Branching Decompositions of Necklaces

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Abstract

This work originates from a combinatorial understanding of a branching property of MSS (Metropolis-Stein-Stein) sequences in symbolic dynamics. It is known that MSS sequences are in one-to-one correspondence with equivalence classes of primitive necklaces on two colors under the exchange of colors. We present a branching property of primitive self-complementary necklaces, leading to a combinatorial explanation of an analogous property of MSS sequences. The branching property of necklaces may have further applications to the combinatorial understanding of discrete dynamic systems and the theory of chaos.

Key Words. necklaces, MSS sequences, symbolic dynamics

AMS Subject Classification. 05A05, 68R15

1 Introduction

Symbolic dynamics has been an efficient machinery in the study of discrete dynamic systems and the theory of chaos. MSS (Metropolis-Stein-Stein) sequences is a fundamental ingredient in symbolic dynamics because it serves as a topological description of periodic orbits of unimodal maps on the unit interval. It was first observed by Metropolis-Stein-Stein [**??**] that the number of MSS sequences of length n equals the numbers of complementary pairs (unordered) of primitive (or aperiodic) necklaces of length n with two colors, say 0 and 1, where two necklaces are regarded as complementary to each other if one can be obtained from the other by a reflection of colors, or cyclic permutation on exchange of 0 and 1. Such a numerical evidence was later confirmed by Sun and Helmberg [**??**] and Brucks [**??**]. As is always in bijective combinatorics there must be a good reason for such a numerical coincidence. At a philosophical level, MSS sequences have similar properties as equivalence classes of necklaces under complementation. First, both correspond to cyclic structures because MSS sequences characterize periodic orbits. Second, MSS sequences possess certain reflection property. In the work of Sun and Helmberg, it is already visible that MSS sequences have certain branching property though not explicitly realized. In [**??**] Brucks also discovered the reflection property of MSS sequences in terms of an explicit bijection between MSS sequences and a pair of complementary necklaces.

The main goal of this paper is to establish a branching property of necklaces based on the notion of self-complementary necklaces introduced in [**??**]. This property is described by a branching decomposition of the sets of necklaces (see Theorems 4.1 and 4.4). We begin with a theorem asserting that self-complementary necklaces play the same role as MSS sequences from the point view of combinatorics. It is our hope that further studies will follow to bring up a closer relationship between combinatorics and discrete dynamical systems.

2 MSS Sequences and Necklaces

In 1973, N. Metropolis, M. L. Stein and P. R. Stein [**??**] developed a theory of iteration of unimodal maps of $[0,1]$ into itself. Given such a map f and a parameter λ between zero and one they associate it with a finite or possibly infinite sequence $b_1b_2\ldots$ of R's and L's, by considering the iterates $(\lambda f)^i$ of the map λf at $1/2$. For $i \geq 1$, set

$$b_i = \begin{cases} R, & \text{if} \quad (\lambda f)^i(1/2) > 1/2, \\ L, & \text{if} \quad (\lambda f)^i(1/2) < 1/2, \\ C, & \text{if} \quad (\lambda f)^i(1/2) = 1/2. \end{cases}$$

If $b_i = C$ for some i, then the sequence stops. Finite sequences of R's , L's and C obtained in this manner are called MSS sequences. The reader is referred to [**??**-**??**, **??**]. Let MSS_n denote the set of MSS sequences of length n. The ordering of MSS sequences is essential. First, the letters L, C, R are endowed with the order $L < C < R$, signifying the symbols L as left, C as center and R as right. In general, a sequence w of L, R and C is said to be admissible if w is an infinite sequence of L's and R's or w is a finite sequence of L's and R's followed by a C as the ending symbol. Admissible sequences are ordered by the parity-lexicographic order. The difference between parity-lexicographic order and the usual lexicographic order lie in the following notion of parity: A finite sequence on L, R and C is said to be even if it contains an even number of R's; otherwise, it is said to be odd. Let $w = w_1 \cdots w_k w_{k+1} \cdots$ and $u = u_1 \cdots u_k u_{k+1} \cdots$ be two admissible sequences such that $w_1 \cdots w_k = u_1 \cdots u_k$, but $w_{k+1} \neq u_{k+1}$. When $w_1 \cdots w_k$ is even, then the order relation of w and v is just like in usual lexicographic order. However, when $w_1 \cdots w_k$ is odd, the order of w and u is the reverse of the lexicographic order: $w < u$ if $w_{k+1} > u_{k+1}$, and $w > u$ if $w_{k+1} < u_{k+1}$. An admissible sequence $w = w_1 \cdots w_n \cdots$ is said to be shift maximal if it is equal or greater than all its proper right factors $w_{i+1} \cdots w_n \cdots$, where $i \geq 1$. In the language of the parity-lexicographic order, a finite admissible sequence is a MSS sequence if it is shift maximal.

The first attempt to make the connection between MSS sequences and necklaces were made by Sun and Helmberg [**??**], and Brucks [**??**]. A key idea in this direction was due to Sun and Helmberg, which is the notion of extended sequences: An extended sequence may be an admissible sequence or a finite sequence of L's and R's. The parity-lexicographic order is extended by the following rule: Let $w = w_1 \cdots w_k$ and $u = w_1 \cdots w_k\, w_{k+1} \cdots$ be two extended sequences, where $w_i = L$ or R. Then $w < u$ if $w_1 \cdots w_k$ is odd, otherwise, $w > u$. However, for the purpose of this paper, we will use the term *words* solely for finite sequences of L's and R's. The notion of shift maximal sequences can be analogously defined for words: a word w is shift maximal if it is greater than all its proper right factors in the extended parity-lexicographic order. It is quite remarkable that the extension of parity-lexicographic order plays a substantial role in the enumerative studies of MSS sequences.

