# Disposition Polynomials and Plane Trees

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**Abstract.** We define the disposition polynomial $R_m(x_1, x_2, \ldots, x_n)$ as $\prod_{k=0}^{m-1}(x_1 + x_2 + \cdots + x_n + k)$. When $m = n-1$, this polynomial becomes the generating function of plane trees with respect to the number of younger children and the number of elder children obtained by Guo and Zeng. They asked for a combinatorial proof of the formula. We find a combinatorial interpretation of the disposition polynomials in terms of the number of right-to-left minima of each linear order in a disposition. Then we establish a bijection between plane trees on $n$ vertices and dispositions from $\{1, 2, \ldots, n-1\}$ to $\{1, 2, \ldots, n\}$ in the spirit of the Prüfer correspondence, which gives an answer to the question of Guo and Zeng. This bijection also provides an answer to another question of Guo and Zeng concerning an identity on the plane tree expansion of a polynomial introduced by Gessel and Seo.

**Keywords:** disposition, disposition polynomial, plane tree, Prüfer correspondence

**AMS Classification:** 05A15, 05A19

## 1 Introduction

The notion of dispositions was introduced by Mullin and Rota [4], see also, Joni, Rota and Sagan [3]. Assume that $x$ is a nonnegative integer. Then the rising factorial $x(x+1)\cdots(x+n-1)$ can be interpreted as the number of dispositions from $[n] = \{1, 2, \ldots, n\}$ to $[x] = \{1, 2, \ldots, x\}$, where a disposition from $[n]$ to $[x]$ is a function from $[n]$ to $[x]$ in which the pre-image of each $i \in [x]$ is endowed with a linear order. In other words, a disposition from $[n]$ to $[x]$ can be viewed as a decomposition of a permutation of $[n]$ into $x$ parts with empty parts allowed.

In this paper, we introduce the disposition polynomials $R_m(x_1, x_2, \ldots, x_n)$ as a multivariate extension of the rising factorials by considering the number of right-to-left minima of each linear order in a disposition from $[m]$ to $[n]$. More precisely,

$$R_m(x_1, x_2, \ldots, x_n) = \prod_{k=0}^{m-1}(x_1 + x_2 + \cdots + x_n + k). \tag{1.1}$$

We are led to the above definition of the disposition polynomials by the special case of $R_m(x_1, x_2, \ldots, x_n)$ for $m = n - 1$ as introduced by Guo and Zeng [2] in connection with the enumeration of plane trees.

Let $\mathcal{P}_n$ denote the set of plane trees on $[n]$, where a plane tree on $[n]$ is a labeled rooted tree on $[n]$ for which the children of each vertex are linearly ordered, and let $\mathcal{P}_n^{(r)}$ denote the set of plane trees on $[n]$ with root $r$. For $T \in \mathcal{P}_n$, let $i$ be a vertex of $T$ and $j$ be a child of $i$. Assume that $e_1, e_2, \ldots, e_k$ are the brothers of $j$ lying on its right. Denote by $\beta_j(T)$ the smallest descendant of $j$ in $T$. If $\beta_j(T)$ is smaller than $\beta_{e_t}(T)$ for any $1 \leq t \leq k$, then we say that $j$ is a younger child of $i$. Otherwise, $j$ is called an elder child of $i$.

Denote by $\mathrm{young}_T(i)$ the number of younger children of $i$ in $T$, and denote by $\mathrm{eld}(T)$ the total number of elder children in $T$. Guo and Zeng [2] obtained the following formulas

$$\sum_{T \in \mathcal{P}_n} t^{\mathrm{eld}(T)} \prod_{i=1}^{n} x_i^{\mathrm{young}_T(i)} = \prod_{k=0}^{n-2} (x_1 + x_2 + \cdots + x_n + kt) \tag{1.2}$$

and

$$\sum_{T \in \mathcal{P}_n^{(r)}} t^{\mathrm{eld}(T)} \prod_{i=1}^{n} x_i^{\mathrm{young}_T(i)} = x_r \prod_{k=1}^{n-2} (x_1 + x_2 + \cdots + x_n + kt). \tag{1.3}$$

Guo and Zeng proved the above formulas (1.2) and (1.3) by induction and asked for combinatorial proofs. In answer to their questions, we first give a combinatorial interpretation of the disposition polynomials. Then, for the case $m = n-1$, we establish a bijection between plane trees and dispositions in the spirit of the Prüfer correspondence, which implies combinatorial interpretations of both relations (1.2) and (1.3).

