The Flagged Double Schur Function

William Y. C. Chen
Center for Combinatorics
The Key Laboratory of Pure Mathematics and Combinatorics of
Ministry of Education
Nankai University, Tianjin 300071, P. R. China
Email: chenstation@hotmail.com

Bingqing Li
Department of Risk Management and Insurance
Nankai University, Tianjin, 300071, P. R. China

J.D. Louck
T-7, Mail Stop B210
Los Alamos National Laboratory
Los Alamos, NM 87545, USA
Email: jimlouck@lanl.gov

AMS Classification: 05E05, 05E10, 17B10, 17B35, 20C30.

Suggested Running Title: The Flagged Double Schur Function

Keywords: Flagged double Schur function, symplectic characters, orthogonal characters.

Please send correspondence to:

William Y. C. Chen
Center for Combinatorics
Nankai University
Tianjin 300071
P. R. China
Email: chenstation@hotmail.com

Abstract. The double Schur function is a natural generalization of the factorial Schur function introduced by Biedenharn and Louck. It also arises as the symmetric double Schubert polynomial corresponding to a class of permutations called Grassmannian permutations introduced by A. Lascoux. We present a lattice path interpretation of the double Schur function based on a flagged determinantal definition, which readily leads to a tableau interpretation similar to the original tableau definition of the factorial Schur function. The main result of this paper is a combinatorial treatment of the flagged double Schur function in terms of the lattice path interpretations of divided difference operators. Finally, we find lattice path representations of formulas for the symplectic and orthogonal characters for $sp(2n)$ and $so(2n+1)$ based on the tableau representations due to King and El-Shakaway, and Sundaram. Based on the lattice path interpretations, we obtain flagged determinantal formulas for these characters.

AMS Classification: 05E05, 05E10, 17B10, 17B35, 20C30.

Keywords: Double Schur function, flagged double Schur function, symplectic characters, orthogonal characters.

1. Introduction

Since their introduction by Lascoux and Schützenberger in 1982 [25], Schubert polynomials have been extensively studied by combinatorialists [22,24,26,40,41] and remain a thriving subject for new insights and challenges. The notion of Schubert polynomials has been further extended to two sets of variables by Lascoux, called double Schubert polynomials which are related to Chern classes [22], and have been recently studied, for example, in [1,8,9,12,13,36]. We are concerned with a class of double Schubert polynomials also singled out by Lascoux – the symmetric double Schubert polynomials, which we call the double Schur function in comparison with the supersymmetric Schur function. The double Schur function can be viewed as a generalization of the factorial Schur function introduced by Biedenharn and Louck [2,3]. The factorial Schur function can be obtained from the double Schur function by specializing the variable set $Y=\{y_1,y_2,\ldots,y_m\}$ to the value set $\{0,1,\ldots,m-1\}$. M. Mendez [29,30] developed an umbral calculus for symmetric functions including the factorial Schur function and the double Schur function. Molev and Sagan [32] have recently obtained the Littlewood-Richardson rule for the factorial Schur function.

Our first result is a lattice path interpretation of the double Schur function based on a flagged determinantal formula derived from a formula of Lascoux for the symmetric double Schubert polynomial. We start with the definition of the double Schubert polynomial. Such a lattice path construction easily translates into the tableau definition as a natural generalization of the original tableau defintion of the factorial Schur function [7]. Different approaches to the double Schur function have been developed by Goulden and Greene [17], and Macdonald [27]. The double Schur function can be defined in terms of a Jacobi-Trudi type determinant, called the multi-Schur function, and it can also be defined in terms of divided difference operators. We take the approach of establishing a nonintersecting lattice path explanation of the determinantal definition of the double Schur function, and then translate the lattice path formulation into tableau notation. Although it has been a standard practice to construct lattice paths based on a certain kind of binomial type determinant, the origins of the lattice paths corresponding to the double Schur function are not on a horizontal line as in the usual cases; whereas the origins we choose lie slightly off the diagonal and the destinations turn out to be on a vertical line. In our construction, the content function of a tableau comes into play in a quite natural way.

The main result in this paper is a combinatorial treatment of the divided difference operators which can be used to compute the double Schur function from a monomial. We present a combinatorial interpretation of such divided difference operators acting on a dominant double Schubert polynomial. With such a lattice path representation, one easily arrives at the operator definition, the tableau interpretation and the determinantal formula of the double Schur function. Our combinatorial approach also extends to the flagged double Schur function.

Finally, we obtain lattice path representations of the tableau definitions of the symplectic and orthogonal characters of $sp_{2n}(\lambda,X)$ and $so_{2n+1}(\lambda,X)$ based on the tableau representations of King and El-Sharkaway [20], and Sundaram [39]. Based on such lattice path correspondence, we obtain two flagged determinantal formulas for these characters.

2. The Double Schur Function

Let us start with the classical defintion of double Schubert polynomials in terms of divided difference operators. Given a function $f(x_1,x_2,\ldots,x_n)$, the transposition operator s_i is defined by

$$s_i\, f(x_1,x_2,\ldots,x_n) = f(x_1,\ldots,x_{i+1},x_i,\ldots,x_n),$$

and the divided difference operator ∂_i is given by

$$\partial_i f = \frac{f - s_i f}{x_i - x_{i+1}} = \frac{f(\ldots,x_i,x_{i+1},\ldots) - f(\ldots,x_{i+1},x_i,\ldots)}{x_i - x_{i+1}}.$$

The double Schubert polynomial is then defined as the action of a series of divided difference operators on the following maximal double Schubert polynomial:

$$\Delta(X,Y) = \prod_{i+j\leq n} (x_i - y_j),$$

where $X = \{x_1,x_2,\ldots,x_n\}$ and $Y = \{y_1,y_2,\ldots,y_n\}$. Given a permutation $w \in S_n$, let

$$c_i(w) = |\{j : i < j \text{ and } w(i) > w(j)\}|.$$

Then

$$c(w) = (c_1(w),\cdots,c_n(w))$$

is called the code of w or the inversion code of w, and $l(w) = \sum_{i=1}^{n} c_i(w)$ is called the length of w. Note that the codes of permutations on n elements are in one-to-one correspondence with sequences $a_1a_2\cdots a_n$ on the set $\{0,1,\ldots,n-1\}$ such that $a_i \leq n-i$. Double Schubert polynomials, denoted by $\mathfrak{S}_I(X,Y)$, can be defined as polynomials on X and Y indexed by an inversion code I of a permutation on n elements, or equivalently by a permutation w in S_n. The following constructive definition of double Schubert polynomials is given by Lascoux [23,24].