Shift maximal words are analogous to Lyndon words in combinatorics (see [**??**]), however, it seems to be substantially more sophiscticated than a straightforward counterpart. A different representation of MSS sequences is used in [**??**] under the name of lexical sequences.

A word of length n is said to be primitive (or aperiodic) if it can not be written as a power of a word of smaller length, where the product of two words is defined by juxtaposition. If a word w can be written as a power of another word v of length p, then we say that the word w has period p. For most cases, one is concerned with the smallest period of a word. For a word w, we denote by $\langle w \rangle$ the set of words which can be obtained from w by cyclic permutations, and call $\langle w \rangle$ a conjugate class or an equivalence class. Intuitively, a conjugate class can be visualized as a necklace. Two words are called conjugate or equivalent if they are in the same conjugate class. The following is a fundamental property of shift maximal words and conjugate classes of primitive words.

Theorem 2.1. ([**??**], Thm 3.3) *Let $w = w_1 \cdots w_n \in \{L, R\}^n$ be shift maximal and primitive. Then w is the only shift maximal word in $\langle w \rangle$.*

For a word $w = w_1 w_2 \cdots w_n \in \{L, R\}^n$, we define the complementation as $\overline{w} = \overline{w}_1 \overline{w}_2 \cdots \overline{w}_n$, where $\overline{R} = L$ and $\overline{L} = R$. We use $\langle\!\langle w \rangle\!\rangle$ to denote the complementary pair of the necklace $\langle w \rangle$, or we may set $\langle\!\langle w \rangle\!\rangle = \langle w \rangle \cup \langle \overline{w} \rangle$. We call a necklace $\langle w \rangle$ self-complementary if $\overline{w} \in \langle w \rangle$. It is clear that w is self-complementary if and only if $\langle w \rangle = \langle\!\langle w \rangle\!\rangle$.

For the sake of brevity, we call both $\langle w\rangle$ and $\langle\!\langle w\rangle\!\rangle$ *necklaces*. This will not cause confusion.

By the correspondence $L \longleftrightarrow 1$, $R \longleftrightarrow -1$, we may identify words on $\{L, R\}$ with words on $\{1, -1\}$, as we will use below.

Now we may define three other sets of necklaces on $\{L, R\}$ which are related to MSS_n: Let O_n denote the set of all primitive necklaces $\langle w\rangle$ where w is an odd word of length n, F_n the set of all primitive necklaces $\langle\!\langle w\rangle\!\rangle$ of length n, and G_n the set of all self-complementary primitive necklaces of length $2n$.

Brucks [**??**] established a bijection g from MSS_n onto O_n as follows:

$$g(b_1 b_2 \cdots b_{n-1} C) = \begin{cases} \langle e_1, e_2, \ldots, e_{n-1}, -1\rangle, & \text{if } \prod_{i=1}^{n-1} e_i > 0, \\ \langle e_1, e_2, \ldots, e_{n-1},\ 1\rangle, & \text{if } \prod_{i=1}^{n-1} e_i < 0, \end{cases}$$

where

$$e_i = \begin{cases} 1, & \text{if } b_i = L, \\ -1, & \text{if } b_i = R. \end{cases}$$

In order to give an explicit expression of the inverse g^{-1} of g, we let O'_n denote the set of shift maximal words of length n which contain an odd number of R. By Theorem 2.1 there is a natural correspondence between O_n and O'_n. Thus we have

$$g^{-1} : \langle b_1 \cdots b_n\rangle \longrightarrow b_1 \cdots b_{n-1} C,$$

where $b_1 \cdots b_n \in O'_n$.

In [**??**], Brucks also established another bijection h from MSS_n onto F_n: For $B = b_1 b_2 \cdots b_{n-1} C \in \mathrm{MSS}_n$,

$$h(B) = \langle e_1, e_2, \cdots, e_n\rangle,$$

where $e_1 = -1$, for $2 \le i \le n$

$$e_i = \begin{cases} e_{i-1}, & \text{if } b_{i-1} = L, \\ -e_{i-1}, & \text{if } b_{i-1} = R. \end{cases}$$

Furthermore, Chen and Louck [**??**] observed that $|G_n| = |F_n|$. Hence all the above classes have the same cardinality. Guided by this evidence, we proceed to produce bijection between F_n and G_n, and to explore the branching decomposition of F_n based on the self-complementary property of necklaces.

3 Self-Complementary Necklaces

We begin this section with a characterization of self-complementary necklaces.

