Labeled Partitions and the q-Derangement Numbers

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Abstract. Inspired by MacMahon's original proof of his celebrated theorem on the distribution of the major index over permutations, we give a reformulation of his argument in terms of labeled partitions. In this framework, we establish a decomposition theorem for labeled partitions which is analogous to the decomposition of a permutation into derangement points and fixed points. This decomposition implies a reformulation of Wachs' formula concerning the derangement parts and major index on permutations which was derived in order to present a bijective treatment of Gessel's formula on the q-derangement numbers.

Keywords: q-derangement number, major index, bijection, partitions, labeled partitions.

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1 Introduction

We will follow the terminology and notation on permutations and partitions and q-series in Andrews [2] and Stanley [10]. The set of permutations on $\{1, 2, \ldots, n\}$ is denoted by S_n. For any permutation $\pi = \pi_1 \pi_2 \cdots \pi_n \in S_n$, an index i with $1 \le i \le n - 1$ is called a *descent* of π if $\pi_i > \pi_{i+1}$. The major index $\mathrm{maj}(\pi)$ of π, introduced by MacMahon [9], is defined as the sum of all descents of π. The following formula is well-known:

$$\sum_{\pi \in S_n} q^{\mathrm{maj}(\pi)} = [n]! = 1 \cdot (1 + q) \cdot (1 + q + q^2) \cdots (1 + q + \cdots + q^{n-1}). \qquad (1.1)$$

The underlying idea of MacMahon' proof goes as follows. It is easier to consider sequences and partitions than solely permutations for the purpose of studying the major index. MacMahon established (1.1) by proving an equivalent formula

$$\frac{1}{(q)_n} \sum_{\pi \in S_n} q^{\mathrm{maj}(\pi)} = \frac{1}{(1 - q)^n}, \qquad (1.2)$$

where $(q)_n = (1 - q) \cdots (1 - q^n)$, and $(q)_n^{-1}$ is the generating function for partitions with n parts, where zero parts are allowed. We will give a reformulation of MacMahon's proof in Section 2 by using the notion of standard labeled partitions.

The main objective of this paper is to employ MacMahon's method to deal with the major index of derangements. An integer i with $1 \le i \le n$ is said to be a *fixed*

point of $\pi \in S_n$ if $\pi_i = i$, and *derangement point* otherwise. *Derangements* are permutations with no fixed points. Let D_n be the set of all derangements in S_n. The q-derangement numbers are defined by $d_0(q) = 1$ and for $n \geq 1$

$$d_n(q) = \sum_{\pi \in D_n} q^{\mathrm{maj}(\pi)}.$$

The following elegant formula was first derived by Gessel and published in [6] as a consequence of the quasi-symmetric generating function encoding the descents and the cycle structure of permutations:

$$d_n(q) = [n]! \sum_{k=0}^{n} \frac{(-1)^k}{[k]!} q^{\binom{k}{2}}. \tag{1.3}$$

A combinatorial proof of (1.3) has been obtained by Wachs [12]. Let us review the combinatorial settings of Wachs. Suppose that the derangement points of π are $p_1, p_2, \ldots, p_k$. The reduction of π to its *derangement part*, denoted by $dp(\pi)$, is defined as a permutation on $\{1, 2, \ldots, k\}$ induced by the relative order of $\pi_{p_1}, \pi_{p_2}, \ldots, \pi_{p_k}$. For example, the derangement points of $\pi = (1, 5, 3, 7, 6, 2, 9, 8, 4)$ are $2, 4, 5, 6, 7, 9$, and $\pi_2 \pi_4 \pi_5 \pi_6 \pi_7 \pi_9 = (5, 7, 6, 2, 9, 4)$. Then $dp(\pi) = (3, 5, 4, 1, 6, 2)$. Clearly $dp(\pi) \in D_k$ if π has k derangement points. On the other hand, we can insert a fixed point j with $1 \leq j \leq k + 1$ into $\pi \in S_k$ to obtain a permutation

$$\bar{\pi} = \pi'_1 \pi'_2 \cdots \pi'_{j-1} j \, \pi'_j \cdots \pi'_k \in S_{k+1}, \tag{1.4}$$

where $\pi'_i = \pi_i$ if $\pi_i < j$ and $\pi'_i = \pi_i + 1$ if $\pi_i \geq j$. Such an insertion operation produces one extra fixed point and will be used in the proof of the Theorem 2.2. Wachs [12] has established the following relation.

