Partially 2-Colored Permutations and the Boros-Moll Polynomials

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Abstract

We find a combinatorial setting for the coefficients of the Boros-Moll polynomials $P_m(a)$ in terms of partially 2-colored permutations. Using this model, we give a combinatorial proof of a recurrence relation on the coefficients of $P_m(a)$. This approach enables us to give a combinatorial interpretation of the log-concavity of $P_m(a)$ which was conjectured by Moll and confirmed by Kauers and Paule.

Keywords: partially 2-colored permutation, Boros-Moll polynomial, rising factorial, log-concavity, bijection

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1 Introduction

The main objective of this paper is to present a combinatorial approach to the log-concavity of the Boros-Moll polynomials. The Boros-Moll polynomials $P_m(a)$ arise in the evaluation of a quartic integral, see [3–7, 13]. Boros and Moll have shown that for any $a > -1$ and any nonnegative integer m,

$$\int_0^\infty \frac{1}{(x^4 + 2ax^2 + 1)^{m+1}} dx = \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}} P_m(a), \tag{1.1}$$

where

$$P_m(a) = \sum_{j,k} \binom{2m+1}{2j} \binom{m-j}{k} \binom{2k+2j}{k+j} \frac{(a+1)^j (a-1)^k}{2^{3(k+j)}}. \tag{1.2}$$

Boros and Moll also derived a single sum formula for $P_m(a)$:

$$P_m(a) = 2^{-2m} \sum_k 2^k \binom{2m-2k}{m-k} \binom{m+k}{k} (a+1)^k, \tag{1.3}$$

which implies that the coefficients of $P_m(a)$ are positive. More precisely, let $d_i(m)$ be the coefficient of a^i in $P_m(a)$. Then (1.3) gives

$$d_i(m) = 2^{-2m} \sum_{k=i}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{k} \binom{k}{i}. \tag{1.4}$$

Several proofs of the formula (1.3) can be found in the survey of Amdeberhan and Moll [2].

Further positivity properties of $P_m(a)$ have been studied recently. Boros and Moll [5] have shown that the sequence $\{d_i(m)\}_{0 \le i \le m}$ is unimodal for $m \ge 0$. Moll conjectured that this sequence is log-concave, that is, for $m \ge 2$ and $1 \le i \le m-1$,

$$d_i^2(m) \ge d_{i-1}(m) d_{i+1}(m). \tag{1.5}$$

This conjecture has been confirmed by Kauers and Paule [12] based on recurrence relations. Chen and Xia [10] have proved a stronger property of $d_i(m)$, called the ratio monotone property, which implies both the log-concavity and the spiral property. Moll [14,15] posed a conjecture that is stronger than the log-concavity of $P_m(a)$. This conjecture has been proved by Chen and Xia [11]. Chen and Gu [8] established the reverse ultra log-concavity of the Boros-Moll polynomials.

It turns out that the polynomials $P_m(a)$ are closely related to combinatorial structures. The 2-adic valuation of the numbers $i!m!2^{m+i}d_i(m)$ has been studied by Amdeberhan, Manna and Moll [1], and Sun and Moll [16]. By using reluctant functions and an extension of Foata's bijection, Chen, Pang and Qu [9] have found a combinatorial derivation of the single sum formula (1.3) from the double sum formula (1.2). For the special case $a = 1$, we are led to a combinatorial argument for the identity

$$\sum_{k=0}^{m} 2^{-2k} \binom{2k}{k} \binom{2m-k}{m} = \sum_{k=0}^{m} 2^{-2k} \binom{2k}{k} \binom{2m+1}{2k}.$$

However, this combinatorial approach does not seem to apply to recurrence relations for $d_i(m)$ or the log-concavity of $P_m(a)$.

In this paper, we shall consider a variation of the coefficients $d_i(m)$, that is,

$$D_i(m) = \binom{2m}{m-i} m! i! (m-i)! 2^i d_i(m). \tag{1.6}$$

Then the numbers $D_i(m)$ have a combinatorial interpretation in terms of partially 2-colored permutations.