Lemma 3.1 *Assume that $B \in \{L, R\}^m$ is a word of length m with the smallest period p. Then $\langle B \rangle$ is self-complementary if and only if p is even and B can be expressed in the following form:*

$$B = (b_1 \cdots b_{p/2} \bar{b}_1 \cdots \bar{b}_{p/2})^{m/p}. \tag{3.1}$$

Proof. It is clear that if B can be written in the form of (3.1) for an even number p, then $\langle B \rangle$ is self-complementary. Now we assume that $\langle B \rangle$ is self-complementary, and proceed to show that B can be written as in (3.1). Since $\langle B \rangle$ is self-complementary, B and $\overline{B}$ conjugate to each other. In other words, there is a minimal positive integer k such that $\sigma_m^k(B) = \overline{B}$, where σ_m is the cyclic permutation $\sigma_m(x_1x_2 \cdots x_m) = x_2x_3 \cdots x_mx_1$. By taking complementation it follows that $\sigma_m^k(\overline{B}) = B$. Thus we obtain $\sigma_m^{2k}(B) = \sigma_m^k(\overline{B}) = B$. Let p is the smallest period of B, it follows that p divides $2k$. Because $\sigma_m^k(B) = \overline{B} \neq B$, we see that k is not a period of B. Therefore, p does not divide k. Hence, p is even and $r = p/2$ divides k. Set $k = r(2t + 1)$. Then $\overline{B} = \sigma_m^k(B) = \sigma_m^{r+2rt}(B) = \sigma_m^r(B)$, which implies $r = k$. Therefore, we obtain (**??**). ∎

Corollary 3.2. *Let B be a primitive word of length $2n$ on $\{L, R\}$. Then $\langle B \rangle$ is self-complementary if and only if n is the smallest positive integer such that $\sigma_{2n}^n(B) = \overline{B}$, i.e., B can be written in the form $b_1 \cdots b_n \bar{b}_1 \cdots \bar{b}_n$.*

In [**??**, Definition 5.6], Brucks introduced an important map ψ:

$$\psi(b_1b_2 \cdots b_m) = \delta(b_1b_2)\delta(b_2b_3) \cdots \delta(b_{m-1}b_m),$$

where $\delta(xy) = L$ if $x = y$ and $\delta(xy) = R$ if $x \neq y$ for any $x, y \in \{L, R\}$. Clearly, we have $\delta(xy) = \delta(yx) = \delta(\bar{x}\bar{y})$. In particular, the action of ψ on a word of the form: $w_1w_2 \cdots w_mw_1$, which we call cyclic words, was considered in [**??**] and a factorization property of such cyclic words was obtained. We will restate this result in a different form as given in the following Lemma 3.4. For the sake of notation, we find it convenient to use the operation Δ, called the cyclic difference operator, defined below:

$$\Delta(b_1b_2 \cdots b_m) = \psi(b_1b_2 \cdots b_mb_1) = \delta(b_1b_2)\delta(b_2b_3) \cdots \delta(b_mb_1),$$

where the function δ is as before. If we use 0 and 1 modulo 2 to denote L and R then the above definition of cyclic difference coincides with the numerical understanding of taking differences along a cycle. We remark that the above notion of cyclic difference operator is reminisient of the cyclic derative as introduced by Rota and Sagan [**??**].

Lemma 3.3. *For every word B, $\Delta(B)$ is always even. Conversely, for each even word v there is a unique pair of complemetary words B and $\overline{B}$ such that $\Delta(B) = \Delta(\overline{B}) = v$.*

Proof. By induction on m we have that for $b_i \in \{L, R\}$, the word $\delta(b_1b_2)\cdots\delta(b_{m-1}b_m)$ is odd if $b_1 \neq b_m$ and even if otherwise. Therefore for every word B, $\Delta(B)$ is always even.

For an even word $v = v_1 \cdots v_n \in \{L, R\}^n$, define $\Delta^{-1}(v) = \{B = b_1 \cdots b_n \in \{L, R\}^n : \Delta(B) = v\}$. To determine this set, we make the following observations:

For every $x \in \{L, R\}$,

$$\delta(L\delta(Lx)) = \begin{cases} L & \text{if} \quad x = L \\ R & \text{if} \quad x = R, \end{cases}$$

and

$$\delta(R\delta(Rx)) = \begin{cases} L & \text{if} \quad x = L \\ R & \text{if} \quad x = R. \end{cases}$$

Therefore, $\delta(y\delta(yx)) = x$, and similarly, $\delta(\overline{y}\delta(yx)) = \overline{x}$ holds for all $x, y \in \{L, R\}$. We now consider the set

$$\Gamma(v) = \{b_1 \cdots b_n : b_1 = L \text{ or } R,\ b_{i+1} = \delta(b_iv_i),\ 1 \le i \le n-1\}. \tag{3.2}$$

It is not difficult to see that $\Gamma(v)$ consists of a pair of complementary words, and for each one, say $b_1b_2 \cdots b_n$, we have that

$$\begin{aligned} \psi(b_1b_2\cdots b_n) &= \delta(b_1b_2)\delta(b_2b_3)\cdots\delta(b_{n-1}b_n) \\ &= \delta(b_1\delta(b_1v_1))\delta(b_2\delta(b_2v_2))\cdots\delta(b_{n-1}\delta(b_{n-1}v_{n-1})) \\ &= v_1v_2\cdots v_{n-1}, \end{aligned}$$

which yields $\Delta^{-1}(v) = \Gamma(v)$ or $\Delta^{-1}(v) = \emptyset$ depending on whether $\delta(b_1b_n) = v_n$ or not. If there is an even word $v = v_1 \cdots v_{n-1}v_n$ such that $\Delta^{-1}(v) = \emptyset$, then $\Delta^{-1}(v') \neq \emptyset$ where $v' = v_1 \cdots v_{n-1}\overline{v}_n$. But v' is odd, we have known that there is no B with $\Delta(B) = v'$. The contradiction shows that for every even word v there is a unique pair of complementary words B and $\overline{B}$ such that $\Delta(B) = \Delta(\overline{B}) = v$. ∎

The following Lemma gives a factorization property of the image of self-complementary necklaces under the action of Δ. We present a proof of this lemma for the sake of completeness.

