On the Combinatorics of the Pfaff Identity

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Abstract

Recently, there has been a revival of interest in the Pfaff identity on hypergeometric series because of the specialization of Simons and a generalization of Munarini. We present combinatorial settings and interpretations of the specialization and the generalization; one is based on free Dyck paths and free Schröder paths, and the other relies on a correspondence of Foata and Labelle between the Meixner endofunctions and bicolored permutations, and an extension of the technique developed by Labelle and Yeh for the Pfaff identity. Applying the involution on weighted Schröder paths, we derive a formula for the Narayana numbers as an alternating sum of the Catalan numbers.

Keywords: Dyck path; Schröder path; permutation.

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1 Introduction

Simons [23] has established the following identity

$$\sum_{k=0}^{n} \frac{(-1)^{n+k}(n+k)!(1+x)^k}{(n-k)!k!^2} = \sum_{k=0}^{n} \frac{(n+k)!x^k}{(n-k)!k!^2}. \tag{1.1}$$

Chapman [2] and Prodinger [21] gave short proofs by the snake oil method and the Cauchy integral formula. Recently, Wang and Sun [25] showed the same identity by an operator method. Hirschhorn [12] pointed out that the above identity is a special case of the Pfaff identity [1, (2.3.14)] on ${}_2F_1$ hypergeometric series:

$$ {}_2F_1\left(\begin{matrix} -n, b \\ c \end{matrix} ; x \right) = \frac{(c-b)_n}{(c)_n} {}_2F_1\left(\begin{matrix} -n, b \\ b+1-n-c \end{matrix} ; 1-x \right). \tag{1.2} $$

Recall that a ${}_2F_1$ hypergeometric series is defined by

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} ; z \right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \tag{1.3} $$

where $(a)_n$ stands for the rising factorial $(a)_n = a(a+1)\cdots(a+n-1)$. Thus (1.1) can be deduced from (1.2) by setting $b = n+1, c = 1$ and substituting x for $-x$. The Pfaff identity has been given a combinatorial interpretation by Labelle and Yeh [14]. It should be noted that (1.2) can be verified by using Zeilberger's algorithm which gives the following recurrence relation for both sides of (1.2):

$$ (1+n)(x-1)S(n) - (x-2+xn-c+bx-2n)S(n+1) - (1+n+c)S(n+2) = 0. \tag{1.4} $$

As to the special case (1.1) the recurrence relation becomes

$$ (n+2)S(n+2) - (1+2x)(2n+3)S(n+1) + (n+1)S(n) = 0. $$

Munarini [20] derived the following generalization of Simons' identity by using Cauchy's integral formula,

$$ [t^n]\frac{(1+yt)^\alpha}{(1-xt)^\beta} = [t^n]\frac{(1+(y-s)t)^\alpha}{(1-(x+s)t)^\beta}(1-st)^{\beta-\alpha+n-1}, \tag{1.5} $$

which can be restated as

$$ \begin{aligned} &\sum_{i+j=n} \binom{\beta+i-1}{i}\binom{\alpha}{j} x^i y^j \\ &\quad = \sum_{i+j+k=n} \binom{\alpha}{i}\binom{\beta-\alpha+n-1}{j}\binom{-\beta}{k}(y-s)^i(-s)^j(-(x+s))^k. \end{aligned} \tag{1.6} $$

By Zeilberger's algorithm and the telescoping method developed by Chen, Hou and Mu [4], we can get the same recurrence relation for the summations on the left and right sides of the above identity:

$$ (n+2)S(n+2) - (nx - yn + \beta x + x - y + y\alpha)S(n+1) - xy(\beta-\alpha+n)S(n) = 0. \tag{1.7} $$

We notice that Munarini's identity (1.6) reduces to Pfaff's identity (1.2) by setting $x = 1$, $y = x-1$, $s = -1$, $\beta = c-b$ and $\alpha = -b$. The aim of this work is to present the combinatorial structures and transformations for the above specialization and generalization of Pfaff's identity. It turns out that the Simons identity can be formulated in terms of free Schröder paths and free Dyck paths, whereas the Munarini identity is related to refined structures of reluctant functions. Our combinatorial interpretation of the Munarini identity relies on the correspondence of Foata and Labelle between Meixner endofunctions and bicolored permutations [10] and the technique developed by Labelle and Yeh for the Pfaff identity [14].

