On the Gosper-Petkovšek Representation of Rational Functions

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Abstract

We show that the uniqueness of the Gosper-Petkovšek representation of rational functions can be utilized to give a simpler version of Gosper's algorithm. This approach also applies to Petkovšek's generalization of Gosper's algorithm, and its q-analogues by Abramov-Paule-Petkovšek and Böing-Koepf.

Keywords: Gosper's algorithm, GP representation, q-Gosper's algorithm, q-GP representation, hypergeometric term

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1 Introduction

Gosper's algorithm has been extensively studied and widely used to verify hypergeometric identities, see, for example, [4, 5, 6, 8, 10, 12, 13]. A key idea behind Gosper's algorithm lies in the representation of a rational function which is called the Gosper representation. As usual, we assume that subject to normalization the gcd (greatest common divisor) of two polynomials always takes a value as a monic polynomial, namely, a polynomial with the leading coefficient being 1. Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{K}$ be a field of characteristic zero, $\mathbb{K}(n)$ be the field of rational functions of n over $\mathbb{K}$, and $\mathbb{K}[n]$ be the ring of polynomials of n over $\mathbb{K}$. Recall that a nonzero term t_n is called a hypergeometric term over $\mathbb{K}$ if there exists a rational function $r \in \mathbb{K}(n)$ such that

$$\frac{t_{n+1}}{t_n} = r(n).$$

Gosper showed that any rational function $r(n)$ can be written in the following form, called the Gosper representation:

$$r(n) = \frac{a(n)}{b(n)} \frac{c(n+1)}{c(n)},$$

where a, b and c are polynomials over $\mathbb{K}$ and

$$\gcd(a(n), b(n+h)) = 1, \quad \text{for all } h \in \mathbb{N}.$$

Petkovšek [11] realized that the Gosper representation becomes unique, which is called the Gosper-Petkovšek representation, or GP representation, for short, if we further require that b, c are monic polynomials such that

$$\begin{aligned} \gcd(a(n), c(n)) &= 1, \\ \gcd(b(n), c(n+1)) &= 1. \end{aligned}$$

Given a hypergeometric term t_n, Gosper's algorithm is a procedure to find a hypergeometric term z_n satisfying

$$z_{n+1} - z_n = t_n, \tag{1.1}$$

if it exists, or claiming the nonexistence of any solution of (1.1). In another paper [10], Paule and Strehl give derivation of Gosper's algorithm by using the GP representation.

The main result of this paper is the observation that if we express both t_{n+1}/t_n and z_{n+1}/z_n in terms of their GP representations, then the solution of (1.1) reduces to a polynomial difference equation. Hence the mystery of Gosper's algorithm disappears from the perspective of the uniqueness of the GP representation. We also show that this idea can be applied to Pekovšek's generalization of Gosper's algorithm to find hypergeometric solutions of linear recurrences with the additional restriction that the leading and trailing coefficients are constants [12], and to the q-analogues by Abramov-Paule-Petkovšek [1], and later by Böing-Koepf [2].

2 The GP Representation

In this section, we present a simplified version of the Gosper algorithm which is similar to the version of Paule and Strehl [10]. However, it seems to have been neglected that the GP representation of the unknown rational function in Gosper's algorithm plays a key role in reducing Gosper's equation to a polynomial difference equation. Given a hypergeometric term t_n, we suppose that there exists a hypergeometric term z_n satisfying the equation (1.1). Let

$$r(n) = \frac{t_{n+1}}{t_n}, \quad R(n) = \frac{z_{n+1}}{z_n}. \tag{2.1}$$

It follows that $r(n)$ and $R(n)$ are rational functions of n. By using (1.1), we find

$$\frac{z_n}{t_n} = \frac{z_n}{z_{n+1} - z_n} = \frac{1}{z_{n+1}/z_n - 1} = \frac{1}{R(n) - 1}. \tag{2.2}$$