Lemma 3.4. *Let $B \in \{L, R\}^m$ be a primitive word. Then, we have*

(i) $\Delta(B)$ is primitive if B is not self-complementary.

(ii) $\Delta(B) = v^2$ where v is odd and primitive if B is self-complementary.

Proof. Let $B = b_1 b_2 \cdots b_m$ and $\Delta(B) = (v_1 \cdots v_p)^{m/p}$, where p is the smallest period of $\Delta(B)$. By (**??**) we have that

$$b_{ip+1} = \delta(b_{ip}v_p),\ b_{ip+2} = \delta(b_{ip+1}v_1), \ldots, b_{ip+p} = \delta(b_{ip+p-1}v_{p-1}), i \geq 1. \tag{3.3}$$

If $b_{p+1} = b_1$, then $b_{p+2} = b_2, \ldots, b_{ip+j} = b_j$, thus we have $B = (b_1 \cdots b_p)^{m/p}$. Since B is primitive, it follows that $p = m$, that is, $\Delta(B)$ is primitive. If $\Delta(B)$ is not primitive, then there must be $b_{p+1} = \bar{b}_1$. By (**??**) we have

$$B = b_1 \cdots b_p \bar{b}_1 \cdots \bar{b}_p b_1 \cdots b_p \bar{b}_1 \cdots .$$

Since $v = v_1 \cdots v_p = \delta(b_1 b_2) \cdots \delta(b_p \bar{b}_1)$, the observation in Lemma 3.1 tells us that v is odd. Since $\Delta(B) = v^{m/p}$ is even, so is m/p. The primitivity of B implies that $m/p = 2$, i.e., $B = b_1 \cdots b_p \bar{b}_1 \cdots \bar{b}_p$ is self-complementary. Conversely, if B is of this form, i.e., it is self-complementary, then $\Delta(B) = v^2$. ∎

Let $p(w)$ denote the smallest period of a word w on $\{L, R\}$. Clearly, $\Delta(w^k) = (\Delta(w))^k$, which yields that $p(\Delta(B))$ divides $p(B)$ for every word B on $\{L, R\}$. Combining Lemmas 3.1 and 3.4 we can obtain the following characterizations of the smallest period of $\Delta(B)$ and self-complementary necklaces in terms of the cyclic difference operator Δ.

Theorem 3.5. *For every $B \in \{R, L\}^m$,*

(i) $p(\Delta(B)) = p(B)|\{\langle B\rangle, \langle \overline{B}\rangle\}|/2$, where $|\cdot|$ denotes the cardinality of a set.

(ii) B is self-complementary if and only if $\Delta(B) = v^2$ and the smallest period of B is the double of v's.

Notice that for $B = b_1 \cdots b_n \bar{b}_1 \cdots \bar{b}_n$

$$\Delta(\sigma_{2n}^i(B)) = (\sigma_n^i(v))^2, \tag{3.4}$$

where v is given by $\Delta(B) = v^2$. So, we may define a map $f : G_n \to O_n$ as follows:

$$f : \langle B \rangle \to \langle v \rangle,$$

where v is given by $\Delta(B) = v^2$. The following theorem shows that the map f is in fact a bijection.

Theorem 3.6. *The map f is a bijection from G_n onto O_n.*

Proof. For each $\langle v \rangle \in O_n$, since v^2 is even, Lemma 3.3 implies that there is a unique pair of complementaty words B and $\overline{B}$ of length $2n$ such that $\Delta(B) = \Delta(\overline{B}) = v^2$. Theorem 3.5 tells us that B is self-complementary and primitive. Hence from (**??**), there is unique $\langle B \rangle \in G_n$ such that $f(\langle B \rangle) = \langle v \rangle$. ∎

Thus, we obtain a bijection from G_n onto F_n as follows

$$G_n \xrightarrow{f} O_n \xrightarrow{g^{-1}} \mathrm{MSS}_n \xrightarrow{h} F_n. \tag{3.5}$$

Analogous to the Lyndon word representation of cycles, or the shift maximal word representation of O_n, it would be interesting to give a word representation of self-complementary necklaces. To this end, we introduce the following definition of cyclic complementation:

Let $B = b_1 b_2 \cdots b_n$ be a word on $\{L, R\}$. The words in the following set $[B]$ are called the cyclic complementations of B:

$$[B] = \{B, b_2 \cdots b_n \bar{b}_1, \ldots, b_n \bar{b}_1 \cdots \bar{b}_{n-1},\ \bar{b}_1 \cdots \bar{b}_n,\ \bar{b}_2 \cdots \bar{b}_n b_1,\ \ldots,\ \bar{b}_n b_1 \cdots b_{n-1}\}.$$

A word $B = b_1 b_2 \cdots b_n$ is said to be complementary shift maximal, or c-shift maximal for short, if B is greater than each of its cyclic complementations.

Notice that B being c-shift maximal implies that all the $2n$ words listed in $[B]$ are distinct because otherwise each word in it would appear the same times ($> 1!$), and there would no words which are greater than all others.

