hook tableau contained in P with contents smaller than (resp., greater than) r. We set $V = P_2$.

To define U, we assume that τ is the shape of H, and λ' is the shape of P_1. Let $(\tau_0, \tau_1, \tau_2, \ldots)$ be a sequence of m-rim hooks, where $\tau_0 = \tau$ and $\tau_i = \mathit{slitherup}(\emptyset, \tau_{i-1})$ for $i \geq 1$. Moreover, we assume that m_0 is the smallest integer such that τ_{m_0} has no intersection with λ'. We still need another sequence $(\sigma_0, \sigma_1, \sigma_2, \ldots)$ of m-rim hooks contained in the outside border of λ', where $\sigma_0 = \tau_{m_0}$ and $\sigma_i = \mathit{slitherup}(\lambda', \sigma_{i-1})$ for $i \geq 1$. Assume that n_0 is the smallest integer such that σ_{n_0} has a legal head with respect to λ'. Then U is set to be $P_1 \cup \tau(r)$.

Based on the m-rim hook tableaux U and V, the m-rim hook tableau $P \leftarrow H$ can be defined as $\mathit{Combine}(U, V)$. It can be shown that the above rim hook insertion algorithm is invertible, see [9].

The rim hook insertion algorithm leads to a one-to-one correspondence between m-hook permutations and pairs of m-rim hook tableaux of the same shape. Let $i_1, i_2, \ldots, i_n$ be n positive integers with $i_1 < i_2 < \cdots < i_n$. An m-hook permutation on $\{i_1, i_2, \ldots, i_n\}$ is a permutation of n m-hook tableaux such that the contents of these m-hook tableaux read off from left to right form a permutation on $\{i_1, i_2, \ldots, i_n\}$. For example, Figure 2.7 illustrates a 4-hook permutation on $\{1, 4, 6, 9\}$.

As remarked in [8], an m-hook permutation on $[n]$ can be viewed as an element of the wreath product $C_m \wr S_n$, where C_m is the cyclic group of order m, or equivalently, as an m-colored permutation on $[n]$ in the sense that each element in the permutation is assigned one of the colors $c_1, c_2, \ldots, c_m$.

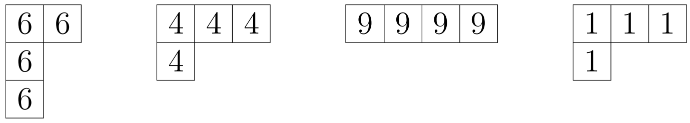


Figure 2.7: A 4-hook permutation

Given an m-hook permutation $\mathcal{H} = H_1 H_2 \cdots H_n$, one can construct an m-rim hook tableaux $P = (\cdots((\emptyset \leftarrow H_1) \leftarrow H_2) \leftarrow \cdots) \leftarrow H_n$ by inserting $H_1, H_2, \ldots, H_n$ one after another, see [8] for details. We call P the insertion tableau of $\mathcal{H}$.

Theorem 2.1 (Stanton and White [8]) *There is a bijection* Sch *between m-hook permutations on $\{1, 2, \ldots, n\}$ and all pairs of m-rim hook tableaux of the same shape with content $\{1, 2, \ldots, n\}$.*

The bijection Sch in Theorem 2.1 inherits many important properties of the ordinary Schensted correspondence. For our purpose, we need the property on the lengths of the longest increasing and decreasing subsequences in an m-hook permutation. An increasing subsequence in an m-hook permutation $\mathcal{H} = H_1 H_1 \cdots H_n$ is a subsequence $H_{i_1} H_{i_2} \cdots H_{i_s}$ such that $H_{i_1}, H_{i_2}, \ldots, H_{i_s}$ are of the same shape and $\text{content}(H_{i_1}) < \text{content}(H_{i_2}) < \cdots < \text{content}(H_{i_s})$. A decreasing subsequence in $\mathcal{H}$ can be defined analogously. The following theorem is due to Stanton and White [8], where $\lceil x \rceil$ is the usual ceiling function meaning the smallest integer greater than or equal to x.

Theorem 2.2 *Let $\mathcal{H}$ be an m-hook permutation, and let P be the insertion tableau of $\mathcal{H}$. Suppose that P has r rows and c columns. Then the length of the longest increasing (resp., decreasing) subsequence in $\mathcal{H}$ is $\lceil r/m \rceil$ (resp., $\lceil c/m \rceil$).*

The following proposition will be used in the proof of Theorem 3.1.