Definition 2.1 *Given an inversion code $I = (i_1, \dots, i_n)$ of a permutation $w \in S_n$, the polynomial $\mathfrak{S}_I(X,Y)$ is constructed by the following procedure. Let K be the inversion code of the longest permutation w_0 in S_n, namely, $w_0 = n\,(n-1)\cdots 2\,1$ and $K = (n-1, n-2, \dots, 0)$. Then the polynomial*

$$\mathfrak{S}_{w_0}(X,Y) = \mathfrak{S}_K(X,Y) = \Delta(X,Y).$$

Suppose $I = (i_1, \dots, i_n)$ is an inversion code of w such that $i_k > i_{k+1}$. Then the double Schubert polynomial corresponding to the inversion code

$$I' = (i_1, \dots, i_{k-1}, i_{k+1}, i_k - 1, i_{k+2}, \dots, i_n), \tag{1}$$

is given by

$$\mathfrak{S}_{I'}(X,Y) = \partial_k \mathfrak{S}_I(X,Y).$$

Suppose that w is the permutation with inversion code I as in the above definition. Then the permutation w' corresponding to I' in (1) can be obtained from w by transposing the elements in the k-th and $(k+1)$-th positions. Thus, we may compute the Schubert polynomial $\mathfrak{S}_I(X,Y)$ for any inversion code I as successive actions on the maximal double Schubert polynomial $\Delta(X,Y)$. It can be verified that the above definition is indeed equivalent to the original definition in terms of reduced words on transpositions. Given any inversion code I, it can be reached from the code of the longest permutation by the lowering operations in the above definition. Note that after each step the length of the resulting code decreases by one. Note that the procedure to arrive at an inversion code from that of the longest permutation may not be unique. However, because of the braid relations:

$$\begin{aligned} \partial_i \cdot \partial_j &= \partial_j \cdot \partial_i \quad \text{for } |i-j| > 1, \\ \partial_i \cdot \partial_{i+1} \cdot \partial_i &= \partial_{i+1} \cdot \partial_i \cdot \partial_{i+1}, \quad \text{for all } i, \end{aligned}$$

the double Schubert polynomial is uniquely defined (see also [26]). In general, we may use the standard route as described below. Let $I = (i_1, i_2, \cdots, i_n)$ be the inversion code of $w \in S_n$, that is $i_k \le n-k$. Then we can obtain $\mathfrak{S}_I(X,Y)$ from $\Delta(X,Y) = \mathfrak{S}_K(X,Y)$, where K is the inversion code of the longest permutation of S_n. If $i_1 = n-k$, then we have

$$\partial_1(\partial_2(\cdots(\partial_{k-1}\Delta(X,Y)))) = \mathfrak{S}_{i_1,\ n-2,\cdots,k+1,\ k,\ k-1,\cdots,2,\ 1,\ 0}(X,Y),$$

and if $i_2 = n-l \neq n-2$, then we have

$$\partial_2(\partial_3(\cdots(\partial_{l-1}\mathfrak{S}_{i_1,n-2,\cdots,2,1,0}(X,Y)))) = \mathfrak{S}_{i_1,i_2,n-3,\cdots,l+1,l,l-1,\cdots,1,0}(X,Y).$$

Iterating this process, we may compute $\mathfrak{S}_I(X,Y)$. For example, Let $I = (1,\ 2,\ 0,\ 0)$ for $n = 4$. We have

$$\mathfrak{S}_{1,2}(X,Y) = \partial_3(\partial_1(\partial_2\Delta(X,Y))).$$

Consider next the class of permutations w of S_n such that the inversion code of w is a non-decreasing sequence by disregarding any string of zeros at the right-hand end of $c(w)$. Such permutations are called Grassmannian permutations. Moreover, a permutation in this class is called Grassmannian permutation of shape

$$\lambda\colon\ \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_m \ge 0,$$

where $m \leq n$, and λ is the reverse of the sequence of I, namely,

$$\lambda_1 = i_m \geq i_{m-1} \geq \cdots \geq i_1 \geq 0.$$

A symmetric double Schubert polynomial is defined as a double Schubert polynomial indexed by the inversion code of a Grassmannian permutation, or by a partition λ.

A different perspective of the symmetric double Schubert polynomials is to view them as supersymmetric Schur functions in X and Y, although these two classes are not quite the same. However, they share a common feature of the supersymmetric complete function for $X = \{x_1, x_2, \ldots\}$ and $Y = \{y_1, y_2, \ldots\}$:

$$h_n(X - Y) = [t^n] \frac{\prod_{y \in Y}(1 - yt)}{\prod_{x \in X}(1 - xt)} = \sum_{k=0}^{n} e_k(X)\, h_{n-k}(-Y), \tag{2}$$

where $[t^n]\, f(t)$ means the coefficient of t^n in $f(t)$, $e_{n-k}(-Y)$ denotes the elementary symmetric function $e_{n-k}(-y_1, -y_2, \ldots)$ and $h_k(X)$ denotes the ordinary complete symmetric function in X. It is important to note that if we change the signs of every variable in Y, then $h_n(X+Y)$ coincides with the supersymmetric function used by Golden and Greene [17] in the notation $H_n(X,Y)$. It is necessary in the context of double Schubert polynomials to define $h_n(X - Y)$ as in (2) for which the variables in Y carry the minus signs in the numerator. If we set $y_i = i - 1$, then $h_n(X,Y)$ becomes the factorial complete symmetric function as defined in [7].

Let $I = (i_1, i_2, \ldots, i_m)$, $i_m > 0$, be an inversion code of a Grassmannian permutation $w \in S_n$. Then $\mathfrak{S}_I(X,Y)$ can be expressed as the following determinant:

$$\mathfrak{S}_I(X,Y) = \det \begin{pmatrix} h_{i_1}(X_m - Y_{i_1}) & h_{i_2+1}(X_m - Y_{i_2+1}) & \cdots & h_{i_m+m-1}(X_m - Y_{i_m+m-1}) \\ h_{i_1-1}(X_m - Y_{i_1}) & h_{i_2}(X_m - Y_{i_2+1}) & \cdots & h_{i_m+m-2}(X_m - Y_{i_m+m-1}) \\ \cdots & \cdots & \cdots & \cdots \\ h_{i_1-m+1}(X_m - Y_{i_1}) & h_{i_2-m+2}(X_m - Y_{i_2+1}) & \cdots & h_{i_m}(X_m - Y_{i_m+m-1}) \end{pmatrix}, \tag{3}$$

where $X_m = \{x_1, x_2, \ldots, x_m\}$ and $Y_m = \{y_1, y_2, \ldots, y_m\}$.

The above determinant can be recast in terms of the divided difference operator as:

$$\mathfrak{S}_I(X,Y) = (\partial_{m-1}\partial_{m-2}\ldots\partial_1)\cdot(\partial_{m-1}\partial_{m-2}\ldots\partial_2)\cdots(\partial_{m-1})\mathfrak{S}_J(X,Y).$$

where $J = (j_1, j_2, \ldots j_m) = (i_m + m - 1, i_{m-1} + m - 2, \cdots, i_1)$ and

$$\mathfrak{S}_J(X,Y) = \prod_{k=1}^{m}\prod_{l=1}^{j_k}(x_k - y_l).$$

The above double Schubert polynomial is called a dominant double Schubert polynomial [24, 26]. If we set $Y = 0$, then the above definition of $\mathfrak{S}_I(X,Y)$ reduces to the following expression of the Schur function:

$$s_\lambda(X) = (\partial_{m-1}\partial_{m-2}\ldots\partial_1)\cdot(\partial_{m-1}\partial_{m-2}\ldots\partial_2)\cdots(\partial_{m-1})\, x_1^{\lambda_1+m-1} x_2^{\lambda_2+m-2}\cdots x_m^{\lambda_m}.$$

We remark that the product of operators in the above equation is an important special case in the theory of Schubert polynomials for the longest permutation w_0 in S_m:

$$\partial_{w_0} = (\partial_{m-1}\partial_{m-2}\ldots\partial_1)\cdot(\partial_{m-1}\partial_{m-2}\ldots\partial_2)\cdots(\partial_{m-1}),$$

as described in Definition 2.1.