It is easy to see that a c-shift maximal word is shift maximal, but the converse is not true. For example, $RLLRL$ is shift maximal, but not c-shift maximal (see Example 3.8). We remark that there may be more than two shift maximal words in a class $[B]$, for example, in $[RLLRR]$, there are three shift maximal words: $RLLRR, RLRRL, RLLRL$.

Theorem 3.7. *Let B be in $\{L, R\}^n$. Then, B is c-shift maximal if and only if $B\overline{B}$ is shift maximal (hence, primitive).*

Proof. We first prove that if $B\overline{B}$ is shift maximal then it is primitive. In fact, if $B\overline{B} = v^k$, $k > 1$, then k is an odd number since $B\overline{B}$ can not be written as a square. Clearly, v^{k-1} is a right factor of $B\overline{B}$ and an even word as well. Therefore, $B\overline{B} < v^{k-1}$, which contradicts that $B\overline{B}$ is shift maximal. This contradiction proves that $B\overline{B}$ is primitive.

Next, let us notice that $\langle B\overline{B}\rangle = \{v\overline{v} : v \in [B]\}$. Therefore, the two sets have the same cardinality. It is easily seen that for any $v, w \in [B]$, $v < w$ if and only if $v\overline{v} < w\overline{w}$, which implies the theorem immediately. ∎

Let G'_n denote the set of all c-shift maximal words of length n. Then, by Theorems 2.1 and 3.7, there is a natural correspondence between G_n and G'_n. For $B = b_1 \cdots b_n \in G'_n$, define

$$f'(B) = \langle \delta(b_1 b_2) \cdots \delta(b_n \bar{b}_1)\rangle.$$

Then $f'(B) = f(\langle B\overline{B}\rangle) \in O_n$ and f' is a bijection from G'_n onto O_n.

Example 3.8. $G'_5 = \{RLLLL, RLLLR, RLLRR\}$. We have

$$\begin{array}{ccccccc}
G'_5 & \xrightarrow{f'} & O_5 & \xrightarrow{g^{-1}} & MSS_5 & \xrightarrow{h} & F_5 \\
RLLLL & & \langle RLLLL\rangle & & RLLLC & & \langle\langle RLLLL\rangle\rangle \\
RLLLR & & \langle RLLRR\rangle & & RLLRC & & \langle\langle RLLLR\rangle\rangle \\
RLLRR & & \langle RLRLR\rangle & & RLRRC & & \langle\langle RLLRL\rangle\rangle
\end{array}$$

Example 3.9. $G'_6 = \{RLLLLL, RLLLLR, RLLLRR, RLLLRL, RLLRLR\}$

$$\begin{array}{ccccccc}
G'_6 & \xrightarrow{f'} & O_6 & \xrightarrow{g^{-1}} & MSS_6 & \xrightarrow{h} & F_6 \\
RLLLLL & & \langle RLLLLL\rangle & & RLLLLC & & \langle\langle RLLLLL\rangle\rangle \\
RLLLLR & & \langle RLLLRR\rangle & & RLLLRC & & \langle\langle RLLLLR\rangle\rangle \\
RLLLRR & & \langle RLLRLR\rangle & & RLLRLC & & \langle\langle RLLLRR\rangle\rangle \\
RLLLRL & & \langle RLLRRL\rangle & & RLLRRC & & \langle\langle RLLLRL\rangle\rangle \\
RLLRLR & & \langle RLRRRR\rangle & & RLRRRC & & \langle\langle RLLRLR\rangle\rangle
\end{array}$$

4 Continuous Decompositions

Let $F_n^{(0)}$ and F_n^{sc} consist of such $\langle\langle B\rangle\rangle \in F_n$ that B is not self-complementary, and self-complementary, respectively. Then we have a partition of F_n as follows

$$F_n = F_n^{(0)} \cup F_n^{\mathrm{sc}}. \tag{4.1}$$

If n is odd, then $F_n^{\mathrm{sc}} = \emptyset$, i.e., $F_n = F_n^{(0)}$.

Note that F_2 has only one element $\langle\langle RL \rangle\rangle$ which is self-complementary. However, for convenience to state we let it be in $F_2^{(0)}$ rather than in F_n^{sc}. Thus we have that $F_2^{(0)} = F_2$. In order to make this assumption compatible with (**??**) we put $F_1 = \emptyset$.

In the sequel of this section, we assume that n is an even number greater than 2.

It is clear that $F_n^{\mathrm{sc}} = G_{n/2}$, and it is bijection-equivalent to $F_{n/2}$, which means that there is a bijection between two sets. We will use the notation "$\sim$" to denote this relation. Then (**??**) implies that

$$F_n \sim F_n^{(0)} \cup F_{n/2}. \tag{4.2}$$

This process can be continued if $n/2$ is still an even number greater than 2. Note that $G_{n/2} \sim F_{n/2}$ as stated (**??**). Therefore, there is a bijection ψ_1 from F_n^{sc} to $F_{n/2}$. We say a $\langle\langle w \rangle\rangle$ in F_n^{sc} is a *self-complementary necklace of degree 1* if its image under ψ_1 is not self-complementary. Let $F_n^{(1)}$ denote the set of all self-complementary necklaces of degree 1 in F_n. By (**??**) and (**??**) we have

$$F_n \sim F_n^{(0)} \cup F_n^{(1)} \cup F_{n/2}^{\mathrm{sc}} \sim F_n^{(0)} \cup F_n^{(1)} \cup F_{n/2^2} \sim F_n^{(0)} \cup F_{n/2}^{(0)} \cup F_{n/2^2}$$

where

$$F_n^{(1)} \sim F_{n/2}^{(0)}.$$

Now suppose that 2^k divides n. Then we may define the self-complementary necklaces of degree k in the similar way, and denote by $F_n^{(k)}$ the set of all self-complementary class of degree k in F_n.

