# Proof of Moll's Minimum Conjecture

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**Abstract.** Let $d_i(m)$ denote the coefficients of the Boros-Moll polynomials. Moll's minimum conjecture states that the sequence $\{i(i+1)(d_i^2(m)-d_{i-1}(m)d_{i+1}(m))\}_{1\le i\le m}$ attains its minimum at $i=m$ with $2^{-2m}m(m+1)\binom{2m}{m}^2$. This conjecture is stronger than the log-concavity conjecture proved by Kauers and Paule. We give a proof of Moll's conjecture by utilizing the spiral property of the sequence $\{d_i(m)\}_{0\le i\le m}$, and the log-concavity of the sequence $\{i!d_i(m)\}_{0\le i\le m}$.

**Keywords:** ratio monotonicity, log-concavity, Boros-Moll polynomials.

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## 1 Introduction

The objective of this note is to give a proof of Moll's conjecture on the minimum value of a sequence involving the coefficients of the Boros-Moll polynomials which arise in the evaluation of the following quartic integral, see, [1–6, 11]. It has been shown that for any $a>-1$ and any nonnegative integer $m$,

$$\int_0^\infty \frac{1}{(x^4+2ax^2+1)^{m+1}}dx = \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}}P_m(a),$$

where

$$P_m(a) = 2^{-2m}\sum_k 2^k\binom{2m-2k}{m-k}\binom{m+k}{k}(a+1)^k. \tag{1.1}$$

Write $P_m(a)$ as

$$P_m(a) = \sum_{i=0}^m d_i(m)a^i. \tag{1.2}$$

The polynomials $P_m(a)$ are called the Boros-Moll polynomials. By (1.2), $d_i(m)$ can be expressed as

$$d_i(m) = 2^{-2m}\sum_{k=i}^m 2^k\binom{2m-2k}{m-k}\binom{m+k}{k}\binom{k}{i}. \tag{1.3}$$

From the above formula (1.3) one sees that the coefficients $d_i(m)$ are positive. Boros and Moll [3,4] have proved that for $m \geq 2$ the sequence $\{d_i(m)\}_{0\leq i\leq m}$ is unimodal and the maximum entry appears in the middle, that is,

$$d_0(m) < d_1(m) < \cdots < d_{[\frac{m}{2}]-1}(m) < d_{[\frac{m}{2}]}(m) > d_{[\frac{m}{2}]+1}(m) > \cdots > d_m(m).$$

Moll [11] conjectured that the sequence $\{d_i(m)\}_{0\leq i\leq m}$ is log-concave for $m \geq 2$. Kauers and Paule [9] have proved this conjecture by using a computer algebra approach. Chen and Xia [8] have shown that the sequence $\{d_i(m)\}_{0\leq i\leq m}$ satisfies the strongly ratio monotone property which implies the log-concavity and the spiral property. Chen and Gu [7] have proved that the sequence $\{d_i(m)\}_{0\leq i\leq m}$ satisfies the reverse ultra log-concavity. They have also proved that the sequence $\{i!d_i(m)\}_{0\leq i\leq m}$ is log-concave.

In fact, Moll [10,12] proposed a stronger conjecture than the log-concavity conjecture. He formulated his conjecture in terms of the numbers $b_i(m)$ as defined by

$$b_i(m) = \sum_{k=i}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{k}\binom{k}{i}. \tag{1.4}$$

Clearly, $b_i(m) = 2^{2m} d_i(m)$ and the log-concavity of $d_i(m)$ is equivalent to that of $b_i(m)$.

**Conjecture 1.1.** *Given $m \geq 2$, for $1 \leq i \leq m$,*

$$(m+i)(m+1-i)b_{i-1}^2(m) + i(i+1)b_i^2(m) - i(2m+1)b_{i-1}(m)b_i(m),$$

*attains its minimum at $i = m$ with $2^{2m}m(m+1)\binom{2m}{m}^2$.*

We will give a proof of the above conjecture by using the spiral property of $\{d_i(m)\}_{0\leq i\leq m}$ and the log-concavity of $\{i!d_i(m)\}_{0\leq i\leq m}$.