Theorem 1.1. *Let $0 \leq k \leq n$ and $\sigma \in D_k$. Then we have*

$$\sum_{\substack{dp(\pi)=\sigma \\ \pi \in S_n}} q^{\mathrm{maj}(\pi)} = q^{\mathrm{maj}(\sigma)} \begin{bmatrix} n \\ k \end{bmatrix}, \tag{1.5}$$

where $\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}$ is the q-binomial coefficient.

Summing over all derangements $\sigma \in D_k$ and $0 \leq k \leq n$, and applying (1.1), we can deduce from (1.5) that

$$[n]! = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} d_k(q).$$

Thus (1.3) follows from the q-binomial inversion [1, Corollary 3.38].

In order to justify the relation (1.5), Wachs found a bijection on S_n by rearranging a permutation π according to *excedant* (where $\pi_i > i$), fixed point, and *subcedant* (where $\pi_i < i$). She showed that this bijection preserves the major index. Then a result of Garsia-Gessel [4, Theorem 3.1] on shuffles of permutations is applied to establish Theorem 1.1.

Inspired by MacMahon's original proof of (1.1), we present an alternative approach to Wachs' formula (1.5) based on the following reformulation:

$$\frac{1}{(q)_n} \sum_{\substack{dp(\pi)=\sigma \\ \pi \in S_n}} q^{\mathrm{maj}(\pi)} = \frac{1}{(q)_k (q)_{n-k}} q^{\mathrm{maj}(\sigma)}. \tag{1.6}$$

We will use the terminology of labeled partitions and will introduce the notion of standard labeled partitions. In this framework, MacMahon's proof can be easily stated. Moreover, a combinatorial justification of (1.6) reduces to a decomposition theorem which is analogous to the decomposition of a permutation by separating the derangement points and the fixed points.

2 Labeled Partitions

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. We say that λ is a partition with n parts, or of length n, and we write $|\lambda| = \lambda_1 + \cdots + \lambda_n$. A *labeled partition* of length n is defined as a pair (λ, π) (or $\binom{\lambda}{\pi}$), where λ is a partition with n parts and π is a permutation $\pi = \pi_1 \pi_2 \cdots \pi_n$ on $[n] = \{1, 2, \ldots, n\}$. A labeled partition is also represented in the following two row form as in Andrews [2, p. 43]:

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \pi_1 & \pi_2 & \cdots & \pi_n \end{pmatrix}.$$

A labeled partition (λ, π) is said to be *standard* if $\pi_i > \pi_{i+1}$ implies $\lambda_i > \lambda_{i+1}$. For example, the labeled partition in (2.1) is standard.

The following Lemma 2.1 is straightforward to verify, which is MacMahon's original approach to study the major index with the aid of partitions, see MacMahon [9], Andrews [2, Theorem 3.7], Knuth [8, p. 18] or [7]. This method was further extended by Stanley [11]. For other applications, see [4].