Proposition 2.3 *Let $\mathcal{H} = H_1 \cdots H_r \cdots H_n$ be an m-hook permutation with H_r containing the maximum content, and let $\hat{\mathcal{H}} = H_1 \cdots \hat{H}_r \cdots H_n$ be the m-hook permutation obtained from $\mathcal{H}$ by deleting H_r. Suppose that P_1 (resp., P_2) is the insertion tableau of $\mathcal{H}$ (resp., $\hat{\mathcal{H}}$). Then P_2 is the tableau obtained from P_1 by removing the m-rim hook filled with the maximum content.*

3 Oscillating rim hook tableaux

In this section, we present a bijection between oscillating m-rim hook tableaux and m-colored matchings. Recall that for an oscillating m-rim hook tableaux λ, $r(\lambda)$ (resp., $c(\lambda)$) is the maximum number of rows (resp., columns) of shapes appearing in λ.

Theorem 3.1 *There is a bijection ϕ between oscillating m-rim hook tableaux of length $2n$ and m-colored matchings on $[2n]$. Moreover, for any oscillating m-rim hook tableau λ we have*

$$\lceil r(\lambda)/m \rceil = \mathrm{ne}(\phi(\lambda)) \tag{3.1}$$

and

$$\lceil c(\lambda)/m \rceil = \mathrm{cr}(\phi(\lambda)). \tag{3.2}$$

Proof. We first describe the bijection ϕ from oscillating m-rim hook tableaux of length $2n$ to m-colored matchings on $[2n]$. Let

$$\lambda = (\lambda^0, \lambda^1, \ldots, \lambda^{2n})$$

be an oscillating m-rim hook tableau. We shall recursively define a sequence

$$(M_0, T_0), (M_1, T_1), \ldots, (M_{2n}, T_{2n}),$$

where M_i is a set of m-colored arcs, and T_i is an m-rim hook tableau of shape λ^i. Let M_0 be the empty set, and let T_0 be the empty tableau. To obtain (M_i, T_i) for $i \geq 1$, we have the following two cases.

Case 1: λ^i is obtained from λ^{i-1} by adding an m-rim hook outside λ_{i-1}. In this case, let $M_i = M_{i-1}$ and let T_i be the m-rim hook tableau obtained from T_{i-1} by filling the m-rim hook λ^i/λ^{i-1} with the element i.

Case 2: λ^i is obtained from λ^{i-1} by deleting an outer m-rim hook of λ_{i-1}. In this case, let T_i be the m-rim hook tableau of shape λ^i such that T_{i-1} is obtained from T_i by inserting an m-hook tableau H. Note that H is uniquely determined since the generalized Schensted algorithm is invertible. Suppose that H has shape $(t, 1, 1, \ldots, 1)$ and content$(H) = \{j\}$. It is easy to see that $j < i$. Let M_i be the set obtained from M_{i-1} by adding an arc (j, i) colored with c_t.

From the above construction, it is easy to check that M_{2n} is an m-colored matching on $[2n]$. We set $\phi(\lambda) = M_{2n}$.

The reverse map ψ, from m-colored matchings to oscillating m-rim hook tableaux, can be described as follows. Let M be an m-colored matching on $[2n]$. We shall construct a sequence $T(M) = (T_0, T_1, \ldots, T_{2n})$ of m-rim hook tableaux by a recursive procedure.

Let T_{2n} be the empty tableau. For $j \leq 2n-1$, T_j can be constructed based on the following two cases.

Case 1: $j+1$ is the right-hand endpoint of an arc $(i, j+1)$ colored with c_t. In this case, let T_j be the m-rim hook tableau obtained from T_{j+1} by inserting an m-hook tableau of shape $(t, 1, \ldots, 1)$ filled with i.

Case 2: $j+1$ is the left-hand endpoint of an arc $(j+1, k)$. Note that in this case $j+1$ is the largest content of T_{j+1}. Let T_j be the m-rim hook tableau obtained from T_{j+1} by deleting the m-rim hook filled with $j+1$.

Set $\lambda = (\lambda^0, \lambda^1, \ldots, \lambda^{2n})$, where λ^i is the shape of T_i. From the above procedure, it can be seen that λ is an oscillating m-rim hook tableau. Define $\psi(M) = \lambda$. It is not hard to check that ψ is the reverse map of ϕ.