Using this combinatorial setting, we give an explanation of the following recurrence relation of $d_i(m)$ derived independently by Kauers and Paule [12] and Moll [14]:

$$i(i+1)d_{i+1}(m) = i(2m+1)d_i(m) - (m-i+1)(m+i)d_{i-1}(m). \tag{1.7}$$

The reasoning of the above recurrence relation also implies a simple combinatorial interpretation of the log-concavity of the Boros-Moll polynomials.

2 A combinatorial setting for $D_i(m)$

In this section, we shall give a combinatorial interpretation of $D_i(m)$ by introducing the structure of partially 2-colored permutations. Throughout this paper, we shall adopt the notation $(x)_n$ for rising factorials, that is, $(x)_0 = 1$ and for $n > 0$,

$$(x)_n = x(x+1)\cdots(x+n-1).$$

From the expression (1.4) for $d_i(m)$, we have

$$\begin{aligned}
d_i(m) &= 2^{-2m}\sum_{k=i}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{k}\binom{k}{i} \\
&= 2^{-2m}\sum_{j=0}^{m-i} 2^{j+i} \binom{2m-2i-2j}{m-i-j}\binom{m+i+j}{i+j}\binom{i+j}{i} \\
&= 2^{-2m}\sum_{j=0}^{m-i} 2^{j+i} \frac{(2m-2i-2j)!}{(m-i-j)!(m-i-j)!}\cdot\frac{(m+i+j)!}{(i+j)!m!}\cdot\frac{(i+j)!}{j!i!} \\
&= 2^{-2m}\sum_{j=0}^{m-i} 2^{j+i} \frac{2^{2m-2i-2j}(m-i-j-\frac{1}{2})!}{(m-i-j)!}\cdot\frac{(m+i+j)!}{(i+j)!m!}\cdot\frac{(i+j)!}{j!i!}.
\end{aligned}$$

It follows that

$$\begin{aligned}
m!i!(m-i)!2^i d_i(m) &= (m-i)!\sum_{j=0}^{m-i}\left(\frac{1}{2}\right)^j \frac{(m-i-j-\frac{1}{2})!}{(m-i-j)!}\cdot\frac{(m+i+j)!}{j!}, \\
&= \sum_{j=0}^{m-i}\binom{m-i}{j}\left(\frac{1}{2}\right)^j\left(\frac{1}{2}\right)_{m-i-j}(m+i+j)!,
\end{aligned}$$

which yields

$$D_i(m) = \binom{2m}{m-i}\sum_{j=0}^{m-i}\binom{m-i}{j}\left(\frac{1}{2}\right)^j\left(\frac{1}{2}\right)_{m-i-j}(m+i+j)!. \tag{2.1}$$

We proceed to give a combinatorial interpretation of $D_i(m)$ according to the expression (2.1). It is well known that $(x)_n$ equals the generating function for permutations on $[n]$ with respect to the number of cycles. Let σ be a permutation on $[n]$. The weight of σ is defined as x^k, where k is the number of cycles in σ. So $(x)_n$ is the weighted count of permutations on $[n]$.

Suppose that (A, B, C) is a composition of $[2m] = \{1, 2, \ldots, 2m\}$, namely, any A, B and C are disjoint and $A \cup B \cup C = [2m]$, where A, B and C are allowed to be empty. A permutation on $[2m]$ associated with a composition (A, B, C) of $[2m]$ is called a partially 2-colored permutation on $[2m]$ if it can be written as $(\pi|\sigma)$, where π is a permutation on $A \cup B$ and σ is a permutation on C. We assume that the elements in A are white, the elements in B are black and written in boldface, while the elements in C are uncolored.

Moreover, we need to use two different representations for the permutations π and σ in a partially 2-colored permutation $(\pi|\sigma)$. To be precise, we shall write π in the one-line notation in the form of a sequence. For example, $5, 7, 8, 2, 1, 6, 4, 3$ is the one-line representation of a permutation. On the other hand, we shall express σ in terms of the cycle decomposition. For instance, the permutation in the above example has cycle decomposition $(1, 5)(2, 7, 4)(3, 8)(6)$.