Lemma 2.1. *Given $\pi \in S_n$, there is a bijection $\psi_\pi \colon \lambda \mapsto \mu$ between the set of partitions λ with n parts and the set of partitions μ with n parts such that (μ, π) is a standard labeled partition and $|\lambda| + \mathrm{maj}(\pi) = |\mu|$. Formally, we write $\psi(\lambda, \pi) = (\psi_\pi(\lambda), \pi)$.*

The bijection ψ_π (or simply ψ when π is understood from the context) is given as follows:

$$\mu = \psi_\pi(\lambda) = (\lambda_1 + \phi_1, \lambda_2 + \phi_2, \ldots, \lambda_n + \phi_n),$$

where ϕ_i is the number of descents in $\pi_i \pi_{i+1} \cdots \pi_n$. One may also view ψ as the operation of adding 1 to $\lambda_1, \ldots, \lambda_i$ whenever i is a descent of π. The inverse map ψ^{-1} can be easily described.

We now give a restatement of MacMahon's proof of (1.2) in the terminology of labeled partitions.

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Proof of (1.2). Given a sequence $a_1 a_2 \cdots a_n$ of nonnegative integers, we associate it with weight $q^{a_1 + a_2 + \cdots + a_n}$. Let us construct a two row array

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ 1 & 2 & \cdots & n \end{pmatrix}.$$

Note that a labeled partition (λ, π) is standard if $\lambda_i = \lambda_{i+1}$ implies $\pi_i < \pi_{i+1}$. Therefore, by permuting the columns of the above array, one can get a unique standard labeled partition (μ, π) with $|\mu| = a_1 + a_2 + \cdots + a_n$. Applying Lemma 2.1, we obtain a partition λ with $|\lambda| + \mathrm{maj}(\pi) = |\mu|$. Clearly, the above steps are reversible. This completes the proof. $\qquad \square$

Let $n = 9$ and $a_1 a_2 \ldots a_9$ be given as the first row in the following array

$$\begin{pmatrix} 3 & 6 & 8 & 3 & 1 & 3 & 6 & 4 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{pmatrix}.$$

By permuting the columns we get the standard labeled partition:

$$\begin{pmatrix} \mu \\ \pi \end{pmatrix} = \begin{pmatrix} 8 & 8 & 6 & 6 & 4 & 3 & 3 & 3 & 1 \\ 3 & \underline{9} & 2 & 7 & \underline{8} & 1 & 4 & \underline{6} & 5 \end{pmatrix}, \tag{2.1}$$

where we have underlined those elements π_i whenever i is a descent of π.

Applying the inverse map ψ^{-1} we obtain

$$\begin{pmatrix} \lambda \\ \pi \end{pmatrix} = \begin{pmatrix} 5 & 5 & 4 & 4 & 2 & 2 & 2 & 2 & 1 \\ 3 & \underline{9} & 2 & 7 & \underline{8} & 1 & 4 & \underline{6} & 5 \end{pmatrix},$$

which is the corresponding labeled partition.

We remark that the idea of standard labeled partitions appeared in [4, p. 292], though it was not used to prove (1.2).

We are now ready to present a decomposition theorem on standard labeled partitions with respect to the fixed points. Let σ be a given derangement in D_k and π a permutation in S_n such that $dp(\pi) = \sigma$. Assume that (μ, π) is a standard labeled partition of length n. Let $i_1 < i_2 < \cdots < i_{n-k}$ be the fixed points, and $j_1 < j_2 < \cdots < j_k$ the derangement points of π. We now define the following decomposition of a standard labeled partition:

$$\Delta \colon (\mu, \pi) \mapsto (\beta, \gamma), \tag{2.2}$$

where $\beta = \mu_{j_1} \mu_{j_2} \cdots \mu_{j_k}$ and $\gamma = \mu_{i_1} \mu_{i_2} \cdots \mu_{i_{n-k}}$ are the partitions corresponding to the derangement points and fixed points, respectively. Evidently, μ consists of the parts from β and γ, or in the common notation, $\mu = \beta \cup \gamma$.