# 2 Proof of Moll's Minimum Conjecture

As pointed out by Moll [12], his conjecture implies that $\{d_i(m)\}_{0\leq i\leq m}$ is log-concave for $m \geq 2$. To see this, we may employ a recurrence relation to reformulate his conjecture by using the three terms $d_{i-1}(m)$, $d_i(m)$ and $d_{i+1}(m)$. Recall that Kauers and Paule [9] and Moll [12] have independently derived the following recurrence relation for $1 \leq i \leq m$,

$$i(i-1)d_i(m) = (i-1)(2m+1)d_{i-1}(m) - (m+2-i)(m+i-1)d_{i-2}(m). \tag{2.1}$$

Note that we have adopted the convention that $d_i(m) = 0$ for $i < 0$ or $i > m$. From (2.1) and the relation $d_i(m) = 2^{-2m}b_i(m)$, it follows that

$$\begin{aligned}(m+i)(m+1-i)b_{i-1}^2(m) + i(i+1)b_i^2(m) - i(2m+1)b_{i-1}(m)b_i(m)\\ = i(i+1)\left(b_i^2(m) - b_{i+1}(m)b_{i-1}(m)\right).\end{aligned}$$

Thus, Moll's conjecture can be restated as follows.

**Theorem 2.1.** *Given $m \geq 2$, for $1 \leq i \leq m$, $i(i+1)\left(d_i^2(m) - d_{i+1}(m)d_{i-1}(m)\right)$ attains its minimum at $i = m$ with $2^{-2m}m(m+1)\binom{2m}{m}^2$.*

Chen and Xia [8] have shown that the Boros-Moll polynomials satisfy the ratio monotone property which implies the log-concavity and the spiral property.

**Theorem 2.2.** *Let $m \geq 2$ be an integer. The sequence $\{d_i(m)\}_{0 \leq i \leq m}$ is strictly ratio monotone, that is,*

$$\frac{d_m(m)}{d_0(m)} < \frac{d_{m-1}(m)}{d_1(m)} < \cdots < \frac{d_{m-i}(m)}{d_i(m)} < \frac{d_{m-i-1}(m)}{d_{i+1}(m)} < \cdots < \frac{d_{m-\left[\frac{m-1}{2}\right]}(m)}{d_{\left[\frac{m-1}{2}\right]}(m)} < 1,$$

$$\frac{d_0(m)}{d_{m-1}(m)} < \frac{d_1(m)}{d_{m-2}(m)} < \cdots < \frac{d_{i-1}(m)}{d_{m-i}(m)} < \frac{d_i(m)}{d_{m-i-1}(m)} < \cdots < \frac{d_{\left[\frac{m}{2}\right]-1}(m)}{d_{m-\left[\frac{m}{2}\right]}(m)} < 1.$$

As a consequence of Theorem 2.2, the spiral property of $\{d_i(m)\}_{0 \leq i \leq m}$ can be stated as follows.

**Corollary 2.3.** (Chen and Xia [8]) *For $m \geq 2$, the sequence $\{d_i(m)\}_{0 \leq i \leq m}$ is spiral, that is,*

$$d_m(m) < d_0(m) < d_{m-1}(m) < d_1(m) < d_{m-2}(m) < \cdots < d_{\left[\frac{m}{2}\right]}(m). \tag{2.2}$$

Chen and Gu [7] have shown that $\{i!d_i(m)\}_{0 \leq i \leq m}$ is log-concave. This property can be recast in the following form.

**Theorem 2.4.** *For $m \geq 2$ and $1 \leq i \leq m-1$,*

$$id_i^2(m) > (i+1)d_{i+1}(m)d_{i-1}(m). \tag{2.3}$$

We are now ready to present a proof of Theorem 2.1.