When we restrict our attention to the involution on ordinary Schröder paths, we obtain an expression of the Narayana numbers as an alternating sum of the Catalan numbers. As a consequence, it follows an identity on the alternating sum of the Catalan numbers.

2 The Simons identity and lattice paths

In this section, we give a lattice path interpretation of the Simons identity (1.1). Substituting $n-k$ for k on the left hand side of (1.1) gives the following equivalent form:

$$\sum_{k=0}^{n}(-1)^k\binom{2n-k}{k}\binom{2n-2k}{n-k}(1+x)^{n-k}=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}x^k. \tag{2.1}$$

A free Dyck path of semilength n is a lattice path from the origin to $(2n,0)$ consisting of up steps $(1,1)$ and down steps $(1,-1)$, whereas a Dyck path is a free Dyck path that does not go below the x-axis. Free Dyck paths have been studied in [5] in connection with the enumeration of plane trees. A free Schröder path of semilength n is a lattice path from $(0,0)$ to $(2n,0)$ with up steps $(1,1)$, horizontal steps $(2,0)$ and down steps $(1,-1)$. A free Schröder path that does not go below the x-axis is called a Schröder path. Figure 1 is an illustration of a free Dyck path and a free Schröder path.

A *weighted path* is a path for which every step is endowed with a weight. The weight of a path is the product of the weights of the steps; the weight of a set of paths means the sum of the weights of the paths. Weighted lattices have been used to give combinatorial interpretations of combinatorial identities, see, for example [5,6]. To connect the Simons identity to lattice paths, we use the following rule to assign the weight of a free Schröder path:

- A horizontal step is given the weight -1.
- A down step is given the weight 1.
- An up step is given the weight $1+x$. Equivalently, we may say that an up step has either weight 1 or weight x.

For free Dyck paths, the weight assignment is related to the peaks. Recall that a peak in a free Dyck path consists of an up step immediately followed by a down step. The weight of a free Dyck path inherits the weights of steps when the path is viewed as a free Schröder path. However, there is a special constraint for weighted Dyck path, namely, the up step in a peak is weighted by x. For this reason, a peak in a weighted free Schröder path is said to have weight 1 if the up step has weight 1.

Let us use $\mathcal{FD}_n$ and $\mathcal{FS}_n$ to denote the sets of weighted free Dyck paths and weighted free Schröder paths of semilength n, respectively. We now give the weighted counting of free Dyck paths and free Schröder paths of semilength n.

Proposition 2.1. *The sum of weigths of free Dyck paths of semilength n with exactly k up steps with weight x equals $\binom{n}{k}\binom{n+k}{k}x^k$.*

Proof. Subject to the condition that every peak in a weighted Dyck path has weight x, the $n-k$ up steps with weight 1 are not followed by down steps. We proceed to count the number of such paths by the following procedure. We use X to represent the up steps with weight x, and use U to denote the up steps with weight 1. First, consider the relative positions of X and U, they form a sequence of k X's and $n-k$ U's. So there are $\binom{n}{k}$ such sequences. Second, we need to insert the n down steps D. Keep in mind that D cannot appear to the right of U in the sequence. So D can be inserted either at the beginning of the sequence or after X. For example, given a sequence $UUXUXXU$, the bars in $|UUX|UX|X|U$ signifies where D can be inserted. In other words, as far as the insertions are concerned, the three segments UUX, UX and X can be regarded as inseparable segments. Therefore, the number of ways to insert n D's equals $\binom{n+k}{k}$. This implies the desired assertion. ∎

Proposition 2.2. *The number of free Schröder paths of semilength n with k horizontal steps equals $\binom{2n-k}{k}\binom{2n-2k}{n-k}$.*

Proof. Clearly, if there are k horizontal steps in a free Schröder path, then there are $n-k$ up steps and $n-k$ down steps. First, there are $\binom{2n-k}{k}$ ways to place the k horizontal steps. After the positions of the horizontal steps are determined, there are $\binom{2n-2k}{n-k}$ ways to place the $n-k$ up steps, and the down steps can be fitted into the path in a unique way. This completes the proof. ∎

From the above proposition, one sees that the summation on the left hand side of (2.1) equals the sum of weights of free Schröder paths of semilength n. With the above interpretations of two sides of the identity (2.1), the equality is a consequence of the following involution on the set $\mathcal{FS}_n \backslash \mathcal{FD}_n$.