Theorem 2.2. *Let $\sigma \in D_k$. There is a bijection between the set of standard labeled partitions (μ, π) of length n with $dp(\pi) = \sigma$ and the set of pairs of partitions (β, γ) such that β is a partition with k parts, γ is a partition with $n - k$ parts, $\mu = \beta \cup \gamma$, and (β, σ) is a standard labeled partition.*

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For example, let
$$\begin{pmatrix}\lambda\\\pi\end{pmatrix}=\begin{pmatrix}5\;4\;4\;4\;4\;3\;2\\\underline{5}\;\underline{2}\;1\;4\;\underline{7}\;3\;6\end{pmatrix}.$$
Applying ψ, we get
$$\begin{pmatrix}\mu\\\pi\end{pmatrix}=\begin{pmatrix}8\;\mathbf{6}\;5\;\mathbf{5}\;5\;3\;2\\5\;\mathbf{2}\;1\;\mathbf{4}\;7\;3\;6\end{pmatrix}.$$

The fixed points of π are $2,4$, as signified in boldface. Hence $\sigma=dp(\pi)=(3\;1\;5\;2\;4)$. Applying Δ on (μ,π) gives $(\beta,\gamma)=((8\;5\;5\;3\;2),(6\;5))$. Finally, applying ψ^{-1} to

$$\begin{pmatrix}\beta\\\sigma\end{pmatrix}=\begin{pmatrix}8\;5\;5\;3\;2\\3\;1\;5\;2\;4\end{pmatrix},$$

we obtain
$$\begin{pmatrix}\alpha\\\sigma\end{pmatrix}=\begin{pmatrix}6\;4\;4\;3\;2\\3\;1\;5\;2\;4\end{pmatrix}.$$

We note that the above theorem and Lemma 2.1 lead to a combinatorial interpretation of the relation (1.6). Given a labeled partition (λ,π), we first use the map ψ to transform it into a standard labeled partition (μ,π). Let $\sigma=dp(\pi)$. Using the above decomposition for (μ,π), we obtain a pair of partitions (β,γ) such that (β,σ) is a standard labeled partition and γ is a partition with $n-k$ parts. Moreover, we can find a partition α with k parts such that $\psi_\sigma(\alpha,\sigma)=(\beta,\sigma)$. Thus we obtain the following relation

$$|\lambda|+\mathrm{maj}(\pi)=|\alpha|+|\gamma|+\mathrm{maj}(\sigma),\qquad(2.3)$$

which yields (1.6).

Proof of Theorem 2.2. We first verify that (β,σ) is standard. Suppose there exists i such that $\sigma_i>\sigma_{i+1}$. Then we need to show that $\beta_i>\beta_{i+1}$. Clearly, σ_i and σ_{i+1} correspond to two elements π_j and π_k ($j<k$) in π such that $\pi_j>\pi_k$ and the points $j+1,j+2,\ldots,k-1$, if any, are fixed points of π. In other words, it is necessary to show that $\mu_j>\mu_k$ since $\beta_i=\mu_j$ and $\beta_{i+1}=\mu_k$ by the decomposition. If $j=k-1$, since (μ,π) is standard, we have $\mu_j>\mu_k$. For the case $j<k-1$, we see that either $\pi_j>\pi_{j+1}=j+1$ or $\pi_{k-1}=k-1>\pi_k$. Otherwise, we have $\pi_j<j+1\leq k-1<\pi_k$, a contradiction. Therefore, we have either $\mu_j>\mu_{j+1}$ or $\mu_{k-1}>\mu_k$. This implies that $\mu_j>\mu_k$. So we conclude that (β,σ) is a standard labeled partition.

We now proceed to construct the inverse map φ which corresponds to the procedure to recover π by inserting the fixed points to the derangement σ on $\{1,2,\ldots,k\}$. It turns out that the order of insertions reflects the property of standard labeled partitions.

We begin with $(\mu^0,\pi^0)=(\beta,\sigma)$ and assume that (μ^i,π^i) is obtained from (μ^{i-1},π^{i-1}) by *inserting* γ_i, where γ_i is the i-th part of γ. Let r be the first position such that the insertion of γ_i produces a partition. In other words, if μ^{i-1} already contains some parts equal to γ_i, then we insert γ_i as the first occurrence. This partition is denoted by μ^i. We need to determine the corresponding fixed point of π^i caused by the insertion of γ_i.