*Proof.* First, it follows from (1.3) that

$$m(m+1)d_m^2(m) = 2^{-2m}m(m+1)\binom{2m}{m}^2. \tag{2.4}$$

We now proceed to show that for $1 \leq i \leq m-1$,

$$i(i+1)\left(d_i^2(m) - d_{i+1}(m)d_{i-1}(m)\right) > m(m+1)d_m^2(m). \tag{2.5}$$

We first consider the case $1 \leq i \leq m-2$. By (2.3), we find that

$$i(i+1)\left(d_i^2(m) - d_{i+1}(m)d_{i-1}(m)\right) > i(i+1)d_i^2(m) - i^2d_i^2(m) = id_i^2(m). \tag{2.6}$$

Using the spiral property (2.2), we see that for $1 \leq i \leq m-2$,

$$i d_i^2(m) \geq d_1^2(m) > d_{m-1}^2(m). \tag{2.7}$$

Combining (2.6) and (2.7), we get

$$i(i+1)\left(d_i^2(m) - d_{i+1}(m)d_{i-1}(m)\right) > d_{m-1}^2(m). \tag{2.8}$$

On the other hand, by direct computation we may deduce from (1.3) that

$$d_{m-1}(m) = \frac{2m+1}{2} d_m(m). \tag{2.9}$$

By (2.8) and (2.9), we have for $1 \leq i \leq m-2$,

$$\begin{aligned} & i(i+1)\left(d_i^2(m) - d_{i+1}(m)d_{i-1}(m)\right) \\ & \qquad > \left(\frac{2m+1}{2}\right)^2 d_m^2(m) > m(m+1)d_m^2(m), \end{aligned} \tag{2.10}$$

and hence (2.5) is true for $1 \leq i \leq m-2$. It remains to consider the case $i = m-1$. Again, by (1.3) we find that

$$d_{m-1}(m) = 2^{-m-1}(2m+1)\binom{2m}{m}, \tag{2.11}$$

$$d_{m-2}(m) = 2^{-m-2}\frac{(m-1)(4m^2+2m+1)}{2m-1}\binom{2m}{m}. \tag{2.12}$$

From (2.4), (2.11) and (2.12), we deduce that

$$\begin{aligned} & m(m-1)\left(d_{m-1}^2(m) - d_m(m)d_{m-2}(m)\right) \\ & \quad = m(m-1)2^{-2m}\binom{2m}{m}^2\left(\frac{(2m+1)^2}{4} - \frac{(m-1)(4m^2+2m+1)}{4(2m-1)}\right) \\ & \quad = \frac{m(4m^2+6m-1)}{4(2m-1)} m(m-1)2^{-2m}\binom{2m}{m}^2 \\ & \quad > m(m+1)2^{-2m}\binom{2m}{m}^2 = m(m+1)d_m^2(m). \end{aligned} \tag{2.13}$$

Thus (2.5) holds for $i = m-1$, and so it holds for $1 \leq i \leq m-1$. This completes the proof. ∎

We conclude with the following ratio monotonicity conjecture. If it is true, it would imply that the sequence $\{i(i+1)(d_i^2(m) - d_{i+1}(m)d_{i-1}(m))\}_{1 \leq i \leq m}$ is both spiral and log-concave for $m \geq 2$.

**Conjecture 2.5.** *The sequence* $\{i(i+1)\left(d_i^2(m) - d_{i+1}(m)d_{i-1}(m)\right)\}_{1\le i\le m}$ *is strongly ratio monotone.*

For example, for $m = 8$, we have

$$\begin{aligned}P_8(a) =& \frac{4023459}{32768} + \frac{3283533}{4096}a + \frac{9804465}{4096}a^2 + \frac{8625375}{2048}a^3 + \frac{9695565}{2048}a^4 \\ &+ \frac{1772199}{512}a^5 + \frac{819819}{512}a^6 + \frac{109395}{256}a^7 + \frac{6435}{128}a^8.\end{aligned}$$

Let $c_i = i(i+1)\left(d_i^2(8) - d_{i+1}(8)d_{i-1}(8)\right)$ for $1 \le i \le 8$. One can verify that

$$\frac{c_8}{c_1} < \frac{c_7}{c_2} < \frac{c_6}{c_3} < \frac{c_5}{c_4} < 1 \qquad \text{and} \qquad \frac{c_1}{c_7} < \frac{c_2}{c_6} < \frac{c_3}{c_5} < 1.$$

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