Let $\mathcal{D}_i(m)$ denote the set of all partially 2-colored permutations $(\pi|\sigma)$ on $[2m]$ such that the 2-colored permutation π has $m + i$ black elements. For example, consider the partially 2-colored permutation

$$(\mathbf{2}, \mathbf{12}, 8, \mathbf{11}, \mathbf{5}, \mathbf{9}, \mathbf{7}, 1, \mathbf{4}, \mathbf{3}|(6, 10))$$

in $\mathcal{D}_2(6)$. Then we have $A = \{1, 8\}$, $B = \{2, 3, 4, 5, 7, 9, 11, 12\}$, and $C = \{6, 10\}$. From the definition, we see that for a partially 2-colored permutation $(\pi|\sigma)$ in $\mathcal{D}_i(m)$, we have $|A \cup C| = m - i$.

We are now ready to give a combinatorial interpretation of $D_i(m)$. With respect to the weight a partially 2-colored permutation $(\pi|\sigma)$ in $\mathcal{D}_i(m)$, we impose the following rules:

(1) An element in A is given a weight $\frac{1}{2}$;

(2) A cycle in σ is given a weight $\frac{1}{2}$.

The weight $(\pi|\sigma)$ is defined as the product of the weights of the white elements and the cycles. In light of the above weight assignment, $D_i(m)$ can be viewed as a weighted count of partially 2-colored permutations. The weight of a set S means to be the sum of weights of its elements, and is denoted by $w(S)$.

Theorem 2.1. *For $m \geq 1$, $D_i(m)$ equals the weight of $\mathcal{D}_i(m)$.*

Proof. Given a composition (A, B, C) of $[2m]$ such that $|B| = m+i$ and $|A \cup C| = m-i$. Assume that there are j elements in A. It is clear that there are $m-i-j$ elements in C. Now, there are $\binom{2m}{m-i}$ ways to distribute $2m$ elements into B and $A \cup C$. Moreover, there are $\binom{m-i}{j}$ ways to distribute $m-i$ elements into A and C.

Consider partially 2-colored permutations in $\mathcal{D}_i(m)$ associated with composition (A, B, C) of $[2m]$. Since $|A \cup B| = m+i+j$, the sum of weights of permutations on $A \cup B$ equals

$$\left(\frac{1}{2}\right)^j \cdot (m+i+j)!.$$

The weighted sum of permutations on C equals $\left(\frac{1}{2}\right)_{m-i-j}$. This completes the proof. ∎

3 Combinatorial proof of the recurrence relation

Using the interpretation of $D_i(m)$ in terms of partially 2-colors permutation, we give a combinatorial proof for the following recurrence relation of the coefficients $d_i(m)$ of the Boros-Moll polynomials

$$i(i+1)d_{i+1}(m) = i(2m+1)d_i(m) - (m-i+1)(m+i)d_{i-1}(m). \tag{3.1}$$

This recurrence was independently derived by Kauers, Paule [12] and Moll [14].

Utilizing (1.6), the recurrence relation (3.1) can be restated as

$$\frac{1}{2}(m+i+1)D_{i+1}(m) + 2(m-i+1)D_{i-1}(m) = (2m+1)D_i(m). \tag{3.2}$$

To give a combinatorial proof of (3.2), we need to introduce some notation. Let $\mathcal{A}_i(m)$ (resp. $\mathcal{B}_i(m)$ and $\mathcal{C}_i(m)$) denote the set of all partially 2-colored permutations $(\pi|\sigma)$ in $\mathcal{D}_i(m)$ such that exactly one element in A (resp. B and C) is underlined. Obviously, the four sets $\mathcal{A}_i(m)$, $\mathcal{B}_i(m)$, $\mathcal{C}_i(m)$ and $\mathcal{D}_i(m)$ are disjoint. For example,

$$(\mathbf{2}, \mathbf{12}, 8, \mathbf{11}, \mathbf{5}, \underline{\mathbf{9}}, \mathbf{7}, 1, \mathbf{4}, \mathbf{3}|(6, 10))$$

is an underlined partially 2-colored permutation belonging to $\mathcal{B}_2(6)$. By definition and Theorem 2.1, we have

$$\begin{aligned} (m+i)D_i(m) &= w(\mathcal{B}_i(m)), & (3.3)\\ (m-i)D_i(m) &= w(\mathcal{A}_i(m) \cup \mathcal{C}_i(m)). & (3.4) \end{aligned}$$