Clearly, $\mu_{r-1}^i > \mu_r^i = \gamma_i$. We may assume that $\mu_r^i = \cdots = \mu_t^i > \mu_{t+1}^i$ for some $t \geq r$. As in the proof of (1.2), in order to get a standard labeled partition after each insertion of γ_i, we should insert s such that the subsequence

$$\pi_r^{i-1\,\prime}, \cdots, \pi_{s-1}^{i-1\,\prime}, s, \pi_s^{i-1\,\prime}, \cdots, \pi_{t-1}^{i-1\,\prime}$$

of the permutation obtained by inserting s into π^{i-1} is increasing. We see that the position s is uniquely determined. If $r = t$ then we set $s = r$. Otherwise we find the first position s ($r \leq s \leq t$) such that $\pi_{s-1}^{i-1} < s \leq \pi_s^{i-1}$. Strictly speaking, we have adopted the convention that $\pi_{r-1}^{i-1} = -\infty$ and $\pi_t^{i-1} = \infty$. Consequently, we can insert s into π^{i-1} as a fixed point of π^i.

It remains to show that $\pi^{n-k} = \pi$. For notational simplicity, we write π^{n-k} as $\bar{\pi}$. By removing the common fixed points, we may assume that the first fixed point f of π is different from the first fixed point $\bar{f}$ of $\bar{\pi}$. We find that f satisfies the condition $\pi_{f-1} < f \leq \pi_{f+1} - 1$. Furthermore, by the insertion procedure, $\bar{f}$ is the first position we like to locate, hence we may assume that $\bar{f} < f$. Clearly, $\mu_f = \mu_{\bar{f}}$. Since (μ, π) and $(\mu, \bar{\pi})$ are standard labeled partitions, we have

$$\pi_{\bar{f}} < \pi_{\bar{f}+1} < \cdots < \pi_f, \text{ and } \bar{\pi}_{\bar{f}} < \bar{\pi}_{\bar{f}+1} < \cdots < \bar{\pi}_f.$$

Now we see that $\pi_f = f$ and $\bar{\pi}_{\bar{f}} = \bar{f}$. But $\pi_{\bar{f}} < \pi_{\bar{f}+1} < \cdots < \pi_f$ and $\pi_f = f$, we can deduce that $\pi_{\bar{f}} \leq \bar{f}$. Since f is the first fixed point of π, we obtain $\sigma_{\bar{f}} = \pi_{\bar{f}} < \bar{f}$. From the construction of $\bar{\pi}$, it follows that $\bar{\pi}_{\bar{f}} \leq \sigma_{\bar{f}} < \bar{f}$ which contradicts $\bar{\pi}_{\bar{f}} = \bar{f}$. This completes the proof. $\square$

For example, given $\sigma = (3\ 1\ 5\ 2\ 4)$, $\beta = (8, 5, 5, 3, 2)$ and $\gamma = (6, 5)$, we may recover (μ, π) through the following steps:

$$\begin{pmatrix} \beta \\ \sigma \end{pmatrix} = \begin{pmatrix} 8\ 5\ 5\ 3\ 2 \\ 3\ 1\ 5\ 2\ 4 \end{pmatrix} \xrightarrow{\gamma_1 = 6} \begin{pmatrix} 8\ \underline{6}\ 5\ 5\ 3\ 2 \\ 4\ \underline{2}\ 1\ 6\ 3\ 5 \end{pmatrix} \xrightarrow{\gamma_2 = 5} \begin{pmatrix} 8\ 6\ 5\ \underline{5}\ 5\ 3\ 2 \\ 5\ 2\ 1\ \underline{4}\ 7\ 3\ 6 \end{pmatrix} = \begin{pmatrix} \mu \\ \pi \end{pmatrix}.$$

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