Replacing $n$ by $n+1$, $t$ by $t-z$ and setting $r = 1$, $x_1 = x$ and $x_i = z$ for $2 \leq i \leq n+1$, the right hand side of (1.3) becomes the polynomial

$$x \prod_{k=1}^{n-1} (x + (n-k)z + kt),$$

which is the polynomial $P_n(t, z, x)$ introduced by Gessel and Seo [1] for the enumeration of labeled trees by the number of proper vertices. By using generating functions, several expansions of the polynomial $P_n(t, z, x)$ have been given by Gessel and Seo [1] in terms of rooted trees with proper vertices, $k$-ary trees with proper vertices, $k$-colored ordered forests with proper vertices and parking functions with lucky cars. Combinatorial proofs of some of these relations have been found by Seo and Shin [5], Shin [6] and Shin and Zeng [7].

With the above substitutions, (1.3) reduces to the relation

$$\sum_{T \in \mathcal{P}_{n+1}^{(1)}} x^{\mathrm{young}_T(1)} (t-z)^{\mathrm{eld}(T)} z^{n-\mathrm{young}_T(1)-\mathrm{eld}(T)} = x \prod_{k=1}^{n-1} (x + (n-k)z + kt). \tag{1.4}$$

Guo and Zeng [2] obtained the above formula as another combinatorial interpretation of the polynomial $P_n(t, z, x)$ of Gessel and Seo, and they raised the question of finding a combinatorial interpretation of (1.4).

Our correspondence between plane trees and dispositions can be used to give a combinatorial interpretation of (1.4). Moreover, the above relation holds for plane trees with a given root $r$, that is,

$$\sum_{T \in \mathcal{P}_{n+1}^{(r)}} x^{\mathrm{young}_T(r)} (t-z)^{\mathrm{eld}(T)} z^{n-\mathrm{young}_T(r)-\mathrm{eld}(T)} = x \prod_{k=1}^{n-1} (x + (n-k)z + kt). \tag{1.5}$$

This paper is organized as follows. In Section 2, we give a combinatorial explanation of the disposition polynomials. Section 3 provides a bijection between plane trees and dispositions which leads to combinatorial interpretations of (1.2) and (1.3). Section 4 is devoted to a combinatorial proof of (1.5).

# 2 The generating function of dispositions

In this section, we give a combinatorial interpretation of the disposition polynomials

$$R_m(x_1, x_2, \ldots, x_n) = \prod_{k=0}^{m-1} (x_1 + x_2 + \cdots + x_n + k).$$

Recall that a *disposition* is a function from $[m]$ to $[n]$ for which the pre-image of each $i \in [n]$ is endowed with a linear order. We denote by $\mathcal{D}_{m,n}$ the set of dispositions from $[m]$ to $[n]$. For example, a disposition from [9] to [8] is given in Figure 2.1.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 2,9 | 7,4 | | 5 | | 6,1,8 | 3 | |
| $D_1$ | $D_2$ | $D_3$ | $D_4$ | $D_5$ | $D_6$ | $D_7$ | $D_8$ |

Figure 2.1: A disposition from [9] to [8].

For a disposition $D$ from $[m]$ to $[n]$, we may write $D$ as $(D_1, D_2, \ldots, D_n)$, where $D_1 D_2 \cdots D_n$ forms a permutation of $[m]$ and each segment $D_i$ is allowed to be empty. Recall that, for a permutation $\pi = \pi_1 \pi_2 \cdots \pi_k$ of $k$ elements, $\pi_i$ is said to be a *right-to-left minimum* if $\pi_i < \pi_j$ for any $j > i$. We denote by $\mathrm{RLmin}(D_i)$ the number of right-to-left minima in $D_i$.