Lascoux introduced the Lagrange operator L_m which extends a polynomial in one variable, say x_1, to a symmetric function in m variables $x_1, x_2, \ldots, x_m$:

$$L_m f(x_1) = \sum_{i=1}^{m} f(x_i) \Big/ \prod_{j \neq i} (x_i - x_j).$$

The Lagrange operator L_m can be expressed in terms of divided difference operators:

$$L_m = \partial_{m-1} \partial_{m-2} \cdots \partial_1.$$

The above operator L_m coincides with the classical higher order divided difference operator, and is denoted by Δ with parameters $[x_1, x_2, \ldots, x_m]$ in [7]. Moreover, the product $\partial_{m-1} \partial_{m-2} \cdots \partial_i$, denoted by L_i^m, can also be regarded as a Lagrange operator extending a polynomial in x_i to a symmetric function in $x_i, x_{i+1}, \ldots, x_m$. It is important to mention that the divided difference operator corresponding to the reduction from the longest permutation to the identity permutation can be written as the product of Lagrange operators:

$$(\partial_{m-1}\partial_{m-2} \ldots \partial_1) \cdot (\partial_{m-1}\partial_{m-2} \ldots \partial_2) \cdots (\partial_{m-1}) = L_1^m L_2^m \cdots L_{m-1}^m.$$

The action of the above operator can be expressed in terms of determinants. For any polynomial $f(X) = f(x_1, x_2, \ldots, x_m)$, we have

$$(\partial_{m-1}\partial_{m-2} \ldots \partial_1) \cdot (\partial_{m-1}\partial_{m-2} \ldots \partial_2) \cdots (\partial_{m-1}) f(X) = \sum_{\sigma \in S_m} (-1)^{|\sigma|} f^{\sigma}(X) / \Delta(X), \tag{4}$$

where $f^{\sigma}(X)$ denotes the action of the permutation σ on the indices of the variables $x_1, \ldots, x_m$, and $\Delta(X)$ is the Vandermonde determinant in $x_1, x_2, \ldots, x_m$:

$$\Delta(X) = \prod_{i<j} (x_i - x_j).$$

The above formula can be understood as the equivalence between the alternant definition of the Schur function and the Jacobi-Trudi identity as described by Stanley [37], or in [7] for the case of the factorial Schur function.

We employ the following notation as in [17, 27]:

$$(x|Y)_n = \prod_{1 \le i \le n} (x - y_i), \tag{5}$$

and extend to

$$(x|Y)_{[i,n]} = \prod_{k=i}^{n} (x - y_k). \tag{6}$$

If we set $f = \prod_{k=1}^{m} (x_{m-k+1}|Y)_{i_k+k-1}$ in (4), then we are led to the following expression given by Lascoux [24] in the terminology of symmetric double Schubert polynomials:

$$\mathfrak{S}_I(X,Y) = \frac{\det\Big((x_i|Y)_{m+i_{m-j+1}-j} \Big)_{m \times m}}{\Delta(X)},$$

where I is the inversion code of a Grassmannian permutation. If we rewrite the above formula in terms of a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$,

$$s_\lambda(X,Y) = \frac{\det\Big((x_i|Y)_{\lambda_j+m-j} \Big)_{m\times m}}{\Delta(X)},$$

then we arrive at the 6th variation of the Schur function as given by Macdonald [26] as a natural generalization of the factorial Schur function.

Note that the above divided difference operator definition of the double Schur function differs from the divided difference operator definition based the maximal double Schubert polynomial $\Delta(X,Y)$. Nevertheless, the equivalence of the two can be viewed as a duality between the operators and the polynomials. The proof of this equivalence can be found in [24, 26].

The factorial Schur function can be obtained from the double Schur function, or the symmetric double Schubert polynomial by specifying y_i to $i-1$. On the other hand, the factorial Schur function possesses almost the same properties as the double Schur function because parameters $0,1,2,\ldots$ in the factorial Schur function basically play a role as indeterminates $y_1, y_2, \ldots$. The idea of using lattice path methods for the factorial Schur function was first pointed out in [7] because of the binomial type property of the entries in the Jacobi-Trudi formula, and later explicitly given by Goulden-Hammel [18], Goulden and Greene [17]. However, as we shall see, there is still something to be said about such a general idea, particularly about the origins of lattice paths, as we shall see in the next section.

3. A Lattice Path Interpretation

There is some advantage of using the index of the double Schur function as an inversion code, instead of a partition. With respect to the factorial Schur function, the number of parts including zero components is important when it is used as an index, although for the ordinary Schur function the zero components can be ignored. For this reason, the usage of Gelfand pattern in the physics literature is a good way to avoid such an ambiguity. Therefore, we use a sequence instead of a partition to index a double Schur function. As a first step to give a lattice path interpretation of the double Schur function $\mathfrak{S}_I(X,Y)$, we prefer the following variation of (3), which can be regarded as a triangulation or a flagged form. As we shall see, such a flagged form leads to nice properties for constructing the corresponding lattices:

$$\det \begin{pmatrix} h_{i_1}(E_{n-1}-Y_{i_1}) & h_{i_2+1}(E_{n-1}-Y_{i_2+1}) & \cdots & h_{i_n+n-1}(E_{n-1}-Y_{i_n+n-1}) \\ h_{i_1-1}(E_{n-2}-Y_{i_1}) & h_{i_2}(E_{n-2}-Y_{i_2+1}) & \cdots & h_{i_n+n-2}(E_{n-2}-Y_{i_n+n-1}) \\ \cdots & \cdots & \cdots & \cdots \\ h_{i_1-n+1}(E_0-Y_{i_1}) & h_{i_2-n+2}(E_0-Y_{i_2+1}) & \cdots & h_{i_n}(E_0-Y_{i_n+n-1}) \end{pmatrix}, \tag{1}$$

where $E_i = X_n \setminus X_i = \{x_{i+1}, \cdots x_n\}$.

The transformation from the determinant (3) to (1) easily follows from a property of the multi-Schur function [24, 26]:

Lemma 3.1 *Let $J = (j_1, j_2, \ldots, j_n)$ be a sequence of integers, and let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be sets of variables. Then the multi-Schur function*

$$S_J(X_1 - Y_1, \ldots, X_n - Y_n) = \det\Big(h_{j_k+k-l}(X_k - Y_k) \Big)_{n\times n}$$

can also be rewritten as the determinant

$$\det\left(h_{j_k+k-l}(X_k - Y_k - D_{n-l})\right)_{n\times n}$$

for any family $D_0, D_1, \ldots, D_{n-1}$ of variables such that $|D_i| \le i$.

We now proceed to give a lattice path realization of the flagged determinant (1). As usual, a lattice path in the plane is a path P from an origin to a destination in which every step is either going up (vertical step) or going right (horizontal step). The weight of each step is defined as follows:

1. For a vertical step from (i, j) to $(i, j+1)$, the weight is $x_i - y_{i+j}$.
2. For a horizontal step from (i, j) to $(i+1, j)$, the weight is 1.
3. The weight of a path P is the product of the weights of the steps in the path, denoted by $w(P)$.