It remains to prove the relations (3.1) and (3.2). For a m-colored matching M, let $T(M) = (T_0, T_1, \ldots, T_{2n})$ be the sequence of m-rim hook tableaux constructed from M. We now give a recursive procedure to generate a sequence of m-hook permutations $(\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_{2n})$ from the sequence $T(M)$ of m-rim hook tableaux. Let $\mathcal{H}_{2n}$ be the empty hook permutation. Suppose that the m-hook permutation $\mathcal{H}_i$ has been constructed. To generate $\mathcal{H}_{i-1}$, we consider the following two cases.

Case 1: T_{i-1} is obtained from T_i by inserting an m-Hook tableau H_j. In this case, let $\mathcal{H}_{i-1} = \mathcal{H}_i H_j$.

Case 2: T_i is obtained from T_{i-1} by adding an m-rim hook filled with i. In this case, let $\mathcal{H}_{i-1}$ be the hook permutation obtained from $\mathcal{H}_i$ by deleting the m-hook tableau with content i.

We claim that T_i is the insertion tableau of $\mathcal{H}_i$. This can be shown by induction. Clearly, the claim holds for $i = 2n$. Suppose that it is true for i, where $1 < i \leq 2n$. We shall show that the claim holds for $i-1$. If $\mathcal{H}_{i-1} = \mathcal{H}_i H_j$, then the statement is obvious. So it suffices to consider the case when $\mathcal{H}_{i-1}$ is the hook permutation obtained from $\mathcal{H}_i$ by deleting the m-hook tableau with content i. Observe that in this case, i must be the largest content appearing in the m-hook tableaux of $\mathcal{H}_i$. In view of Proposition 2.3, we see that T_{i-1} is the insertion tableau of $\mathcal{H}_{i-1}$.

To finish the proof of (3.1), we proceed to show that M has a k-crossing if and only if there exists an m-hook permutation $\mathcal{H}_i$ $(1 \leq i \leq 2n)$ that contains a decreasing subsequence of length k. Suppose that $\mathcal{H}_i = H_{i1} H_{i2} \cdots H_{it_i}$ $(1 \leq i \leq 2n)$ contains a decreasing subsequence of length k. By the construction of $\mathcal{H}_i$, there are t_i m-colored arcs of M whose left-hand endpoints are $\mathrm{content}(H_{i1}), \mathrm{content}(H_{i2}), \ldots, \mathrm{content}(H_{it_i})$. For $1 \leq s \leq t_i$, let w_{is} be the right-hand endpoint of the arc with left-hand endpoint $\mathrm{content}(H_{is})$. Again, by the construction of the sequence $(\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_{2n})$, it can be easily checked that

$$w_{i1} > w_{i2} > \cdots > w_{it_i}. \tag{3.3}$$

Denote by $H_{is_1} H_{is_2} \cdots H_{is_k}$ the decreasing subsequence of $\mathcal{H}_i$. Clearly, the arcs

$$(\mathrm{content}(H_{is_1}), w_{is_1}), \ldots, (\mathrm{content}(H_{is_k}), w_{is_k})$$

have the same color. Moreover, from (3.3) it follows that they form a k-crossing of M.

Finally, we need to consider the other direction of the above statement. Suppose that M has a k-crossing consisting the arcs

$$(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)$$

with $i_1 < i_2 < \cdots < i_k < j_1 < j_2 < \cdots < j_k$. By the construction of ϕ, it can be easily checked that $i_1, \ldots, i_k$ are contained in $\mathrm{content}(T_{j_1 - 1})$. Consider the m-hook permutation

$\mathcal{H}_{j_1-1}$. Replacing i with j_1-1 in (3.3), we obtain that the subsequence of $\mathcal{H}_{j_1-1}$ consisting of the m-hook tableaux with contents $i_1, i_2, \ldots, i_k$ is a decreasing subsequence of length k. This proves the above claim. Thus, relation (3.1) can be deduced from Theorem 2.2.

To prove (3.2), it suffices to show that M has a k-nesting if and only if there exists an m-hook permutation $\mathcal{H}_i$ $(1 \le i \le 2n)$ that contains an increasing subsequence of length k. The proof is omitted since it is analogous to the case for k-crossings. ∎

We now give an example to illustrate the bijection ϕ in Theorem 3.1. Let λ be the oscillating domino tableau in Figure 3.1.