For example, for the disposition in Figure 2.1, we have $\mathrm{RLmin}(D_1) = 2$, $\mathrm{RLmin}(D_2) = 1$, $\mathrm{RLmin}(D_3) = 0$, $\mathrm{RLmin}(D_4) = 1$, $\mathrm{RLmin}(D_5) = 0$, $\mathrm{RLmin}(D_6) = 2$, $\mathrm{RLmin}(D_7) = 1$, $\mathrm{RLmin}(D_8) = 0$.

As will be seen, disposition polynomials can be interpreted as the generating functions of dispositions with respect to the statistics $\mathrm{RLmin}(D_i)$. The proof of the following theorem is essentially the same argument for the combinatorial interpretation of the rising factorials.

**Theorem 2.1** *For $n \geq 1$, we have*

$$\sum_{D\in\mathcal{D}_{m,n}} \prod_{i=1}^{n} x_i^{\mathrm{RLmin}(D_i)} = \prod_{k=0}^{m-1} (x_1 + x_2 + \cdots + x_n + k). \tag{2.1}$$

*Proof.* We use induction on $m$. For $m = 1$, the assertion is obvious. Assume that (2.1) holds for $m-1$, that is,

$$\sum_{D\in\mathcal{D}_{m-1,n}} \prod_{i=1}^{n} x_i^{\mathrm{RLmin}(D_i)} = \prod_{k=0}^{m-2} (x_1 + x_2 + \cdots + x_n + k). \tag{2.2}$$

We proceed to show that the theorem holds for $m$. A disposition from $[m]$ to $[n]$ can be obtained by inserting the element $m$ in a segment of a disposition from $[m-1]$ to $[n]$. Let $(D_1, D_2, \ldots, D_n)$ be a disposition from $[m-1]$ to $[n]$. Write $D_i = a_1a_2\cdots a_{r_i}$. There are $r_i + 1$ possible positions for the insertion of $m$ into $D_i$. Here are two cases. Case 1. The element $m$ is inserted at the end of $D_i$. Let $D_i' = a_1a_2\cdots a_{r_i}m$. It is clear that $D_i'$ has one more right-to-left minima than $D_i$, that is,

$$\mathrm{RLmin}(D_i') = \mathrm{RLmin}(D_i) + 1.$$

Case 2. The element $m$ is inserted before an element in $D_i$. Let $D_i' = a_1a_2\cdots a_{t-1}ma_t\cdots a_{r_i}$, where $1 \leq t \leq r_i$. In this case, we have

$$\mathrm{RLmin}(D_i') = \mathrm{RLmin}(D_i).$$

Since $r_1+r_2+\cdots+r_n = m-1$, considering all possible insertions of $m$ into $(D_1, D_2, \ldots, D_n)$, we obtain that

$$\begin{aligned}\sum_{D\in\mathcal{D}_{m,n}} \prod_{i=1}^{n} x_i^{\mathrm{RLmin}(D_i)} &= (x_1 + r_1 + \cdots + x_n + r_n) \sum_{D\in\mathcal{D}_{m-1,n}} \prod_{i=1}^{n} x_i^{\mathrm{RLmin}(D_i)} \\ &= (x_1 + x_2 + \cdots + x_n + m - 1) \sum_{D\in\mathcal{D}_{m-1,n}} \prod_{i=1}^{n} x_i^{\mathrm{RLmin}(D_i)}.\end{aligned}$$

Thus, by the induction hypothesis, we find that the theorem holds for $m$. This completes the proof. ∎

We define the homogenous disposition polynomials as follows

$$Q_m(x_1, x_2, \ldots, x_n, t) = \prod_{k=0}^{m-1} (x_1 + x_2 + \cdots + x_n + kt).$$

For $m = n-1$, Guo and Zeng [2] has shown that the homogenous disposition polynomials are the generating functions of plane trees with respect to the number of younger children and the number of elder children.