For a set of paths $P_1, P_2, \ldots, P_m$, the weight is defined to be the product of all the weights. Let $A = (A_1, A_2, \ldots, A_m)$ and $B = (B_1, B_2, \ldots, B_m)$ be sequences of lattice points, we say that $(P_1, P_2, \ldots, P_m)$ is a group of nonintersecting lattice paths from A to B if P_i's are nonintersecting and P_i is a lattice path with origin A_i and destination B_i. Moreover, we use $w(A,B)$ to denote the sum of weights of all nonintersecting lattices paths from A to B. We now can state the first theorem of this paper:

Theorem 3.2 *Let $I = (i_1, i_2, \ldots, i_m)$ be a non-decreasing sequence. Then the double Schur function $\mathfrak{S}_I(X,Y)$ can be evaluated by $w(A,B)$ for $A_k = (k, -k+1)$ and $B_k = (m, i_{m-k+1} - k + 1)$.*

For the first step in proving the above theorem, we need to give a lattice path interpretation of the entries in the determinant (1). They are the supersymmetric functions, and we may express them as the action of divided difference operators on the polynomial $h_n(x_1 - Y_n)$ which turns out to be a product $(x_1 - y_1)\cdots(x_1 - y_n)$. This leads to a lattice path interpretation of the entries in the determinant.

Lemma 3.3 (Lascoux [24]) *For the complete double Schur function $h_n(x_1 - Y)$, we have*

$$L_r h_n(x_1 - Y) = L_r(x_1|Y)_n = h_{n-r+1}(X_r - Y). \tag{2}$$

Proof. While the following identity is straightforward to verify, it is a fundamental idea in dealing with divided differences of generating functions:

$$\partial_i \frac{1}{1 - tx_i} = t \cdot \frac{1}{(1 - tx_i)(1 - tx_{i+1})}. \tag{3}$$

Iterating the same argument, we arrive at the following identity:

$$L_r h_n(x_1 - Y) = [t^n] L_r \left(\frac{\prod_{y\in Y}(1 - ty)}{1 - x_1 t} \right) = [t^{n-r+1}] \frac{\prod_{y\in Y}(1 - ty)}{\prod_{1\le i\le r}(1 - tx_i)} = h_{n-r+1}(X_r - Y),$$

which completes the proof. ∎

As a critical case of the above lemma, we have the following relation, as noted in [24]:

$$h_n(x_1 - Y_n) = \sum_{i=0}^{n} (-1)^{n-i} x_1^i e_{n-i}(y_1, \cdots, y_n) = (x_1 - y_1)\cdots(x_1 - y_n).$$

Note that if we set $y_i = i-1$, then $h_n(x_1 - Y_n)$ turns out to the factorial $(x_1)_n$. With the above formula for $h_n(x_1 - Y_n)$ and the formula for $h_{n-r+1}(X_r - Y_n)$, we may obtain the following lattice path interpretation of the function $h_m(X_n - Y_{n+m-1})$:

Lemma 3.4 *The double complete symmetric function $h_m(X_n - Y_{n+m-1})$ can be described by the sum of weights over lattice paths from $(1,0)$ to (n,m).*

Proof. By Lemma 3.3, we have

$$h_m(X_n - Y_{n+m-1}) = L_n\,(x_1|Y)_{n+m-1}.$$

Iterating the following identity [7]:

$$\frac{(x_1|Y)_{m+1} - (x_2|Y)_{m+1}}{x_1 - x_2} = \sum_{0\le k\le m} \prod_{1\le l\le k} (x_1 - y_l) \prod_{k+2\le l\le m+1} (x_2 - y_l), \tag{4}$$

it follows that

$$h_m(X_n - Y_{n+m-1}) = \sum_{i_1+i_2+\cdots+i_n=m} \left(\prod_{1\le r\le n} \left(\prod_{t=i_1+\cdots+i_{r-1}+r}^{i_1+\cdots+i_r+r-1} (x_r - y_t) \right)\right), \tag{5}$$

which is the sum of the weights over all lattice paths from $(1,0)$ to (n,m). ∎

In general, all the entries in the determinant (1) can be interpreted by lattice paths. Here we only consider those nonzero entries.

Lemma 3.5 *Suppose that $i_k + j \geq 0$ and $j < k$. Then the following entry*

$$h_{i_k+j}(X_n \backslash X_{n+j-k} - Y_{i_k+k-1}) \tag{6}$$

equals the sum of weights of all lattice paths from $(n+j-k+1, -(n+j-k))$ to $(n,\ i_k+k-n)$.

In the notation of divided differences, the function (6) can be expressed as

$$L^n_{n+j-k+1}\,(x_{n+j-k+1}|Y)_{i_k+k-1}.$$

We are now ready to give an involutional proof of Theorem 3.2 in the spirit of the Gessel-Viennot methodology [15, 16].

Proof of Theorem 3.2: Recall that $A_l = (l, -l+1)$, $B_l = (m, i_{m-l+1} - l + 1)$. Let $\pi = \pi_1\pi_2\cdots\pi_m$ be a permutation on $\{1, 2, \ldots, m\}$. Suppose that P_l is a lattice path from A_l to B_{π_l}, $1 \le l \le m$. The sign of the configuration $(P_1, P_2, \ldots, P_m)$ is defined to be the sign of the permutation π. We need to find the smallest index j such that P_j intersects with a path P_k ($j < k$). We choose k to be the smallest if P_j intersects with more than one path. Let v be the intersection point of P_j and P_k. Then we may switch the segments from v to P_{π_j} and P_{π_k}, leading to lattice paths $(P_1, \ldots, P'_j, \ldots, P'_k, \ldots, P_m)$. This construction is a sign-reversing and weight preserving involution. It is illustrated in Figures 1 and 2. ∎

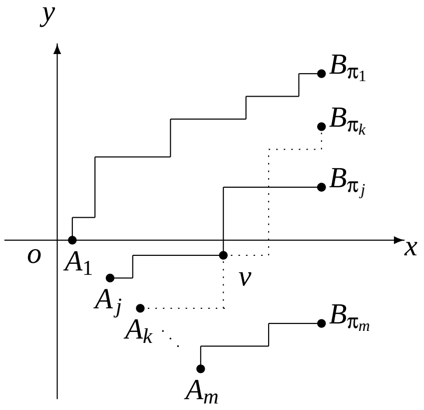

Figure 1: Before the involution

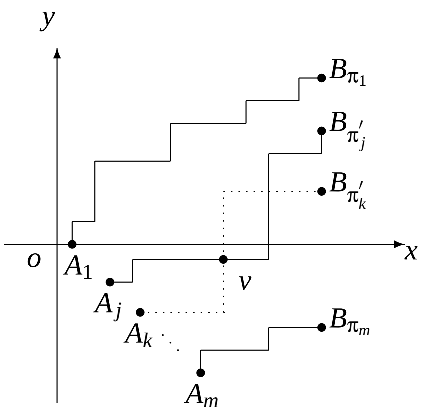

Figure 2: After the involution.

Once the lattice path interpretation of the determinant (1) is obtained, it is straightforward to translate it into a Young tableau representation as given by Biedenharn and Louck for the factorial Schur function [2, 3], and for the double Schur function as given by Goulden and Greene [17] and Macdonald [27].