Theorem 2.3. *There is a sign reversing involution on $\mathcal{FS}_n \backslash \mathcal{FD}_n$.*

Proof. Observe that a weighted free Schröder path is in $\mathcal{FS}_n \backslash \mathcal{FD}_n$ if and only if it contains either a horizontal step or a peak with weight 1. Then consider the first occurrence of a horizontal step or a peak with weight 1, whichever comes first. If we first encounter a horizontal step, then change it to a peak with weight 1. If we first encounter a peak with

Figure 1: A free Dyck path and a free Schröder path

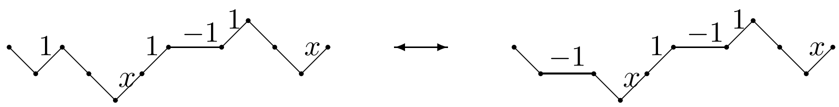

Figure 2: The involution on $\mathcal{FS}_n \backslash \mathcal{FD}_n$

weight 1, then change it to a horizontal step. Clearly, the above operation gives a sign reversing involution on $\mathcal{FS}_n \backslash \mathcal{FD}_n$. ∎

Figure 2 is an illustration of the involution. Thus, we are led to a combinatorial interpretation of the identity (2.1).

By using the above involution applied to Schröder paths and Dyck paths, we may derive an identity on the Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n},$$

and the Narayana numbers

$$N_{n,k} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

Theorem 2.4. *For $n \geq 1$, we have*

$$\sum_{k=0}^{n} N_{n,k} y^k = \sum_{k=0}^{n} \binom{2n-k}{k} C_{n-k}(y-1)^k, \tag{2.2}$$

$$\sum_{k=0}^{n} N_{n,k} y^{2k}(y+1)^{2(n-k)} = \sum_{k=0}^{n} (-1)^k \binom{2n-k}{k} C_{n-k}(1+2y)^k (y+1)^{2(n-k)}. \tag{2.3}$$

Proof. It is well known that the set of Dyck paths of semilength n with k peaks is counted by $N_{n,k}$, see, for example, [8]. Note that the number of Schröder paths of semilength n with $n-k$ horizontal steps, k up steps and k down steps equals $\binom{n+k}{n-k} C_k$ because there are $\binom{n+k}{n-k}$ choices to place the $(n-k)$ horizontal steps relative to the positions of the up steps and down steps. Now the remaining $2k$ steps simply form a Dyck path of semilength k.

Inheriting the weights from the free Dyck paths and free Schröder paths and applying the above involution to Dyck paths and Schröder paths, it follows that

$$\sum_{k=0}^{n} N_{n,k} x^k (x+1)^{n-k} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n+k}{n-k} C_k (x+1)^k, \tag{2.4}$$

or equivalently,

$$\sum_{k=0}^{n} N_{n,k} x^k (x+1)^{n-k} = \sum_{k=0}^{n} (-1)^{k} \binom{2n-k}{k} C_{n-k} (x+1)^{n-k}. \tag{2.5}$$

Setting $x = y/(1-y)$ or $x = y^2/(1+2y)$ in (2.5), we obtain (2.2) or (2.3). ∎

We notice that a general version of (2.2) is derived by Mansour and Sun [18, Example 2.13]. From (2.2) it follows that the Narayana number $N_{n,k}$ can be expressed as an alternating sum of the Catalan numbers:

$$N_{n,k} = \sum_{i=0}^{n} (-1)^{i-k} \binom{2n-i}{i} \binom{i}{k} C_{n-i}. \tag{2.6}$$