To give a combinatorial interpretation of the homogenous disposition polynomials, we recall a permutation statistic introduced by Guo and Zeng [2]. Given a permutation $\pi = \pi_1\pi_2\cdots\pi_m$, a *general descent* is defined as an index $i$ such that $\pi_i > \pi_j$ for some $j > i$. Let $\mathrm{gdes}(\pi)$ denote the number of general descents of $\pi$. For a disposition $D = (D_1, D_2, \ldots, D_n)$ from $[m]$ to $[n]$, we define the statistic $\mathrm{gdes}(D)$ to be the total number of general descents of $D_i$ for $1 \leq i \leq n$, that is,

$$\mathrm{gdes}(D) = \sum_{i=1}^{n} \mathrm{gdes}(D_i).$$

It is easily checked that

$$\mathrm{gdes}(D) = m - \sum_{i=1}^{n} \mathrm{RLmin}(D_i).$$

By Theorem 2.1, the homogeneous disposition polynomials have the following combinatorial interpretation

$$Q_m(x_1, x_2, \ldots, x_n, t) = \sum_{D \in \mathcal{D}_{m,n}} t^{\mathrm{gdes}(D)} \prod_{i=1}^{n} x_i^{\mathrm{RLmin}(D_i)}. \tag{2.3}$$

In the next section, we shall construct a bijection between the two interpretations of the homogeneous disposition polynomials.

# 3 A bijection between plane trees and dispositions

The objective of this section is to establish a correspondence between plane trees and dispositions. Let us give an overview of notation and terminology. Given two vertices $i$ and $j$ of a plane tree $T$, we say that $j$ is a *descendant* of $i$ if $i$ lies on the unique path from the root to $j$. In particular, each vertex is a descendant of itself. Denote by $\beta_T(i)$ the smallest descendant of $i$. A child of $i$ means a descendant $j$ such that $(i, j)$ is an edge of $T$. A vertex $i$ is called the father of a vertex $j$ if $j$ is a child of $i$. Vertices having the same father are called brothers of each other. A child $j$ of $i$ in a plane tree $T$ is called an elder child if $j$ has a brother $k$ to its right such that $\beta_T(k) < \beta_T(j)$; Otherwise,

$j$ is called a younger child of $i$. Denote by $\mathrm{eld}_T(v)$ the number of elder children of $v$ in $T$, and denote by $\mathrm{young}_T(v)$ the number of younger children of $v$ in $T$. It is not difficult to see that $\mathrm{young}_T(v)$ equals the number of right-to-left minima of the sequence $\{\beta_T(v_1), \beta_T(v_2), \ldots, \beta_T(v_m)\}$, where $v_1, v_2, \ldots, v_m$ are the children of $v$ listed from left to right. Moreover, we denote by $\mathrm{eld}(T)$ the total number of elder children in $T$.

For example, in Figure 3.1, each younger child of a vertex is represented by a square, whereas each elder child of a vertex is represented by a solid dot.

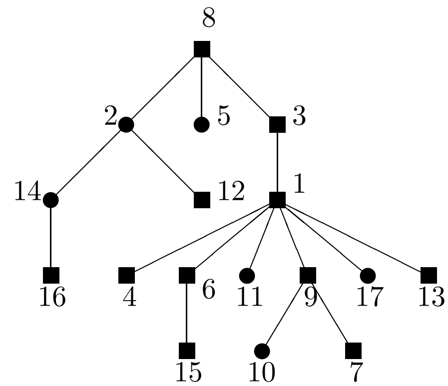


Figure 3.1: Younger and elder children in a plane tree.

**Theorem 3.1** *There is a bijection $\varphi$ between plane trees on $[n]$ and dispositions from $[n-1]$ to $[n]$. Let $T$ be a plane tree in $\mathcal{P}_n$, and let $D = (D_1, D_2, \ldots, D_n)$ be the corresponding disposition under the bijection $\varphi$. Then for any $1 \leq i \leq n$,*

$$\mathrm{young}_T(i) = \mathrm{RLmin}(D_i).$$

*Proof.* The map $\varphi$ from $\mathcal{P}_n$ to $\mathcal{D}_{n-1,n}$ can be described as follows. Let $T$ be a plane tree in $\mathcal{P}_n$. We proceed to construct a disposition $D = (D_1, D_2, \ldots, D_n)$ through the following procedure.