Theorem 3.6 *Let $I = (i_1, i_2, \cdots, i_m)$ be a code of a Grassmannian permutation, and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ be a partition with $\lambda_k = i_{m+1-k}$, $1 \le k \le m$. Then the double Schur function $\mathfrak{S}_I(X,Y)$ equals the function $s_\lambda(X,Y)$ defined on column strict tableaux T on $\{1,2,\ldots,m\}$ of shape λ with the following weight function*:

$$(x_{T(\alpha)} - y_{T(\alpha)+C(\alpha)}),$$

where $T(\alpha)$ is the entry of T in the cell α, and $C(\alpha)$ is the content of α which equals $j-i$ if α falls in the i-th row and j-th column.

Proof. For any column strict tableau T with shape λ on the set $\{1,2,\ldots,m\}$, we associate it with a sequence $(P_1, P_2, \cdots, P_m)$ of nonintersecting paths such that P_k has origin $A_k = (k, -k+1)$ and destination $B_k = (m, \lambda_k + 1 - k)$. Let us consider the k-th row of T. If the first cell is u ($u \ge k$), then we draw a line from $(u, -k+1)$ to $(u, -k+2)$. Suppose that the second cell in the k-th row is v, then we may draw a line from $(v, -k+2)$ to $(v, -k+3)$, and so on. Thus, we have λ_k vertical lines and we can add some horizontal lines to get a path P_k from $(k, -k+1)$ to $(m, \lambda_k - k + 1)$. Moreover, these paths $P_1, P_2, \ldots, P_m$ are nonintersecting because the tableau T is column strict. The above procedure is reversible. Hence we obtain a bijection.

A cell α in the k-th row and l-th column has content $l-k$ and corresponds to the lth vertical step in P_k from $(T(\alpha), -k+l)$ to $(T(\alpha), -k+l+1)$, this step has weight

$$x_{T(\alpha)} - y_{T(\alpha)-k+l} = x_{T(\alpha)} - y_{T(\alpha)+C(\alpha)}.$$

It follows that

$$\prod_{\alpha\in\lambda} (x_{T(\alpha)} - y_{T(\alpha)+C(\alpha)}) = \prod_k w(P_k),$$

where $w(P_k)$ is the weight of P_k. This completes the proof. ∎

It is worth mentioning the following formula of Pragacz and Thorup [34] for the supersymmetric Schur function indexed by a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$:

$$S_\lambda(X_m, Y_n) = \det\left(S_{\lambda_i - i + j}(X_m/Y_n) \right)_{l\times l},$$

where

$$S_n(X/Y) = [t^n] \frac{\prod_{y\in Y}(1+yt)}{\prod_{x\in X}(1-xt)} = h_n(X-(-Y)),$$

and $-Y = \{-y_1, -y_2, \ldots\}$. As noted in [17, 27], although the double Schur function is different from the supersymmetric Schur function, the two have a common tableau representation when we extend X and Y to the following infinite sets:

$$X = \{\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots\} \quad \text{and} \quad Y = \{\ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots\}.$$

4. The Flagged Double Schur Function

In this section, we introduce the notion of a flagged double Schur function, which falls into the more general framework of determinantal forms studied by Lascoux [22]. The flagged form of the ordinary Schur function was introduced by Lascoux and Schützenberger [25]. Gessel observed that the tableau definition of the Schur function could be extended to the flagged Schur function, and a detailed study was later carried out by Wachs [41]. The flagged version of the supersymmetric Schur function has been studied by Goulden and Hammel [18,19]. Our main idea is to use lattice paths to characterize the actions of divided difference operators, and then to turn the lattice paths into flagged determinantal formulas. To this end, we start with the divided difference operator definition of the flagged double Schur function, and then establish the lattice path interpretation.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ be a partition with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > 0$ and let $b = (b_1, b_2, \ldots, b_m)$ be a sequence of nondecreasing positive integers. The flagged Schur function with shape λ and flag b is defined as

$$s_\lambda(b) = \det(h_{\lambda_i - i + j}(b_i))_{m\times m},$$

where $h_{\lambda_i-i+j}(b_i) = h_{\lambda_i-i+j}(x_1, x_2, \cdots, x_{b_i})$.

In [41], Wachs gave a combinatorial definition of the flagged Schur function in terms of column strict tableaux. Let $\mathcal{T}(\lambda, b)$ be the set of all column strict tableaux T of shape λ such that the elements in the i-th row of T do not exceed b_i. Then we have

$$s_\lambda(b) = \sum_{T\in\mathcal{T}(\lambda,b)} w(T),$$

where $w(T) = \prod_{\alpha\in T} x_{T(\alpha)}$.

We define the flagged double Schur function as follows.

Definition 4.1 *Given a partition λ and a flag b, the flagged double Schur function is given by*

$$s_{\lambda,b}(X - Y) = \det\Big(h_{\lambda_i-i+j}(X_{b_i} - Y_{\lambda_i+b_i-i})\Big)_{m\times m}.$$

Note that if set $b_1 = b_2 = \cdots = b_m$, then the flagged double Schur function reduces to the double Schur function. We now shift our attention to a divided difference definition of the flagged Schur function, and then pursue a lattice path interpretation based on the divided difference operators. Given a partition λ with m positive parts, and a flag b of length m, set $a_i = \lambda_i + b_i - i$, and

$$L_b = L_{b_1,\ldots,b_m} = (\partial_{b_1-1}\partial_{b_1-2}\cdots\partial_1)(\partial_{b_2-1}\partial_{b_2-2}\cdots\partial_2)\cdots(\partial_{b_m-1}\partial_{b_m-2}\cdots\partial_m).$$

Then we have a lattice path interpretation for the action of $L_{b_1,\cdots,b_m}$ on the polynomial

$$(X|Y)_a = (x_1|Y)_{a_1}(x_2|Y)_{a_2}\cdots(x_m|Y)_{a_m},$$

from which one may easily recover the tableau definition and the determinantal definition of the flagged double Schur function. Hence we arrive at the conclusion that the divided difference definition of the flagged double Schur function coincides with the determinantal definition and the tableau definition.

Theorem 4.2 *The polynomial $L_{b_1,\ldots,b_m}((x_1|Y)_{a_1}(x_2|Y)_{a_2}\cdots(x_m|Y)_{a_m})$ equals the sum of weights of all sequences $(P_1,P_2,\cdots,P_m)$ of nonintersecting paths such that P_i has origin $(i,-i+1)$ and destination (b_i,λ_i-i+1).*

Before we present a proof of the theorem, we make some remarks.

- The above lattice path represenation gives the following determinantal formula:

$$L_{b_1,\ldots,b_m}((x_1|Y)_{a_1}(x_2|Y)_{a_2}\cdots(x_m|Y)_{a_m}) = \det(h_{\lambda_i-i+j}((X_{b_i}\setminus X_{j-1})-Y_{\lambda_i+b_i-i}))_{m\times m}.$$

  By Lemma 3.1, we may rewrite it as our first definition:

$$\det\Big(h_{\lambda_i-i+j}(X_{b_i}-Y_{\lambda_i+b_i-i})\Big)_{m\times m}.$$

- The above lattice path representation can also be translated into the following tableau notation:

$$s_{\lambda,b}(X-Y)=\sum_{T\in\mathcal{T}(\lambda,b)} w(T),$$

  where $\mathcal{T}(\lambda,b)$ is the set of column strict tableau of shape λ such that the elements in the i-th row do not exceed b_i, and

$$w(T)=\prod_{\alpha\in T}(x_{T(\alpha)}-y_{T(\alpha)+C(\alpha)}),$$

  with $C(\alpha)$ being the content function as before.