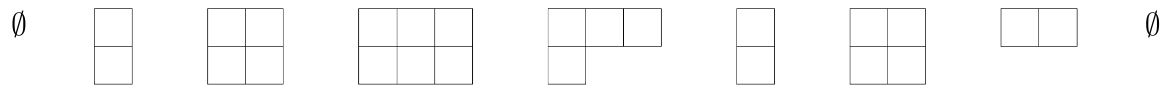


Figure 3.1: An oscillating domino tableau

The sequence $(T_1, T_2, \ldots, T_8)$ of domino tableaux corresponding to λ is given in Figure 3.2.

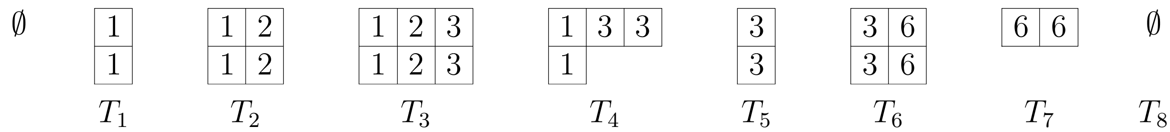


Figure 3.2: An illustration of the bijection ϕ

In view of the construction of the bijection ϕ, to obtain the 2-colored matching corresponding to λ, we need the hook tableaux H_4, H_5, H_7 and H_8 as given in Figure 3.3. Note that for $i = 4, 5, 7, 8$, T_{i-1} is obtained from T_i by inserting H_i.

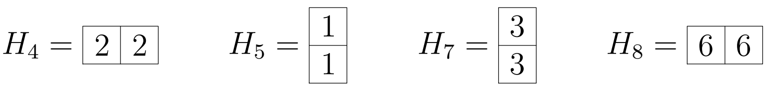


Figure 3.3: H_4, H_5, H_7 and H_8

The 2-colored matching corresponding to λ is given in Figure 3.4, where the dotted lines represent arcs with color c_1, and the solid lines represent arcs with color c_2.

It can be easily checked that

$$\lceil r(\lambda)/2 \rceil = \mathrm{ne}(\phi(\lambda)) = 1 \quad \text{and} \quad \lceil c(\lambda)/2 \rceil = \mathrm{cr}(\phi(\lambda)) = 2.$$

The bijection in Theorem 3.1 leads to a symmetry property with respect to the crossing number and the nesting number of m-colored matchings. Let $\lambda = (\lambda^0, \lambda^1, \ldots, \lambda^{2n})$ be an oscillating m-rim hook tableau. The conjugate oscillating m-rim hook tableau of λ,

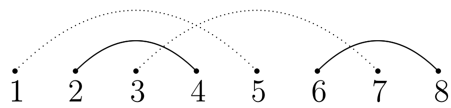


Figure 3.4: The corresponding 2-colored matching

denoted by λ', is defined by $(\mu^0, \mu^1, \ldots, \mu^{2n})$, where μ^i is the conjugate of λ^i. By Theorem 3.1, we get an involution on m-colored matchings satisfying the following symmetry property.

Corollary 3.2 *The crossing number* $\mathrm{cr}(M)$ *and the nesting number* $\mathrm{ne}(M)$ *have a symmetric distribution over all m-colored matchings on* $[2n]$*, that is, for any* $0 \le i, j \le n$*, the number of m-colored matchings on* $[2n]$ *with crossing number i and nesting number j equals the number of m-colored matchings on* $[2n]$ *with crossing number j and nesting number i.*

Remark. We note that the above symmetry property can also be deduced from the bijection between oscillating tableaux and ordinary matchings, see Chen, Deng, Du, Stanley and Yan [3]. More precisely, an m-colored matching M can be considered as a sequence $(M_1, M_2, \ldots, M_m)$ of disjoint matchings whose union is a complete matching on $[2n]$, where M_i is the set of arcs of M colored with c_i. It is obvious that M is k-noncrossing if and only if each M_i is k-noncrossing. Nevertheless, Theorem 3.1 does not seem to be a direct consequence of the correspondence for the case $m = 1$.

We should also note that the above symmetry property holds even if we fix the left-hand and right-hand endpoints of matchings. Recall that for ordinary matchings the symmetry property holds while fixing the left-hand endpoints and the right-hand endpoints of matchings, see [3]. More precisely, let S and T be two disjoint subsets of $[2n]$ with $|S| = |T|$ and $S \cup T = [2n]$. For matchings with left-hand endpoint set S and right-hand endpoint set T, the symmetry property with respect to the crossing number and the nesting number is still valid.