First, we mark the vertices of $T$ according to the Prüfer correspondence. More precisely, we mark the vertices of $T$ by the numbers $0, 1, 2, \ldots, n-1$. We start with the maximum leaf of $T$, and mark it by $n-1$. Then we remove the maximum leaf and repeat the this process until the root is marked by 0. Such marks are called the Prüfer marks of $T$, which represent the order that the vertices are removed in the Prüfer correspondence. For example, Figure 3.2 gives the Prüfer marks of a plane tree, which are represented by subscripts.

Using the Prüfer marks, the disposition $D = (D_1, D_2, \ldots, D_n)$ can be easily constructed by setting $D_i$ to be the set of the Prüfer marks of the children of vertex $i$ endowed with the linear order as in $T$. For example, for the plane tree $T$ in Figure 3.2, we have $D_1 = \{4, 6, 10, 7, 16, 12\}$, $D_2 = \{13, 11\}$, $D_3 = \{2\}$, and so on.

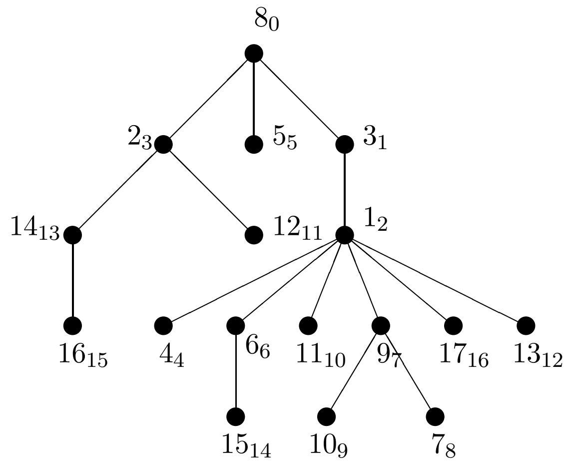


Figure 3.2: A plane tree with Prüfer marks $T \in \mathcal{P}_{17}$.

The above map $\varphi$ turns out to be a bijection. The inverse map can be described as follows. From a disposition $D$, one can easily recover the Prüfer marks. We begin with the rightmost empty segment $D_i$, and set the Prüfer mark of $i$ to be $n-1$. Then we remove the empty segment $D_i$ and the element $n-1$ in the disposition $D$. Repeating this procedure we get the Prüfer marks.

For example, for the disposition in Figure 3.3, the rightmost empty segment is $D_6$, thus, we mark 6 by 5. Deleting $D_6$ and removing 5 from $D_4$, we see that $D_4$ becomes the rightmost empty segment. So we mark 4 by 4. Iterating this process, we obtain the marks $6_5, 4_4, 3_3, 1_2, 5_1, 2_0$, where the subscripts stand for the marks.

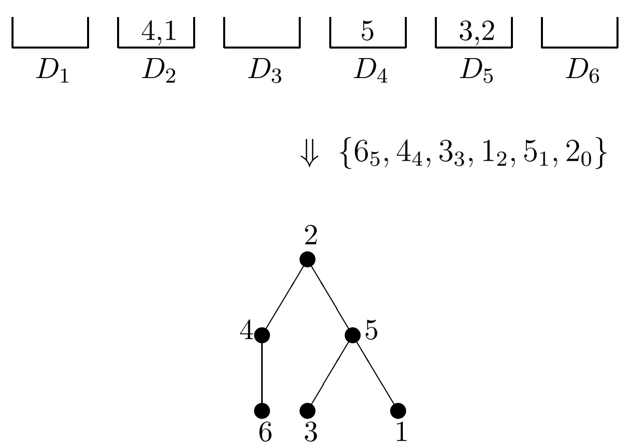


Figure 3.3: An example of the map $\varphi^{-1}$ for $n = 6$.