The following is our first decomposition theorem.

Theorem 4.1. *(i) We have a partition of F_n as follows:*

$$F_n = \bigcup_{k \geq 0} F_n^{(k)}. \tag{4.3}$$

(ii) There is a bijection $f_k : F_n^{(k)} \longrightarrow F_{n/2^k}^{(0)}$.

(iii) $F_n^{(k)} \neq \emptyset$ if and only if 2^k divides n but $2^k \neq n$, and when $F_n^{(k)} \neq \emptyset$, $\cup_{i \geq k} F_n^{(i)}$ is bijection-equivalent to $F_{n/2^k}$.

Proof. (i) and (ii) are obvious. We prove (iii). From (**??**) and induction on k it follows that

$$F_n \sim F_n^{(0)} \cup F_{n/2}^{(0)} \cup \cdots \cup F_{n/2^{k-1}}^{(0)} \cup F_{n/2^k}. \tag{4.4}$$

¿From this we see that (a) if $2^k < n$, then $F_{n/2^k} \neq \emptyset$, and from (**??**) and (**??**) it follows that $F_n^{(k)} \neq \emptyset$ and $\cup_{i \geq k} F_n^{(i)} \sim F_{n/2^k}$; (b) if $2^k = n$, then $F_{n/2^k} = F_1 = \emptyset$. Thus $F_n^{(k)} = \emptyset$. ∎

We continue our decomposition for each $F_{n/2^i}^{(0)}$, $i = 0, 1, \ldots, k$. By similarity, we need only consider $F_n^{(0)}$.

Since n is even, for every $v \in \{L, R\}^n$, v and $\overline{v}$ have the same parity-property, so do all words in $\langle\langle B \rangle\rangle$. Let F_n^{od} and F_n^{ev} be the sets of such $\langle\langle B \rangle\rangle \in F_n$ that B is odd and even, respectively. Then we have another partition of F_n as follows

$$F_n = F_n^{\text{od}} \cup F_n^{\text{ev}}.$$

By Corollary 3.2, it is easy to check that

$$\begin{cases} F_n^{\text{sc}} \subseteq F_n^{\text{od}} & \text{if } n/2 \text{ is odd,} \\ F_n^{\text{sc}} \subseteq F_n^{\text{ev}} & \text{if } n/2 \text{ is even.} \end{cases} \tag{4.5}$$

By Lemma 3.4, for $\langle\langle B \rangle\rangle \in F_n^{(0)}$, $\langle\langle \Delta(B) \rangle\rangle$ belongs to F_n^{ev}. Conversely, for $\langle\langle v \rangle\rangle \in F_n^{\text{ev}}$, put

$$\Delta^{-1}\langle\langle v \rangle\rangle = \{\langle\langle B \rangle\rangle : \Delta(B) \in \langle\langle v \rangle\rangle\}.$$

Then from (**??**) we can see that

$$\Delta^{-1}\langle\langle v \rangle\rangle = \{\langle\langle B \rangle\rangle, \langle\langle D \rangle\rangle : \langle \Delta(B) \rangle = \langle v \rangle, \langle \Delta(D) \rangle = \langle \overline{v} \rangle\}.$$

If $\langle v \rangle \neq \langle \overline{v} \rangle$, i.e., $\langle v \rangle \in F_n^{(0)}$ is non-self-complementary, then Lemma 3.3 implies that $\langle\langle B \rangle\rangle \neq \langle\langle D \rangle\rangle$. In this case, we say that $\langle\langle v \rangle\rangle$ is ramified and $\langle\langle B \rangle\rangle$ and $\langle\langle D \rangle\rangle$ are called conjugate with respect to $\langle\langle v \rangle\rangle$. Otherwise, if $\langle v \rangle = \langle \overline{v} \rangle$, i.e., $\langle v \rangle \in F_n^{\text{sc}}$ is self-complementary, then $\langle\langle B \rangle\rangle = \langle\langle D \rangle\rangle$, and in this case we say that $\langle\langle v \rangle\rangle$ is irramified and $\langle\langle B \rangle\rangle$ is called self-conjugate. By H_n we denote the set of all self-conjugate necklaces in F_n. Clearly, H_n is a subset of $F_n^{(0)}$. The above discussion ((**??**) and Lemma 3.3) yields the following theorem.

Theorem 4.2. *If $n/2$ is odd, then H_n is empty. If $n/2$ is even, then $|H_n| = |G_{n/2}| = |F_{n/2}|$, and*

$$\Delta : \langle\langle B \rangle\rangle \longrightarrow \langle\langle \Delta(B) \rangle\rangle$$

is a bijection from H_n onto $G_{n/2}$.

Now we determine the conjugate pair with respect to a ramified necklace $\langle\langle v \rangle\rangle$.