It is easy to see that for m-colored matchings, the above symmetry property also holds if we fix the set S_i of the left-hand endpoints of arcs with color c_i and fix the set T_i of the right-hand endpoints of arcs with color c_i. However, this stronger symmetry property does not seem to be a consequence of the bijection ϕ in Theorem 3.1, since transposing an oscillating m-rim hook tableau does not preserve the sets S_i and T_i for the corresponding m-colored matchings.

The bijection between oscillating m-rim hook tableaux and m-colored matchings leads to a generating function formation for certain classes of oscillating m-rim hook tableaux. Let $f_{m,k}(n)$ be the number of oscillating m-rim hook tableaux λ of length $2n$ with $c(\lambda) \le m(k-1)$. By Theorem 3.1, it equals the number of k-noncrossing m-colored matchings on $[2n]$. Let $F_{m,k}(x)$ be the generating function of $f_{m,k}(n)$, that is,

$$F_{m,k}(x) = \sum_{i\geq 0} f_{m,k}(i) \frac{x^{2i}}{(2i)!}.$$

When $m = 1$, we simply write $f_k(n)$ and $F_k(x)$ for $f_{m,k}(n)$ and $F_{m,k}(x)$ respectively. It is known that

$$F_k(x) = \det[I_{i-j}(2x) - I_{i+j}(2x)]_{i,j=1}^{k-1}, \tag{3.4}$$

where

$$I_n(2x) = \sum_{i\geq 0} \frac{x^{n+2i}}{i!(n+i)!}$$

is the hyperbolic Bessel function of the first kind of order n, see Grabiner and Magyar [5]. Then we have the following relation.

Corollary 3.3 *For $m \geq 1$ and $k \geq 2$, we have*

$$F_{m,k}(x) = (F_k(x))^m. \tag{3.5}$$

Proof. Let $M = (M_1, M_2, \ldots, M_m)$ be an m-colored matching. It is clear that M is k-noncrossing if and only if each matching M_i is k-noncrossing. It is easy to see that the number of k-noncrossing m-colored matchings on $2n$ can be expressed as

$$\sum_{i_1+i_2+\cdots+i_m=n} \binom{2n}{2i_1, 2i_2, \ldots, 2i_m} f_k(i_1) f_k(i_2) \cdots f_k(i_m), \tag{3.6}$$

from which it follows (3.5). This completes the proof. ∎

4 Oscillating domino tableaux

In this section, we introduce the structure of packings of Dyck paths and establish a connection with oscillating domino tableaux with each shape having at most two columns. By Theorem 3.1, we see that such oscillating domino tableaux are in one-to-one correspondence with noncrossing 2-colored matchings. We show that oscillating domino tableaux with at most two columns are in one-to-one correspondence with Dyck path packings. This means that there is a correspondence between Dyck path packings and noncrossing 2-colored matchings. It is easy to prove that the number of noncrossing 2-colored matchings on $[2n]$ equals C_nC_{n+1}, where C_n is the n-th Catalan number.

As will be seen, noncrossing 2-colored matchings are quite close to Guy's walks [6] in the sense that there is a simple bijection between these two structures. However, it should be mentioned that the problem of counting Dyck path packings does not seem to be as easy as counting noncrossing 2-colored matchings. Using formula (3.6) for the case $m = 2$ and the following identity

$$\sum_{i=0}^{n} \binom{2n}{2i} C_i C_{n-j} = C_n C_{n+1}, \tag{4.1}$$

see [4], we obtain the following formula for number of noncrossing 2-colored matchings on $[2n]$.

Theorem 4.1 *For $n \geq 0$, the number of noncrossing 2-colored matchings on $[2n]$ equals $C_n C_{n+1}$.*

Let $O_{2n,2}$ be the set of oscillating domino tableaux λ of length $2n$ with each shape appearing in λ having at most two columns. By Theorem 4.1, we see that the set $O_{2n,2}$ is also counted by $C_n C_{n+1}$. We shall give a bijection between $O_{2n,2}$ and Dyck path packings. A Dyck path of length $2n$ is a lattice path in the plane from $(0, 0)$ to $(2n, 0)$ with steps $(1, 1)$ and $(1, -1)$ that never passes below the x-axis. A dispersed Dyck path is a concatenation of Dyck paths and some horizontal steps $(1, 0)$ on the x-axis. In other words, a dispersed Dyck path of length $2n$ is a lattice path from $(0, 0)$ to $(2n, 0)$ with up and down steps above the x-axis and with horizontal steps on the x-axis. We say that a dispersed Dyck path E is weakly covered by a Dyck path D if E never goes above D. If D is a Dyck path of length $2n$ and E is a dispersed Dyck path of length $2n$ that is weakly covered by D, then we say that the pair (D, E) is a packing of Dyck paths of length $2n$. Denote by P_{2n} the set of Dyck path packings of length $2n$.