Using the marks, we may construct the plane tree $T$ by setting the root to be the element $r$ marked by 0. If $D_r$ is empty, then $r$ must be 1 and $T$ consists of the single vertex 1. Otherwise, we assume that $D_r = a_1a_2\cdots a_t$, and assume that $u_i$ is marked by $a_i$. Set the children of $r$ to be $u_1, u_2, \ldots, u_t$, listed from left to right. Repeating the above process with respect the vertices $u_1, u_2, \ldots, u_t$, we may get plane trees rooted at $u_1, u_2, \ldots, u_t$, so that we finally obtain a plane tree $T$ on $[n]$.

For example, we consider the disposition from [5] to [6] in Figure 3.3. The Prüfer marks are $6_5, 4_4, 3_3, 1_2, 5_1, 2_0$. Notice that the element 2 is marked by 0, which indicates that 2 is the root of the corresponding plane tree. The elements in $D_2$ are $4, 1$, which are the marks of $4, 5$. Thus, the children of 2 are 4 and 5 (listed from left to right). Now, $D_4$ contains a single element 5, which is the mark of the element 6. Thus, the only child of 4 is 6. Repeating this process, we get the plane tree as shown in Figure 3.3.

We now proceed to show that the above map is the inverse of $\varphi$. It suffices to prove that the marks obtained from the disposition $D$ are the same as the Prüfer marks obtained from the plane tree $T$. Observe that the largest leaf $l$ in a plane tree on $[n]$ is marked by $n-1$. On the other hand, $D_l$ must be the rightmost segment in the corresponding disposition, and so $l$ is marked by $n-1$ as well. We may repeat this argument for the element marked by $n-2$, if there is any segment left in the disposition. Hence we reach the conclusion that the marks obtained from the disposition $D$ are the same as the marks obtained from the plane tree $T$.

Next we verify the relation $\mathrm{young}_T(i) = \mathrm{RLmin}(D_i)$, where $D = (D_1, D_2, \ldots, D_n)$ is the corresponding disposition under the map $\varphi$. It is not difficult to see that the degree of a vertex $i$ in $T$ equals the number of elements of $D_i$. Moreover, let $D_i = a_1a_2\cdots a_m$ and let $u_1, u_2, \ldots, u_m$ be the children of $i$ in $T$. We claim that for $1 \leq j < k \leq m$, $a_j < a_k$ if and only if $\beta(u_j) < \beta(u_k)$. This property follows from the fact that the Prüfer mark of a vertex is the smallest among all its descendants. Hence we deduce that the number of younger children of a vertex $i$ in $T$ equals the number of right-to-left minima of $D_i$. This completes the proof. ∎

It is clear that Theorem 3.1 gives a combinatorial interpretation of the following relation

$$\sum_{T\in\mathcal{P}_n} t^{\mathrm{eld}(T)} \prod_{i=1}^{n} x_i^{\mathrm{young}_T(i)} = \sum_{D\in\mathcal{D}_{n-1,n}} t^{\mathrm{gdes}(D)} \prod_{i=1}^{n} x_i^{\mathrm{RLmin}(D_i)}. \tag{3.1}$$

Combining (3.1) and (2.3), we obtain a combinatorial proof of relation (1.2) in answer to the question posed by Guo and Zeng [2].

Note that the above correspondence can be restricted to plane trees with a specific root $r$. More precisely, a disposition $D$ corresponds to a plane tree $T$ with root $r$ if and only if the element 1 is contained in $D_r$. This leads to a combinatorial interpretation of relation (1.3).

It should also be noted that the correspondence in Theorem 3.1 can be extended to a bijection between dispositions from $[n-r]$ to $[n]$ and forests of $r$ plane trees on $[n]$. To be precise, the plane trees in a forest are not linearly ordered. This more general correspondence is stated in the following theorem. The proof is similar to that of Theorem 3.1, and hence it is omitted.