Proof. From (3.3) and (3.4), we know that

$$(m+i+1)D_{i+1}(m) = w(\mathcal{B}_{i+1}(m)), \tag{3.5}$$

$$(m-i+1)D_{i-1}(m) \;=\; w(\mathcal{A}_{i-1}(m)\cup\mathcal{C}_{i-1}(m)). \tag{3.6}$$

On the other hand, we have

$$(2m+1)D_i(m) = w(\mathcal{A}_i(m)\cup\mathcal{B}_i(m)\cup\mathcal{C}_i(m)\cup\mathcal{D}_i(m)). \tag{3.7}$$

First, we claim that

$$\frac{1}{2}w(\mathcal{B}_{i+1}(m)) = w(\mathcal{A}_i(m)). \tag{3.8}$$

Given $(\pi|\sigma)\in\mathcal{B}_{i+1}(m)$ with underlying composition (A,B,C), where $|B|=m+i+1$ and $|A\cup C|=m-i-1$, by changing the underlined black element in π to an underlined white element, we obtain an underlined partially 2-colored permutation in $\mathcal{A}_i(m)$. Clearly, this operation yields a bijection between $\mathcal{B}_{i+1}(m)$ and $\mathcal{A}_i(m)$. Since the weight of a white element equals $1/2$, we obtain (3.8). Substituting i with $i-1$ in (3.8), we get

$$w(\mathcal{B}_i(m)) = 2w(\mathcal{A}_{i-1}(m)). \tag{3.9}$$

Hence (3.2) simplifies to the following relation

$$2w(\mathcal{C}_{i-1}(m)) = w(\mathcal{C}_i(m)\cup\mathcal{D}_i(m)). \tag{3.10}$$

Assume that $(\pi|\sigma)\in\mathcal{C}_{i-1}(m)$ is a partially 2-colored permutation with underlying composition (A,B,C), that is, $|B|=m+i-1$, $|A\cup C|=m-i+1$, and σ is a permutation with an underlined element. Suppose that σ has cycle decomposition $C_0,C_1,\ldots,C_r$, where C_0 contains the underlined element. Without loss of generality, we may always write C_0 as $(\underline{i_1}i_2\cdots i_k)$. Given $(\pi|\sigma)\in\mathcal{C}_{i-1}(m)$, we define

$$\Delta(\pi|\sigma) = \{\Delta_1,\Delta_2,\ldots,\Delta_k\},$$

where

$$\begin{aligned}
\Delta_1 &= (\pi,\mathbf{i_1}|(\underline{i_2},i_3,\ldots,i_k)C_1C_2\cdots C_r),\\
\Delta_2 &= (\pi,\mathbf{i_1},i_2|(\underline{i_3},\ldots,i_k)C_1C_2\cdots C_r),\\
&\cdots\\
\Delta_{k-1} &= (\pi,\mathbf{i_1},i_2,\ldots,i_{k-1}|(\underline{i_k})C_1C_2\cdots C_r),\\
\Delta_k &= (\pi,\mathbf{i_1},i_2,\ldots,i_{k-1},i_k|C_1C_2\cdots C_r).
\end{aligned}$$