Theorem 4.2 *There is a bijection between the set $O_{2n,2}$ of oscillating domino tableaux and the set P_{2n} of Dyck path packings.*

Proof. Let $\mu = (\mu^0, \mu^1, \ldots, \mu^{2n})$ be an oscillating domino tableau in $O_{2n,2}$, and let $\lambda = (\lambda^0, \lambda^1, \ldots, \lambda^{2n})$ be the conjugate oscillating domino tableau of μ, that is, λ^i is the conjugate of μ^i $(0 \leq i \leq 2n)$. Since each shape λ^i has at most two rows, we may write $\lambda^i = (u_i, v_i)$, where $u_i \geq v_i \geq 0$. Clearly, both $u_i + v_i$ and $u_i - v_i$ are even. For $0 \leq i \leq 2n$, let

$$a_i = \frac{u_i + v_i}{2} \quad \text{and} \quad b_i = \frac{u_i - v_i}{2}.$$

Define two lattice paths D and E by setting

$$D = ((0, a_0), (1, a_1), \ldots, (2n, a_{2n})) \quad \text{and} \quad E = ((0, b_0), (1, b_1), \ldots, (2n, b_{2n})), \tag{4.2}$$

where a lattice path is represented by the lattice points.

We proceed to show that the map $\alpha\colon \mu \longrightarrow (D, E)$ defined by the above construction is a bijection from $O_{2n,2}$ to P_{2n}. As the first step, we prove that (D, E) is a packing of Dyck paths of length $2n$. It can be easily checked that $a_0 = a_{2n} = 0$ and $(i+1, a_{i+1}) - (i, a_i) = (1, 1)$ or $(1, -1)$. Hence D is a Dyck path of length $2n$. Let us consider the possible values of $(i+1, b_{i+1}) - (i, b_i)$. There are two cases.

Case 1: λ^{i+1} is obtained from λ^i by adding or deleting a horizontal domino. In this case, it is easy to check that $(i+1, b_{i+1}) - (i, b_i) = (1, 1)$ or $(1, -1)$.

Case 2: λ^{i+1} is obtained from λ^i by adding or deleting a vertical domino. In this case, both λ^i and λ^{i+1} have two rows with the same number of cells, that is, $u_i = v_i$ and $u_{i+1} = v_{i+1}$. This implies that $b_i = b_{i+1} = 0$ and $(i+1, b_{i+1}) - (i, b_i) = (1, 0)$. So we deduce that if there is a horizontal step on the path E, then it lies on the x-axis. It is clear that $b_i \geq 0$. Thus E is a dispersed Dyck path of length $2n$.

Since $a_i \geq b_i$, we see that E is weakly covered by D. Hence we conclude that (D, E) is a packing of Dyck paths of length $2n$.

The inverse map of α can be described as follows. Let (D, E) be a Dyck path packing in P_{2n}. Write D and E in the forms as in (4.2). Set $\lambda^i = (a_i + b_i, a_i - b_i)$, and let μ^i be the conjugate of λ^i. It is easy to check that $(\mu^0, \mu^1, \ldots, \mu^{2n})$ is an oscillating domino tableau belonging to $O_{2n,2}$. This completes the proof. ∎

For example, let λ be the conjugate of the oscillating domino tableau in Figure 3.2. The corresponding Dyck path packing is given in Figure 4.1.

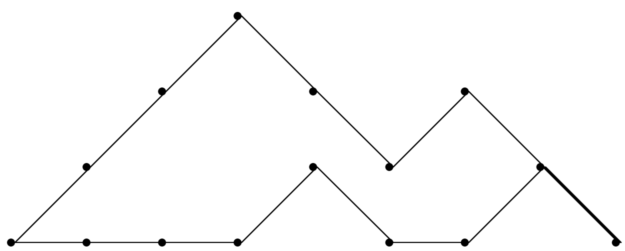


Figure 4.1: An example for Theorem 4.2

By Theorem 4.1 and Theorem 4.2, we obtain the following relations.