From (2.2) it follows that

$$r(n) = \frac{z_{n+1}}{z_n} \frac{z_n/t_n}{z_{n+1}/t_{n+1}} = R(n)\frac{(R(n+1)-1)}{(R(n)-1)}. \tag{2.3}$$

Let us express both $r(n)$ and $R(n)$ in terms of their GP representations:

$$r(n) = \frac{a(n)}{b(n)} \frac{c(n+1)}{c(n)}, \ R(n) = \frac{A(n)}{B(n)} \frac{C(n+1)}{C(n)}. \tag{2.4}$$

Then the following theorem can be easily verified:

Theorem 2.1. *Suppose that $R(n)$ has the above* GP *representation. Then the rational function on the right hand side of* (2.3) *has the following* GP *representation*

$$\frac{A(n)}{B(n+1)} \frac{A(n+1)C(n+2) - B(n+1)C(n+1)}{A(n)C(n+1) - B(n)C(n)}. \tag{2.5}$$

By the uniqueness of the GP representation, we may compare the GP representation of $r(n)$ as given in (2.4) and the GP representation in (2.5) to obtain the following equations: $a(n) = A(n), \ b(n) = B(n+1)$, and

$$A(n)C(n+1) - B(n)C(n) = c(n).$$

It follows that

$$a(n)C(n+1) - b(n-1)C(n) = c(n). \tag{2.6}$$

Therefore, hypergeometric solutions of (1.1) are determined by polynomial solutions of (2.6) as given by the following relation:

$$z_n = \frac{b(n-1)C(n)}{c(n)} \, t_n. \tag{2.7}$$

Let us take an example from [13]:

Example 2.1. *Let $t_n = (4n+1) \cdot \dfrac{n!}{(2n+1)!}$, then*

$$r(n) = \frac{t_{n+1}}{t_n} = \frac{1/4}{(n+3/2)} \frac{(n+5/4)}{(n+1/4)}.$$

Hence $a(n) = 1/4$, $b(n) = n + 3/2$, $c(n) = n + 1/4$. From equation (2.6) *we find*

$$\frac{1}{4} \cdot C(n+1) - \left(n + \frac{1}{2}\right) \cdot C(n) = n + \frac{1}{4}.$$

The constant polynomial $C(n) = -1$ is a solution to this equation. By (2.7)*, we have $z_n = -2 \cdot \dfrac{n!}{(2n)!}$.* □

3 Gosper's Algorithm for Recurrences of Arbitrary Order

In this section, we show that the above approach to Gosper's algorithm can be generalized to find hypergeometric solutions z_n of the recurrence

$$\sum_{k=0}^{d} p_k(n) z_{n+k} = t_n, \tag{3.1}$$

where t_n is a given hypergeometric term and $p_0(n)$, $p_1(n)$, …, $p_d(n)$ are polynomials with the additional constraints that $p_0(n)$ and $p_d(n)$ are constants. Suppose that there exists a hypergeometric solution z_n of (3.1). Let $r(n)$ and $R(n)$ be given as in (2.1). From (3.1) we obtain

$$\frac{z_n}{t_n} = \frac{z_n}{\sum_{k=0}^{d} p_k(n) \cdot z_{n+k}} = \frac{1}{\sum_{k=0}^{d} p_k(n) \cdot \frac{z_{n+k}}{z_n}} = \frac{1}{\sum_{k=0}^{d} p_k(n) \cdot \prod_{j=0}^{k-1} R(n+j)},$$

and

$$r(n) = R(n) \frac{\sum_{k=0}^{d} p_k(n+1) \cdot \prod_{j=0}^{k-1} R(n+j+1)}{\sum_{k=0}^{d} p_k(n) \cdot \prod_{j=0}^{k-1} R(n+j)}. \tag{3.2}$$

Let us express both $r(n)$ and $R(n)$ in terms of their GP representations as in (2.4). Since $p_0(n)$ and $p_d(n)$ are constants, by the definition of the GP representation, we have

$$\gcd\left(A(n), p_0(n) \cdot C(n) \cdot \prod_{j=0}^{d-1} B(n+j)\right) = 1,$$

and

$$\gcd\left(B(n+d), p_d(n+1) \cdot C(n+d+1) \cdot \prod_{j=0}^{d-1} A(n+j+1)\right) = 1.$$

Therefore, we have the following theorem. The proof is a straightforward verification, hence omitted.