**Theorem 3.2** *For $n \geq 1$ and $1 \leq r \leq n-1$, let $\mathcal{D}_{n-r,n}$ denote the set of dispositions from $[n-r]$ to $[n]$ and let $\mathcal{F}_n^r$ denote the set of forests of $r$ plane trees on $[n]$. Then we have*

$$\sum_{F \in \mathcal{F}_n^r} t^{\mathrm{eld}(F)} \prod_{i=1}^{n} x_i^{\mathrm{young}_F(i)} = \binom{n-1}{r-1} \sum_{D \in \mathcal{D}_{n-r,n}} t^{\mathrm{gdes}(D)} \prod_{i=1}^{n} x_i^{\mathrm{RLmin}(D_i)}, \tag{3.2}$$

*where the elder and younger children in forests of plane trees are defined as in the plane trees.*

To conclude this section, we remark that the correspondence $\varphi$ is also valid for labeled rooted trees. In this case, we disregard the linear order in each segment of a disposition. In other words, $\varphi$ becomes a correspondence between labeled rooted trees and decompositions of $[n-1]$ into $n$ components. Under this correspondence, the empty sets in a decomposition correspond to leaves of a labeled rooted tree, and more generally, the number of elements in $D_i$ corresponds to the degree of the vertex $i$ in the corresponding rooted tree.

# 4 The Gessel-Seo polynomials

In this section, we use the correspondence between plane trees and dispositions to give a combinatorial interpretation of relation (1.4) in answer to a question posed by Guo and Zeng [2] concerning an expansion of the Gessel-Seo polynomials. In fact, we obtain a more general relation as given below.

**Theorem 4.1** *For $n \geq 1$ and $1 \leq r \leq n+1$, we have*

$$\sum_{T \in \mathcal{P}_{n+1}^{(r)}} x^{\mathrm{young}_T(r)} (t-z)^{\mathrm{eld}(T)} z^{n-\mathrm{young}_T(r)-\mathrm{eld}(T)} = x \prod_{k=1}^{n-1} (x + (n-k)z + kt), \tag{4.1}$$

*where $\mathcal{P}_{n+1}^{(r)}$ is the set of plane trees on $[n+1]$ with root $r$.*

*Proof.* Replacing $t$ by $t+z$, we may rewrite (4.1) as

$$\sum_{T \in \mathcal{P}_{n+1}^{(r)}} x^{\mathrm{young}_T(r)} t^{\mathrm{eld}(T)} z^{n-\mathrm{young}_T(r)-\mathrm{eld}(T)} = x \prod_{k=1}^{n-1} (x + nz + kt). \tag{4.2}$$

We first give a combinatorial interpretation of the right hand side of (4.2). By the combinatorial interpretation of the disposition polynomials, we see that the Gessel-Seo polynomial $P_n(t+z, z, x)$ is the generating function of dispositions $D = (D_1, D_2, \ldots, D_{n+1})$ from $[n]$ to $[n+1]$ with the element 1 contained in $D_r$, where a right-to-left minimum in $D_r$ is given a weight $x$, a right-to-left minimum in $D_i$ $(i \neq r)$ is given a weight $z$, and any other element is given a weight $t$.

For a disposition $D = (D_1, D_2, \ldots, D_{n+1})$ in which the element 1 appears in $D_r$, let $T$ be the plane tree corresponding to $D$ under the bijection $\varphi$ in Theorem 3.1. It is easily seen that $r$ is the root of $T$, namely, $T \in \mathcal{P}_{n+1}^{(r)}$. Moreover, for $1 \leq i \leq n+1$, a younger child of a vertex $i$ of $T$ corresponds to a right-to-left minimum in $D_i$, and an elder child of a vertex $i$ of $T$ corresponds to an element which is not a right-to-left minimum in $D_i$. Hence the weight of $T$ is given by

$$x^{\mathrm{young}_T(r)} t^{\mathrm{eld}(T)} z^{n-\mathrm{young}_T(r)-\mathrm{eld}(T)}.$$

This completes the proof. ∎

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