For positive integer m and $w = w_1 \cdots w_m \in \{L, R\}^m$, define

$$\phi : w \longrightarrow w_1\overline{w}_2 \cdots w_{2k-1}\overline{w}_{2k} \cdots .$$

Then when $B = b_1 \cdots b_n \in \{L, R\}^n$ and n is even, we have that

$$\Delta(\phi(B)) = \delta(b_1\bar{b}_2)\delta(\bar{b}_2 b_3) \cdots \delta(\bar{b}_n b_1) = \overline{\Delta(B)}. \tag{4.6}$$

Lemma 4.3. *Let $B = b_1 \cdots b_n$ is a primitive word on $\{L, R\}$. We have that*

(i) $\phi(B)$ is primitive if B is non-self-complementary, or B is self-complementary but $n/2$ is even.

(ii) $\phi(B)$ is a square of a primitive word if B is self-complementry and $n/2$ is odd.

Proof. By Lemma 3.4 and (**??**) we have that

$$p(\Delta(\phi(B))) = p(\overline{\Delta(B)}) = p(\Delta(B)) = \begin{cases} n & \text{if } B \text{ is non-self-complementary,} \\ n/2 & \text{if } B \text{ is self-complementary.} \end{cases} \tag{4.7}$$

¿From this and (i) in Theorem 3.5 it follows that $\phi(B)$ is primitive if B is non-self-complementary. Also, one sees that $\phi(B)$ is not primitive if and only if $\phi(B)$ is a square. By a direct verification one has that if $B = D\overline{D}$ is self-complementary, then

$$\phi(B) = \begin{cases} \phi(D)^2 & \text{if } n/2 \text{ is odd,} \\ \phi(D)\overline{\phi(D)} & \text{if } n/2 \text{ is even.} \end{cases}$$

¿From this our assertions follow immediately. ∎

The following is our second decomposition theorem.

Theorem 4.4. *(i) For each ramified necklace $\langle\langle v \rangle\rangle \in F_n^{\mathrm{ev}}$, there exactly exists a pair of conjugate necklaces $\langle\langle B \rangle\rangle$ and $\langle\langle \phi(B) \rangle\rangle$ in $F_n^{(0)}$ satisfying $\langle \Delta(B) \rangle = \langle v \rangle$ and $\langle \Delta(\phi(B)) \rangle = \langle \overline{v} \rangle$. Especially, when $n/2$ is odd, each even necklace in F_n is ramified, and $\phi : \langle\langle B \rangle\rangle \longrightarrow \langle\langle \phi(B) \rangle\rangle$ is a bijection between F_n^{od} and F_n^{ev}.*

(ii) If $n/2$ is even, then for each irramified necklace $\langle\langle v \rangle\rangle \in F_n^{\mathrm{sc}}$, there is a unique self-conjugate necklace $\langle\langle B \rangle\rangle = \langle\langle \phi(B) \rangle\rangle \in F_n^{(0)}$ with $\langle \Delta(B) \rangle = \langle v \rangle$.

Proof. The former part of (i) and (ii) are obvious. We now prove the latter part of (i). Suppose that $n/2$ is odd. It is easily seen that every self-complementary necklace is odd, and for every $\langle\langle B \rangle\rangle \in F_n^{(0)}$, B is odd if and only if $\phi(B)$ is even. Therefore, $\phi : \langle\langle B \rangle\rangle \longrightarrow \langle\langle \phi(B) \rangle\rangle$ is a bijection between F_n^{od} and F_n^{ev}. ∎

The following theorem presents a characterization of self-conjugate necklaces.

Theorem 4.5. *Let $B = b_1 b_2 \cdots b_n \in F_n^{(0)}$. Then $\langle\!\langle B \rangle\!\rangle$ is self-conjugate if and only if $n/2$ is even and $B = D\phi(D)$ or $B = D\phi(\overline{D})$ where $D = b_1 \cdots b_{n/2}$.*

Proof. Since B is self-conjugate, i.e., $\langle B \rangle = \langle \phi(B) \rangle$, one has that $\langle v \rangle = \langle \Delta(B) \rangle$ is self-complementary. Set $n/2 = k$. If k is odd, then B and $\phi(B)$ have the different parities, hence $\langle B \rangle \neq \langle \phi(B) \rangle$. Now suppose k is even. Then

$$\begin{aligned} \Delta(B) = &\ \delta(b_1 b_2) \cdots \delta(b_k b_{k+1}) \delta(b_{k+1} b_{k+2}) \cdots \delta(b_{2k} b_1) \\ = &\ \delta(b_1 b_2) \cdots \delta(b_k b_{k+1}) \overline{\delta(b_1 b_2)} \cdots \overline{\delta(b_k b_{k+1})}. \end{aligned}$$

There are two possible cases:

(a) $b_{k+1} = b_1,\ b_{k+2} = \overline{b}_2, \cdots, b_{2k-1} = b_{k-1}, b_{2k} = \overline{b}_k$, which implies that

$$B = b_1 \cdots b_k b_1 \overline{b}_2 \cdots b_{k-1} \overline{b}_k = D\phi(D).$$

In this case, $\phi(B) = b_1 \overline{b}_2 \cdots b_{k-1} \overline{b}_k b_1 \cdots b_k = \sigma_n^k(B)$.

(b) $b_{k+1} = \overline{b}_1,\ b_{k+2} = b_2, \cdots, b_{2k-1} = \overline{b}_{k-1}, b_{2k} = b_k$, which implies that

$$B = b_1 \cdots b_k \overline{b}_1 b_2 \cdots \overline{b}_{k-1} b_k = D\phi(\overline{D}).$$

In this case, $\phi(B) = b_1 \overline{b}_2 \cdots b_{k-1} \overline{b}_k \overline{b}_1 \overline{b}_2 \cdots \overline{b}_{k-1} \overline{b}_k = \sigma_n^k(\overline{B})$. ∎

Note that there may be both cases that $\langle\!\langle D\phi(D) \rangle\!\rangle = \langle\!\langle D\phi(\overline{D}) \rangle\!\rangle$ or $\langle\!\langle D\phi(D) \rangle\!\rangle \neq \langle\!\langle D\phi(\overline{D}) \rangle\!\rangle$. The reader could check the words $D = RL$ and $D = RLLL$.