Theorem 3.1. *Suppose that $R(n)$ has a* GP *representation as in* (2.4)*. Then the rational function on the right hand side of* (3.2) *has the following* GP *representation*

$$\frac{A(n)}{B(n+d)} \frac{\sum_{k=0}^{d} p_k(n+1) \cdot C(n+k+1) \cdot \prod_{j=0}^{k-1} A(n+j+1) \cdot \prod_{j=k}^{d-1} B(n+j+1)}{\sum_{k=0}^{d} p_k(n) \cdot C(n+k) \cdot \prod_{j=0}^{k-1} A(n+j) \cdot \prod_{j=k}^{d-1} B(n+j)}. \tag{3.3}$$

By the uniqueness of the GP representation, we may compare the GP representation of $r(n)$ as given in (2.4) and the GP representation in (3.3) to obtain the following equations: $a(n) = A(n),\ b(n) = B(n + d)$, and

$$\sum_{k=0}^{d} p_k(n) \cdot C(n+k) \cdot \prod_{j=0}^{k-1} A(n+j) \cdot \prod_{j=k}^{d-1} B(n+j) = c(n).$$

It follows that

$$\sum_{k=0}^{d} p_k(n) \cdot C(n+k) \cdot \prod_{j=0}^{k-1} a(n+j) \cdot \prod_{j=k}^{d-1} b(n+j-d) = c(n). \tag{3.4}$$

Therefore, hypergeometric solutions of (3.1) are determined by polynomial solutions of (3.4) as given by the following relation:

$$z_n = \frac{\prod_{j=1}^{d} b(n-j) \cdot C(n)}{c(n)}\, t_n. \tag{3.5}$$

4 The q-GP Representation

In this section, we consider the q-analogue of Gosper's algorithm, see Böing-Koepf [2], Koornwinder [7], Paule-Riese [9], Paule-Strehl [10]. Let $\mathbb{F}$ denote the transcendental extension of $\mathbb{K}$ by the indeterminate q, i.e., $\mathbb{F}$=$\mathbb{K}(q)$ and let $n = q^k$. A non-zero term f_k is called q-hypergeometric over $\mathbb{F}$ if there exists a rational function $\rho \in \mathbb{F}(n)$ such that

$$\frac{f_{k+1}}{f_k} = \rho(n).$$

If $f_{k+1}/f_k = A(n)/B(n)$, where $A,\ B \in \mathbb{F}[n]$, then the function A/B is called the rational representation of a q-hypergeometric term f_k. If $\gcd(A, B) = 1$, then A/B is called the reduced rational representation of f_k. In [9], Paule and Riese presented the q-GFF (q-greatest factorial factorization) of q-monic polynomials for finding q-hypergeometric solutions of the equation

$$g_{k+1} - g_k = f_k, \tag{4.1}$$

where f_k is a given q-hypergeometric term. In [1], Abramov, Paule, and Petkovšek gave the algorithm q`Hyper` for finding all q-hypergeometric solutions of homogeneous recurrences with polynomial coefficients. In [2], Böing-Koepf obtained an algorithm for the same purpose as the algorithm q`Hyper`.