We restate the identity (4) in terms of lattice paths.

Lemma 4.3 *Let P be the vertical segment from (m,k) to (m,p). Then the action of ∂_m on the weight of P yields the sum of weights of all lattice paths from (m,k) to $(m+1,p-1)$.*

Using the above lemma, we may have the following rule for computing the action of ∂_m.

Lemma 4.4 (Pairing Lemma) *Let $(A_1,A_2,\ldots,A_n)$ be a sequence of the lattice points with $A_i=(m,k_i)$, and let $B=(B_1,B_2,\ldots,B_n)$ be a sequence of lattices points with $B_1=(m,p)$ and $B_i=(m+1,t_i)$ for $i\geq 2$. Suppose $p>k_1>k_2>\cdots>k_n$, $p-1>t_2>\cdots>t_n$, and $k_i\leq t_i$ for $i\geq 2$. Then we have*

$$\partial_m w(A,B)=w(A,B'),$$

where B' is obtained from B by replacing B_1 with $(m+1,p-1)$.

Proof. First, we note that if $w(A,B)$ contains a factor that is symmetric in x_m and x_{m+1}, then this factor can be regarded as a constant when applying the operator ∂_m. We proceed to show that what really matters for the operator ∂_m is the segment of the path from A_1 to B_1 that is above the horizontal line $y=t_2+1$. The polynomial $w(A,B)$ can be computed by the following procedure. Suppose $t_2+1>k_1$. Then every path from A_2 to B_2 must have the segment from $(m+1,k_1-1)$ to $(m+1,t_2)$, and $w(A,B)$ must contain the factor

$$(x_m|Y)_{[m+k_1,m+t_2]}\,(x_{m+1}|Y)_{[m+k_1,m+t_2]}, \tag{1}$$

which is symmetric in x_m and x_{m+1}. If $k_2 > t_3$, then every path from A_3 to B_3 automatically does not intersect with any path from A_2 to B_2. By Lemma 4.3 or Lemma 3.4, the weights of such paths contribute to the factor

$$h(x_m,x_{m+1},Y) \tag{2}$$

which is again symmetric in x_m and x_{m+1}. If $k_2 < t_3+1$, we may repeat the above process to get a factor in the form of (1). Iterating the above process, one may have factors symmetric in x_m and x_{m+1}.

For the case when $t_2+1 \le k_1$, we first take out the factor $w(A_1,B_1)$, and then we may use the above argument to show that the rest factors of $w(A,B)$ are symmetric in x_m and x_{m+1}. For each case, we may apply Lemma 4.3 to reach the desired conclusion. ∎

We can now prove Theorem 4.2.

Proof of Theorem 4.2. We begin with the m vertical lines $P_1,P_2,\cdots,P_m$, where P_i is from $A_i=(i,-i+1)$ to $B_i=(i,a_i-i+1)$. Recall that $a_i=\lambda_i+b_i-i$. Consider the action of ∂_m on $(X|Y)_a$. By Lemma 4.3, $\partial_m(X|Y)_a$ equals the sum of weights of all lattice paths from A to B' where B' is obtained from B by replacing B_1 with $(m+1,a_m-m)$. Next consider the action of ∂_{m+1} on $\partial_m\,(X|Y)_a$. For any group of paths $(Q_1,Q_2,\ldots,Q_m)$ from A to B', ∂_{m+1} affects only the area between the lines $x=m+1$ and $x=m+2$. We may assume that the points of $(Q_1,Q_2,\ldots,Q_m)$ on the lines $x=m+1$ and $x=m+2$ satisfy the conditions in Lemma 4.4, otherwise the action of ∂_{m+1} on the weight of these paths leads to zero. Repeating the same argument, it follows that $L_m^{b_m}\,(X|Y)_a$ equals the sum of weights of all lattice paths (automatically nonintersecting) from A to

$$(1,a_1),\ (2,a_2-1),\ (m-1,a_{m-1}-m+2),\ \cdots,\ (b_m,\lambda_m-m+1). \tag{3}$$

We continue with the action of ∂_{m-1} on the weight of a set of nonintersecting lattice path from A to the destination points (3), and we may still apply Lemma 4.4. Iterating the same argument, we get the desired lattice path interpretation of $L_b(X|Y)_a$. ∎

Setting $Y=0$ in Theorem 4.2, we arrive at the following corollary for the ordinary flagged Schur function.

Corollary 4.5 *Given a flag $b=(b_1,b_2,\ldots,b_m)$ and a partition λ with m parts, let $a_i=\lambda_i+b_i-i$. Then $L_{b_1,b_2,\cdots,b_m}(x_1^{a_1}x_2^{a_2}\cdots x_m^{a_m})$ equals the sum of weights of all nonintersecting paths $P_1,P_2,\cdots,P_m$, such that P_i has origin $(i,-i+1)$ and destination (b_i,λ_i-i+1).*

From this corollary, we obtain the following determinantal formula:

$$\det\Big(h_{\lambda_i-i+j}(X_{b_i}\setminus X_{b_{j-1}})\Big)_{m\times m}.$$

By Lemma 3.1, we may rewrite the above formula as follows:

$$\det\Big(h_{\lambda_i-i+j}(X_{b_i})\Big)_{m\times m},$$

which coincides with the definition of the flagged Schur function $S_\lambda(b)$ given by Wachs [41].

5. Flagged Determinantal Formulas for Sympletic and Orthogonal Characters

Compared with previous lattice path approaches to the double Schur function by Goulden-Greene [17], Krattenthaler [21] and Molev [31], the construction given in the present paper easily leads to the flagged determinantal formula. Moreover, without additional effort these paths can also be translated into a tableau representation. We find another application of this idea to the symplectic and orthogonal characters $sp(2n)$ and $so(2n+1)$ by giving new flagged determinantal formulas for these two kinds of characters. They have been studied via various approaches, see, for example, [11, 39]. Fulmek and Krattenthaler [11] give a proof for the determinant expression

$$sp_{2n}(\lambda,X) = \det\left(h_{\lambda_i-j+1}(X) \vdots h_{\lambda_j-j+i}(X)+h_{\lambda_j-j-i+2}(X)\right)_{r\times r}$$

where $h_n(X) = h_n(x_1,x_1^{-1},x_2,x_2^{-1},\cdots,x_n,x_n^{-1})$ is the ordinary complete symmetric function, and the first expression gives the entries of the first row and the second for the remaining rows.

5.1. The Symplectic Characters

The symplectic characters $sp_{2n}(\lambda,X)$ can also be expressed using $2n$-symplectic tableau introduced by King and El-Sharkaway in [20].

Definition 5.1 *A semi-standard tableau T of shape λ is called a $2n$-symplectic tableau if its entries are elements of $\{1,2,\cdots,2n\}$ and they obey the additional constraint*

$$T_{i,j} \geq 2i-1.$$

Let $\mathcal{S}_{(\lambda)}$ be the set of $2n$-symplectic tableau, then

$$sp_{2n}(\lambda,X) = \sum_T X^T,$$

where $T \in \mathcal{S}_{(\lambda)}$, $X^T = \prod_{l=1}^n x_l^{|\{T_{i,j}=2l-1\}|-|\{T_{i,j}=2l\}|}$.