For $1\le j\le k-1$, we have $\Delta_j\in\mathcal{C}_i(m)$ and

$$w(\Delta_j) = \frac{1}{2^{j-1}}w(\pi|\sigma). \tag{3.11}$$

Moreover, we see that $\Delta_k \in \mathcal{D}_i(m)$ and

$$w(\Delta_k) = \frac{1}{2^{k-2}} w(\pi|\sigma). \tag{3.12}$$

Conversely, any partially colored permutation in $\mathcal{C}_i(m) \cup \mathcal{D}_i(m)$ can be obtained from a partially colored permutation in $\mathcal{C}_{i-1}(m)$ by applying the above operation Δ. Thus, we deduce that

$$\Delta(\mathcal{C}_{i-1}(m)) = \mathcal{C}_i(m) \cup \mathcal{D}_i(m), \tag{3.13}$$

where Δ acts on the partially colored permutations in $\mathcal{C}_{i-1}(m)$. Since

$$\sum_{j=1}^{k-1} \frac{1}{2^{j-1}} + \frac{1}{2^{k-2}} = 2,$$

combining (3.11), (3.12) and (3.13) we obtain (3.2). This completes the proof. ∎

4 Combinatorial proof of the log-concavity

In this section, we shall use the structure of partially 2-colored permutations to give a combinatorial reasoning of the following relation

$$(m+i+1)D_{i+1}(m) \cdot (m-i+1)D_{i-1}(m) < (m+i)(m-i+1)D_i^2(m), \tag{4.1}$$

which implies the log-concavity of the Boros-Moll polynomials. We shall follow the notation introduced in the previous section.

Proof. From (3.5) and (3.6), we see that

$$\begin{aligned}
&(m+i+1)D_{i+1}(m) \cdot (m-i+1)D_{i-1}(m) \\
&\quad= w(\mathcal{B}_{i+1}(m)) \cdot w(\mathcal{A}_{i-1}(m) \cup \mathcal{C}_{i-1}(m)) \\
&\quad= w(\mathcal{B}_{i+1}(m)) \cdot w(\mathcal{A}_{i-1}(m)) + w(\mathcal{B}_{i+1}(m)) \cdot w(\mathcal{C}_{i-1}(m)).
\end{aligned} \tag{4.2}$$

Meanwhile, in view of (3.3) and (3.4), we find

$$\begin{aligned}
&(m+i)(m-i+1)D_i^2(m) \\
&\quad= w(\mathcal{B}_i(m)) \cdot w(\mathcal{A}_i(m) \cup \mathcal{C}_i(m) \cup \mathcal{D}_i(m)) \\
&\quad= w(\mathcal{B}_i(m)) \cdot w(\mathcal{A}_i(m)) + w(\mathcal{B}_i(m)) \cdot w(\mathcal{C}_i(m) \cup \mathcal{D}_i(m)).
\end{aligned} \tag{4.3}$$

Hence (4.1) can be recast as

$$w(\mathcal{B}_{i+1}(m)) \cdot w(\mathcal{A}_{i-1}(m)) + w(\mathcal{B}_{i+1}(m)) \cdot w(\mathcal{C}_{i-1}(m))$$

$$< w(\mathcal{B}_i(m)) \cdot w(\mathcal{A}_i(m)) + w(\mathcal{B}_i(m)) \cdot w(\mathcal{C}_i(m) \cup \mathcal{D}_i(m)). \tag{4.4}$$

Invoking (3.8) and (3.9), we obtain

$$w(\mathcal{B}_{i+1}(m)) \cdot w(\mathcal{A}_{i-1}(m)) = w(\mathcal{B}_i(m)) \cdot w(\mathcal{A}_i(m)). \tag{4.5}$$

Using (4.5) and the fact that

$$2w(\mathcal{C}_{i-1}(m)) = w(\mathcal{C}_i(m) \cup \mathcal{D}_i(m))$$

as given by (3.10), (4.4) simplifies to

$$\frac{1}{2} w(\mathcal{B}_{i+1}(m)) < w(\mathcal{B}_i(m)). \tag{4.6}$$

Applying(3.8), (4.6) is equivalent to the relation

$$w(\mathcal{A}_i(m)) < w(\mathcal{B}_i(m)), \tag{4.7}$$

which can be easily deduced from (3.3) and (3.4), since for $1 \leq i \leq m-1$,

$$w(\mathcal{A}_i(m)) \leq w(\mathcal{A}_i(m) \cup \mathcal{C}_i(m)) = (m-i)D_i(m) < (m+i)D_i(m) = w(\mathcal{B}_i(m)). \tag{4.8}$$

This completes the proof. ∎

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