We adopt the notation of basic hypergeometric series in [3]. The q-shifted factorial is given by

$$(a;q)_k = (1-a)\,(1-aq)\,\cdots\,(1-aq^{k-1}).$$

Any q-hypergeometric term can be written in the following form

$$f_k = c\frac{(a_1;q)_k(a_2;q)_k\cdots(a_r;q)_k}{(b_1;q)_k(b_2;q)_k\cdots(b_s;q)_k}\,\frac{z^k}{(q;q)_k}\,q^{\alpha\binom{k}{2}+\beta k},$$

where $r, s \geq 0$, $a_i, b_j \in \mathbb{F}$, $(1 \leq i \leq r,\ 1 \leq j \leq s)$, $c, z \in \mathbb{F}$ with $z(0) \neq 0$, and α, β are integers. Let $A(n)/B(n)$ be the rational representation of f_k. Thus, we have

$$\frac{f_{k+1}}{f_k} = \frac{A(n)}{B(n)} = \frac{(1-a_1n)(1-a_2n)\ldots(1-a_rn)}{(1-b_1n)(1-b_2n)\ldots(1-b_sn)(1-qn)}\,n^\alpha q^\beta z.$$

Recall that a polynomial $p \in \mathbb{F}[n]$ is said to be q-monic if $p(0) = 1$. Any non-zero rational function $\rho(n) = A(n)/B(n)$ with A, $B \in \mathbb{F}[n]$ can be written in the form

$$\rho(n) = \frac{A(n)}{B(n)} = \frac{A_1(n)}{B_1(n)}\,n^\alpha q^\beta z,$$

where A_1, $B_1 \in \mathbb{F}[n]$ are q-monic, α and β are integers, and z is a rational function in $\mathbb{F}$ with $z(0) \neq 0$. For a reduced rational function s, its numerator and denominator are denoted by $\mathrm{num}(s)$ and $\mathrm{den}(s)$, respectively. Let $\mu(n) = n^\alpha \in \mathbb{K}(n)$ and $\pi(q) = q^\beta \in \mathbb{F}$. Paule and Strehl [10] have shown that any rational function over $\mathbb{F}$ have the following q-Gosper-Petkovšek representation, or q-GP representation, for short. Note that the shift operator ϵ is defined by $\epsilon a(n) = a(qn)$ for a polynomial of n.

Theorem 4.1. *For any non-zero rational function $\rho \in \mathbb{F}(n)$, there exist unique q-monic polynomials $\tilde{a}$, $\tilde{b}$ and $\tilde{c} \in \mathbb{F}[n]$ such that*

$$\frac{A_1}{B_1} = \frac{\tilde{a}}{\tilde{b}}\frac{\epsilon\tilde{c}}{\tilde{c}},$$

$\gcd(\tilde{a}, \tilde{c}) = \gcd(\tilde{b}, \epsilon\tilde{c}) = 1$ *and* $\gcd(\tilde{a}, \epsilon^j\tilde{b}) = 1$ *for all $j \geq 0$, and*

$$\rho = \frac{a}{b}\frac{\epsilon c}{c},$$

where

$$\begin{aligned} a &= \tilde{a}z\ num(\mu(n))/den(\pi(q)),\\ b &= \tilde{b}\ den(\mu(n)),\\ c &= \tilde{c}\ num(\pi(n)). \end{aligned}$$

Given a q-hypergeometric term f_k, suppose that there exists a q-hypergeometric solutions g_k satisfying equation (4.1). Let

$$\rho(n) = f_{k+1}/f_k,\ \tau(n) = g_{k+1}/g_k, \tag{4.2}$$

It follows that $\rho(n)$ and $\tau(n)$ are rational functions of n. From (4.1) it follows that

$$\frac{g_k}{f_k} = \frac{1}{\tau(n)-1},$$

and

$$\rho(n) = \tau(n)\frac{(\tau(qn)-1)}{(\tau(n)-1)}. \tag{4.3}$$

Assume that $\rho(n)$ and $\tau(n)$ have the following q-GP representations:

$$\rho(n) = \frac{a(n)}{b(n)}\frac{c(qn)}{c(n)},\ \tau(n) = \frac{A(n)}{B(n)}\frac{C(qn)}{C(n)}. \tag{4.4}$$

Then we have the following q-GP representation.