Lemma 5.2 *Given a partition $\lambda = (\lambda_1,\cdots,\lambda_r)$, $r \leq n$, there is a bijection between $\mathcal{S}_{(\lambda)}$ and the set of nonintersecting lattice paths $(P_1,P_2,\ldots,P_r)$ such that P_i is from $A_i = (2i-1,-i+1)$ to $B_i = (2n,\lambda_i-i+1)$ for $1 \leq i \leq r$.*

Proof. For a path P_i, let us consider $T_{i,j}$. We can draw a vertical line from $(T_{i,j}, j-i)$ to $(T_{i,j}, j-i+1)$ and complete the path from $(2i-1,-i+1)$ to $(2n,\lambda_i-i+1)$ by adding horizontal lines.

Because each step can be reversed, we obtain a bijection between $\mathcal{S}_{(\lambda)}$ and the nonintersecting lattice paths. ∎

For example, if $\lambda = (4,3,2)$ and $n = 3$, the following Figure 3 is a symplectic tableau of shape λ.

The corresponding lattice paths are shown in Figure 4.

We can now define the weight of a path to ensure that we can compute $sp_{2n}(\lambda,X)$ by lattice paths:

$$T = \begin{array}{|c|c|c|c|}\hline 2 & 3 & 4 & 6 \\ \hline 4 & 4 & 6 \\ \cline{1-3} 5 & 6 \\ \cline{1-2}\end{array}$$

Figure 3: A Symplectic tableau.

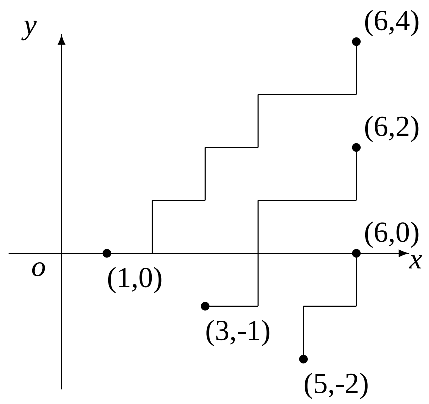

Figure 4: Corresponding lattice paths.

Definition 5.3 *For each path, the weight of each step is given by the following rules*:

1. *for the step from* $(2i-1, j)$ *to* $(2i-1, j+1)$, *the weight is* x_i;
2. *for the step from* $(2i, j)$ *to* $(2i, j+1)$, *the weight is* x_i^{-1};
3. *for the step from* (i, j) *to* $(i+1, j)$, *the weight is* 1.

Thus, the above weight assignment does lead to a lattice path interpretation of $sp_{2n}(\lambda, X)$. Now let us consider the paths from A_i to B_j, as in the above lemma, i.e., $A_i = (2i-1, -i+1)$ and $B_j = (2n, \lambda_j - j + 1)$. If $\lambda_j - j + 1 < -i + 1$, then set the weight to 0; otherwise, the sum over all such weighted paths becomes

$$h_{\lambda_j+i-j}(x_i, x_i^{-1}, x_{i+1}, x_{i+1}^{-1}, \cdots, x_n, x_n^{-1}).$$

Then we have the following theorem:

Theorem 5.4 *We have the following formula for the symplectic characters*:

$$sp_{2n}(\lambda, X) = \det(h_{\lambda_j+i-j}(x_i, x_i^{-1}, x_{i+1}, x_{i+1}^{-1}, \cdots, x_n, x_n^{-1}))_{r\times r}.$$

5.2. The Odd Orthogonal Characters

The charaters $so_{2n+1}(\lambda, X)$ can be interpreted in terms of a set of orthogonal tableau of shape λ, as denoted by $O_{(\lambda)}$ and introduced by Sundaram [39]. The Proctor tableaux [35] also leads to the same character as the Sundaram tableaux, and a weight preserving bijection of these two classes of tableaux is established by Fulmek and Krattenthaler [8]. Let us recall the definition of the Sundaram tableaux.

Definition 5.5 *A semistandard tableau* T *of shape* λ, *with* $l(\lambda) \le n$, *is called a* $so(2n+1)$ *tableau if its entries are elements of*

$$1 < 2 < 3 < \ldots < 2n-1 < 2n < \infty,$$

and obey the additional constaints:

1. $T_{i,j} \ge 2i-1$,
2. *for each row, there is at most one* ∞.

Let $O_{(\lambda)}$ *be the set of such tableau, then*

$$so_{2n+1}(\lambda, X) = \sum_T X^T,$$

where $T \in O_{(\lambda)}$, $X^T = \sum_{l=1}^n x_l^{|T_{i,j}=2l-1|-|T_{i,j}=2l|}$.

The following lemma gives a lattice path representation of odd orthogonal tableaux.

Lemma 5.6 *Given a partition $\lambda = (\lambda_1, \cdots, \lambda_r)$, $r \le n$, there is a bijection between $O_{(\lambda)}$ and the set of nonintersecting lattice paths $(P_1, P_2, \ldots, P_r)$ where P_i has origin $A_i = (2i-1, -i+1)$ and destination $B_i = (2n+i, \lambda_i - i+1)$ such that for the region $x > 2n$ there is at most one vertical step on P_i which is possibly from $(2n+i, \lambda_i - i)$ to $(2n+i, \lambda_i - i+1)$.*

Proof. For each path P_i, let us consider the cells (i, j) such that $T_{i,j} < \infty$. We can draw a vertical line from $(T_{i,j}, j-i)$ to $(T_{i,j}, j-i+1)$ in the lattice path. If $T_{i,j} = \infty$, in which case $j = \lambda_i$ by definition, then we can draw a vertical line from $(2n+i, \lambda_i - i)$ to $(2n+i, \lambda_i - i+1)$. After the vertical lines are drawn, the path P_i can be completed by adding horizontal lines. It is easy to see that this construction is in fact a bijection. ∎

For a lattice path corresponding to an odd orthogonal tableau, its weight is given below:

1. for the step from $(2i-1, j)$ to $(2i-1, j+1)$, the weight is x_i, if $i \le n$;
2. for the step from $(2i, j)$ to $(2i, j+1)$, the weight is x_i^{-1}, if $i \le n$;
3. for the step from $(2n+i, \lambda_i - i)$ to $(2n+i, \lambda_i - i+1)$, the weight is 1;
4. for the step from (i, j) to $(i+1, j)$, the weight is 1.

As for the case of symplectic characters, the above weight assignment yields a lattice path interpretation of $so_{2n+1}(\lambda, X)$. The set of lattice paths from $A_i = (2i-1, -i+1)$ to $B_j = (2n+j, \lambda_j - j+1)$ subject to the conditions in Lemma 5.6 gives the following function for $\lambda_j - j + 1 \ge -i+1$:

$$h_{\lambda_j+i-j}(x_i, x_i^{-1}, x_{i+1}, x_{i+1}^{-1}, \ldots, x_n, x_n^{-1}) + h_{\lambda_j+i-j-1}(x_i, x_i^{-1}, x_{i+1}, x_{i+1}^{-1}, \ldots, x_n, x_n^{-1}). \tag{1}$$

Then we have the following theorem:

Theorem 5.7 *The orthogonal character $so_{2n+1}(\lambda, X)$ can be evaluated by the following determinant*:

$$\det\left(h_{\lambda_j+i-j}(x_i, x_i^{-1}, x_{i+1}, x_{i+1}^{-1}, \ldots, x_n, x_n^{-1}) + h_{\lambda_j+i-j-1}(x_i, x_i^{-1}, x_{i+1}, x_{i+1}^{-1}, \ldots, x_n, x_n^{-1})\right)_{r\times r}.$$

Acknowledgments. This work was performed under the US Department of Energy, the National Science Foundation and the MCSEC of China. We thank A. Lascoux for his lectures on Schubert polynomials given at the Nankai University in June 1996, and for many inspiring discussions and generous suggestions. We also thank I.P. Goulden, C. Greene, B. Sagan, and M. Wachs for helpful comments.