Theorem 4.3 *For* $n \geq 0$,

$$|O_{2n,2}| = |P_{2n}| = C_n C_{n+1}. \tag{4.3}$$

To conclude this paper, we give a bijection between noncrossing 2-colored matchings and Guy's walks. More precisely, a Guy's walk is defined to be a lattice walk within the first quadrant starting and ending at $(0, 0)$ and consisting of the following steps

$$(-1, 0),\ (0, -1),\ (0, 1),\ (1, 0).$$

Figure 4.2 gives a Guy's walk with steps

$$\rightarrow\ \uparrow\ \uparrow\ \downarrow\ \rightarrow\ \uparrow\ \uparrow\ \rightarrow\ \downarrow\ \leftarrow\ \downarrow\ \leftarrow\ \leftarrow\ \uparrow\ \downarrow\ \downarrow.$$

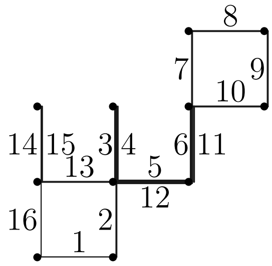


Figure 4.2: A Guy's walk

Theorem 4.4 *There is a bijection between noncrossing 2-colored matchings on* $[2n]$ *and Guy's walks with* $2n$ *steps.*

Proof. We construct a bijection β from noncrossing 2-colored matchings on $[2n]$ to Guy's walks with $2n$ steps. Let M be a noncrossing 2-colored matching on $[2n]$. For any i in $[2n]$, we define a step s_i as follows:

$$s_i = \begin{cases} (1,0), & \text{if } i \text{ is the left-hand endoint of an arc with color } c_1; \\ (-1,0), & \text{if } i \text{ is the right-hand endpoint of an arc with color } c_1; \\ (0,1), & \text{if } i \text{ is the left-hand endpoint of an arc with color } c_2; \\ (0,-1), & \text{if } i \text{ is the right-hand endpoint of an arc with color } c_2. \end{cases}$$

Then define $\beta(M)$ to be the walk $s_1 s_2 \cdots s_{2n}$ starting at the origin.

We claim that $\beta(M)$ is a walk in the first quadrant, starting and ending at the origin. For any $i \in [2n]$, consider the first i steps $s_1, s_2, \ldots, s_i$. Since M is a noncrossing 2-colored matching, it is not hard to check that there are at least as many $(1,0)$ steps as $(-1,0)$ steps in $\{s_1, s_2, \ldots, s_i\}$. Similarly, there are at least as many $(0,1)$ steps as $(0,-1)$ steps in $\{s_1, s_2, \ldots, s_i\}$. Hence $\beta(M)$ is in the first quadrant. It is clear that $\beta(M)$ terminates at the origin. So the claim holds. It is not difficult to see that the map β is invertible. This completes the proof. ∎

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References

[1] A. Bostan and M. Kauers, Automatic classification of restricted lattice walks, Proceedings of FPSAC'09, 201-215.

[2] M. Bousquet-Mélou and M. Mishna, Walks with small steps in the quarter plane, Contemp. Math. 520 (2010), 1-40.

[3] W.Y.C. Chen, E.Y.P. Deng, R.R.X. Du, R.P. Stanley and C.H. Yan, Crossings and nestings of matchings and partitions, Trans. Amer. Math. Soc. 359 (2007), 1555–1575.

[4] R. Cori, S. Dulucq and G. Viennot, Shuffle of parenthesis systems and Baxter permutations, J. Combin. Theory Ser. A 43 (1986), 1-22.

[5] D. Grabiner and P. Magyar, Random walks in Weyl chambers and the decomposition of tensor powers, J. Alg. Combin. 2 (1993), 239-260.

[6] R.K. Guy, Catwalks, sandsteps and Pascal pyramids, J. Integer Seq. 3 (2000), Article 00.1.6.

[7] C. Schensted, Longest increasing and decreasing subsequences, Canad. J. Math. 13 (1961), 179-191.

[8] D. Stanton and D. White, A Schensted algorithm for rim hook tableaux, J. Combin. Theory Ser. A 40 (1985), 211-247.

[9] D. White, A bijection proving orthogonality of the characters of S_n, Adv. Math. 50 (1983), 160-186.