Theorem 4.2. *Suppose that $\tau(n)$ has the above q-GP representation. Then the rational function on the right hand side of* (4.3) *has the following q-GP representation*:

$$\frac{A(n)}{B(qn)}\frac{A(qn)C(q^2n)-B(qn)C(qn)}{A(n)C(qn)-B(n)C(n)}. \tag{4.5}$$

By the uniqueness of the q-GP representation, we may compare the q-GP representation of $\rho(n)$ in (4.4) and the q-GP representation in (4.5) to obtain the following equations: $a(n) = A(n),\ b(n) = B(qn)$, and

$$A(n)C(qn) - B(n)C(n) = c(n).$$

It follows that

$$a(n)C(qn) - b(q^{-1}n)C(n) = c(n). \tag{4.6}$$

Therefore, q-hypergeometric solutions of (4.1) are determined by polynomial solutions of (4.6) as given by the following relation:

$$g_k = \frac{b(q^{-1}n)C(n)}{c(n)}\ f_k, \tag{4.7}$$

5 q-Gosper's Algorithm of Arbitrary Order

This section is concerned with the q-hypergeometric solution g_k of the recurrence

$$\sum_{i=0}^{d} \lambda_i(n)\cdot g_{k+i} = f_k, \tag{5.1}$$

where f_n is a given q-hypergeometric term, and $\lambda_0(n), \lambda_1(n), \dots, \lambda_d(n) \in \mathbb{F}[n]$ are given polynomials, with additional restrictions that $\lambda_0(n)$ and $\lambda_d(n)$ are constants. Suppose that there exists a q-hypergeometric solution g_k of (5.1). Let $\rho(n)$ and $\tau(n)$ be as in (4.2). From (5.1) it follows that

$$\frac{g_k}{f_k} = \frac{1}{\sum_{i=0}^{d}\lambda_i(n)\cdot\prod_{j=0}^{i-1}\tau(q^jn)},$$

and

$$\rho(n)=\tau(n)\frac{\sum_{i=0}^{d}\lambda_i(qn)\cdot\prod_{j=0}^{i-1}\tau(q^{j+1}n)}{\sum_{i=0}^{d}\lambda_i(n)\cdot\prod_{j=0}^{i-1}\tau(q^{j}n)}. \tag{5.2}$$

Let us express $\rho(n)$ and $\tau(n)$ in terms of their q-GP representations as in (4.4). The following theorem can be easily checked.

Theorem 5.1. *Suppose that $\tau(n)$ has q-GP representation as in (4.4). Then the rational function on the right hand side of (5.2) has the following q-GP representation:*

$$\frac{A(n)}{B(q^dn)}\frac{\sum_{i=0}^{d}\lambda_i(qn)\cdot C(q^{i+1}n)\cdot\prod_{j=0}^{i-1}A(q^{j+1}n)\cdot\prod_{j=i}^{d-1}B(q^{j+1}n)}{\sum_{i=0}^{d}\lambda_i(n)\cdot C(q^{i}n)\cdot\prod_{j=0}^{i-1}A(q^{j}n)\cdot\prod_{j=i}^{d-1}B(q^{j}n)}. \tag{5.3}$$

By the uniqueness of the q-GP representation, we may compare the q-GP representation of $\rho(n)$ in (4.4) and the q-GP representation in (5.3) to obtain the following equations: $a(n)=A(n),\ b(n)=B(q^dn)$, and

$$\sum_{i=0}^{d}\lambda_i(n)\cdot C(q^{i}n)\cdot\prod_{j=0}^{i-1}A(q^{j}n)\cdot\prod_{j=i}^{d-1}B(q^{j}n)=c(n).$$

It follows that

$$\sum_{i=0}^{d}\lambda_i(n)\cdot C(q^{i}n)\cdot\prod_{j=0}^{i-1}a(q^{j}n)\cdot\prod_{j=i}^{d-1}b(q^{j-d}n)=c(n). \tag{5.4}$$

Therefore, q-hypergeometric solutions of (5.1) are determined by polynomial solutions of (5.4) as given by

$$g_k=\frac{\prod_{j=1}^{d}b(q^{-j}n)\cdot C(n)}{c(n)}\ f_k, \tag{5.5}$$

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