5 Concluding Remarks

We may understand the branching property of necklaces in such a way: the decomposition in (**??**) is the trunk, while each $F_n^{(k)}$ is a stemlet of the trunk. Since $F_n^{(k)}$ is equivalent to $F_{n/2^k}^{(0)}$, we may take $F_n^{(0)}$ as a sample to describe the branches on it. Every necklace in F_n^{ev} may be regarded as a node of $F_n^{(0)}$, and on each node of this kind there grows a pair of branches (conjugate with respect to it) or a sole branch (self-conjugate with respect to it). Especially, the collection of the sole branches in $F_n^{(0)}$ is bijection-equivalent to $F_n^{\text{sc}} = F_n^{(1)} \cup \cdots \cup F_n^{(k)}$ (that is the bijection Δ from H_n onto F_n^{sc}). We illustrate the property by F_6 and F_8.

Example 5.1. F_6 has 5 elements, in which there are two conjugate pairs and one element in $F_6^{(1)}$.

$$
\begin{array}{lcllll}
F_6^{(0)} & & F_6^{\mathrm{ev}} & & & \\
\left\{\begin{array}{l}\langle\!\langle RL^5\rangle\!\rangle \\ \langle\!\langle RL^3RL\rangle\!\rangle\end{array}\right. & \xrightarrow{\Delta} & \langle\!\langle RL^4R\rangle\!\rangle & & & \\
\left\{\begin{array}{l}\langle\!\langle RL^4R\rangle\!\rangle \\ \langle\!\langle RL^2R^2L\rangle\!\rangle\end{array}\right. & \xrightarrow{\Delta} & \langle\!\langle RL^3RL\rangle\!\rangle & F_6^{(1)} & & F_3^{(0)} \\
 & & & \langle\!\langle RL^3R^2\rangle\!\rangle & \xrightarrow{f_1} & \langle\!\langle RL^2\rangle\!\rangle
\end{array}
$$

Example 5.2. F_8 has 16 elements, in which there are 6 conjugate pairs, two self-conjugate ones, and one element in $F_8^{(1)}$ and $F_8^{(2)}$, respectively.

$$
\begin{array}{lcllllll}
F_8^{(0)} & & F_8^{\mathrm{ev}} & & & & & \\
\left\{\begin{array}{l}\langle\!\langle RL^7\rangle\!\rangle \\ \langle\!\langle RL^3RLRL\rangle\!\rangle\end{array}\right. & \xrightarrow{\Delta} & \langle\!\langle RL^6R\rangle\!\rangle & & & & & \\
\left\{\begin{array}{l}\langle\!\langle RL^6R\rangle\!\rangle \\ \langle\!\langle RL^2R^2LRL\rangle\!\rangle\end{array}\right. & \xrightarrow{\Delta} & \langle\!\langle RL^5RL\rangle\!\rangle & & & & & \\
\left\{\begin{array}{l}\langle\!\langle RL^5R^2\rangle\!\rangle \\ \langle\!\langle RL^2RL^2RL\rangle\!\rangle\end{array}\right. & \xrightarrow{\Delta} & \langle\!\langle RL^4RL^2\rangle\!\rangle & & & & & \\
\left\{\begin{array}{l}\langle\!\langle RL^4R^2L\rangle\!\rangle \\ \langle\!\langle RL^4RLR\rangle\!\rangle\end{array}\right. & \xrightarrow{\Delta} & \langle\!\langle RL^3R^3L\rangle\!\rangle & & & & & \\
\left\{\begin{array}{l}\langle\!\langle RL^4RL^2\rangle\!\rangle \\ \langle\!\langle RL^3R^3L\rangle\!\rangle\end{array}\right. & \xrightarrow{\Delta} & \langle\!\langle RL^3R^2LR\rangle\!\rangle & & & & & \\
\left\{\begin{array}{l}\langle\!\langle RL^3RL^2R\rangle\!\rangle \\ \langle\!\langle RL^3R^2L^2\rangle\!\rangle\end{array}\right. & \xrightarrow{\Delta} & \langle\!\langle RL^2R^2LRL\rangle\!\rangle & F_8^{(1)} & & F_4^{(0)} & & \\
\langle\!\langle RL^5RL\rangle\!\rangle & \multicolumn{2}{c}{- - \;\; - - \overset{\Delta}{- -} \longrightarrow} & \langle\!\langle RL^4R^3\rangle\!\rangle & \xrightarrow{f_1} & \langle\!\langle RL^3\rangle\!\rangle & F_8^{(2)} & F_2^{(0)} \\
\langle\!\langle RL^3R^2LR\rangle\!\rangle & \multicolumn{2}{c}{- - \;\; - - \overset{\Delta}{- -} - - -} & - - - - & - - & \longrightarrow & \langle\!\langle RL^2RLR^2L\rangle\!\rangle \xrightarrow{f_2} & \langle\!\langle RL\rangle\!\rangle
\end{array}
$$

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