References

1. N. Bergeron and S. Billey, RC-graphs and Schubert polynomials, Experiment. Math. **2** (1993) 257-269.
2. L.C. Biedenharn and J.D. Louck, A new class of symmetric polynomials defined in terms of tableaux, Adv. Appl. Math. **9** (1988) 447-464.

3. L.C. Biedenharn and J.D. Louck, Inhomogeneous basis set of symmetric polynomials defined by tableaux, Proc. Nat. Acad. Sci. U.S.A. **87** (1990) 1441-1445.
4. L.C. Biedenharn, A.A. Bincer, M.A. Lohe, and J.D. Louck, New relations and identities for generalized hypergeometric coefficients, Adv. Appl. Math. **13** (1992) 62-121.
5. S.C. Billey, Kostant polynomials and the cohomology ring for G/B, Proc. Natl. Acad. Sci. U.S.A. **94** (1997) 29-32..
6. F. Brenti, Determinants of super Schur functions, lattice paths, and dotted plane partitions, Adv. Math. **98** (1993) 27-64.
7. W.Y.C. Chen and J.D. Louck, The factorial Schur function, J. Math. Phys. **34**(9) (1993) 4144-4160.
8. S. Fomin and A.N. Kirillov, The Yang-Baxter equation, symmetric functions, and Schubert polynomials, Discrete Math. 153 (1996) 123-143.
9. S. Fomin and R.P. Stanley, Schubert polynomials and the nil-Coxeter algebra, Adv. Math. **103** (1994) 196-207.
10. M. Fulmek and C. Krattenthaler, A bijection between Proctor's and Sundaram's odd orthogonal tableaux, Discrete Math. **161** (1996) 101-120.
11. M. Fulmek and C. Krattenthaler, Lattice path proofs for determinantal formulas for symplectic and orthogonal characters, J. Combin. Theory, Series A **77** (1997) 3-50.
12. W. Fulton, Flags, Schubert polynomials, degeneracy loci, and determinantal formulas, Duke Math. J. **55** (1992) 381-420.
13. W. Fulton, Determinantal formulas for orthogonal and symplectic degeneracy loci, J. Differential Geom. **43** (1996) 276-290.
14. W. Fulton and J. Harris, Representation Theory, Springer-Verlag, New York, 1991.
15. I. Gessel and G. Viennot, Binomial determinants, paths, and hook length formulae, Adv. Math. **58** (1985) 300-321.
16. I. Gessel and G. Viennot, Determinants, paths, and plane partitions, preprint.
17. I. Goulden and C. Greene, A new tableau representation for supersymmetric Schur function, J. Algebra **170** (1994) 687-703.
18. A.H. Hamel and I.P. Goulden, Lattice paths and a sergeev-pragacz formula for skew supersymmetric functions, Can. J. Math. Vol. **47** (2) (1995) 364-382.
19. A.H. Hamel and I.P. Goulden, Planar decompositions of tableaux and Schur function determinants, Europ. J. Combin. **16** (1995) 461-477.
20. R.C. King and N.G.I. El-Sharkaway, Standard Young tableaux and weight multiplicities of the classical Lie groups, J. Phys. A: Math. Gen. **16** (1983) 3153-3177.
21. C. Krattenthaler, A bijective proof of the hook-content formula for super Schur functions and a modified jeu de Taquin, Electronic J. Combin. **3** (2) (1996) #R14.
22. A. Lascoux, Puissances extérieures, déterminants et cycles de Schubert, Bull. Soc. Math. France **102** (1974) 161-179.
23. A. Lascoux, Classes de Chern des Variétés de drapeaux, C. R. Acad. Sca. Paris Sér. I Math. **295** (1982) No. 5 393-398.
24. A. Lascoux, Interpolation, unpublished notes.
25. A. Lascoux and M. Schützenberger, Géométrie algébrique-Polynómes de Schubert, C. R. Acad. Sci. Paris **294** (1982) 447-450.
26. I.G. Macdonald, Notes on Schubert polynomials, Publication du LACIM, Université du Québec à Montréal, No. 6, 1991.

27. I.G. Macdonald, Schur functions: theme and variations, Publ. I.R.M.A. Strasbourg, 1992, 498/S-27, Actes 28^e Seminaire Lotharingien, p. 5-39.

28. I.G. Macdonald, Symmetric Functions and Hall Polynomial, Oxford Univ. Press, New York/London, 1979.

29. M.A. Méndez, The umbral calculus of symmetric functions, Adv. Math.**124** (1996) 207-271.

30. M.A. Méndez, Umbral shifts and symmetric functions of Schur type, Mathematical Essays in Honor of Gian-Carlo Rota, B.E. Sagan and R.P. Stanley, Eds., Birkhäuser, Boston, 1998, pp. 285-303.

31. A. Molev, Factorial supersymmetric Schur functions and super Capelli identities, Kirillov's Seminar on Representation Theory, G. I. Olshanski, Eds., Amer. Math. Soc. Iransl., AMS, Providence, R. I., Ser. 2, Vol. 181, 1998, pp. 109-137

32. A. Molev and B.E. Sagan, A Littlewood-Richardson rule for factorial Schur functions, Trans. Amer. Math. Soc. **351** (1999) 4429-4443

33. A. Okounkov, Quantum immanants and higher Capelli identities, Transformation Groups **1** (1996) 99-126.

34. P. Pragacz and A. Thorup, On a Jacobi-Trudi identity for supersymmetric polynomials, Adv. Math. **95** (1992) 8-17.

35. R.A. Proctor, A Schensted algorithm with models tensor representations of the orthogonal group, Can. J. Math. **42** (1990) 28-49.

36. F. Sottile, Pieri's formula for flag manifolds and Schubert polynomials, Ann. Inst. Fourier **46** (1996) 89-110.

37. R. P. Stanley, Theory and application of plane partitions, I, Studies in Appl. Math. **50** (1971) 167-188.

38. R.P. Stanley, On the number of reduced decompositions of elements of Coxter groups, Eur. J. Combin. **5** (1984) 359-372.

39. S. Sundaram, Orthogonal tableaux and an insertion algorithm for $so(2n+1)$, J. Combin. Theory Ser. A **53** (1990) 239-256.

40. S. Veigneau, SP, A package for Schubert polynomials realized with the computer algebra system Maple, J. Symbolic Computation 23 (1997) 413-425.

41. M.L. Wachs, Flagged Schur functions, Schubert polynomials, and symmetrizing operators, J. Combin. Theory, Ser. A **40** (1985) 276-289.