Setting $k=0$, since $N_{n,0}=0$ it follows that

$$\sum_{i=0}^{n} (-1)^{i} \binom{2n-i}{i} C_{n-i} = 0. \tag{2.7}$$

For comparison, let us recall the following two identities of Coker [6, 7]:

$$\sum_{k=0}^{n-1} \frac{1}{n} \binom{n}{k} \binom{n}{k+1} y^k = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1}{2k} C_k y^k (1+y)^{n-2k-1}, \tag{2.8}$$

$$\sum_{k=0}^{n-1} \frac{1}{n} \binom{n}{k} \binom{n}{k+1} y^{2k} (1+y)^{2(n-1-k)} = \sum_{k=0}^{n-1} \binom{n-1}{k} C_{k+1} y^k (1+y)^k. \tag{2.9}$$

We remark that the identity (2.8) is the same as an identity of Simion and Ullman [22, Corollary 3.2]. A bijective proof of (2.8) is given in [3]. Combinatorial interpretations of both (2.8) and (2.9) in terms of weighted 2-Motzkin paths are obtained in [6].

3 The Munarini identity and reluctant functions

The purpose of this section is to present a combinatorial interpretation of the Munarini identity (1.6). We follow the approach of Foata-Labelle [10] and Labelle-Yeh [14, 16] to combinatorial interpretations of hypergeometric series and special function identities including the Pfaff identity. Since we will deal with an extension of the Pfaff identity, we need to introduce the notions of reluctant bi-permutation and partisan permutations. The reluctant bi-permutations are a generalization of the Meixner endofunctions introduced by Foata and Labelle [10]. We will show that the Munarini identity follows from the correspondence between reluctant bi-permutations and the partisan permutations.

As mentioned in the previous section (1.6) is a generalization of the Pfaff identity, it is natural to expect that the combinatorial interpretation of the Pfaff identity as given by Labelle and Yeh [14] can be extended to the Munarini identity. We find that this is indeed the case. In order to fit our combinatorial setting, it is necessary to rewrite (1.6) in the following form after substituting $-\alpha$ by α and $-y$ by y:

$$\begin{aligned}\sum_{i+j=n} &\binom{n}{i,j}(\beta)_i(\alpha)_j x^i y^j \\ &= \sum_{i+j+k=n} \binom{n}{i,j,k}(\alpha)_i(\beta+\alpha+i+k)_j(\beta)_k(y+s)^i(-s)^j(x+s)^k. \qquad (3.1)\end{aligned}$$

A basic ingredient of the combinatorial settings for hypergeometric identities is the interpretation of the rising factorial $(x)_n = x(x+1)\cdots(x+n-1)$, or, in general, of $(x+k)_n$. It is well known that $(x)_n$ can be expanded in terms of the signless Stirling numbers of the first kind. Note that $(x)_n$ can also be interpreted as the number of dispositions from $[n] = \{1, 2, \ldots, n\}$ to a set X with x elements, see [13]. Let us recall that the fiber of a function f from $[n]$ to X is defined as a partition of $[n]$ such that i and j are in the same block if and only if i and j have the same image. Then a disposition from $[n]$ to X can be considered as a function f from $[n]$ to X whose fiber is endowed with the structure that each block is linearly ordered. Intuitively speaking, a disposition can be viewed as a configuration of n people queuing at the x windows of a box office.

In general, the rising factorial $(x+j)_i$ can be explained as the number of reluctant functions from A to B, where $|A| = i$ and $|B| = j$, introduced by Mullin and Rota in their theory of sequences of polynomials of binomial type [19]. A reluctant function f from A to B, where A and B are two disjoint finite sets, is defined as an injective map from A to $A \cup B$. The functional digraph of f is a digraph on $A \cup B$ with arcs $(k, f(k))$ for $k \in A$. The weight of f is defined as a^k, where k is the number of cycles in the functional digraph of f.

Evidently, the functional digraph of any reluctant function f has a unique decomposition into disjoint cycles on elements in A and directed paths ending with an element in B. The ending points in B are called *terminals*. Now we proceed to give a canonical cycle representation of a reluctant function.

Let us recall the canonical cycle representation of a permutation, see, Stanley [24, page 17]. Suppose that π is a permutation on $[n]$ with k cycles $C_1, C_2, \ldots, C_k$. We may always

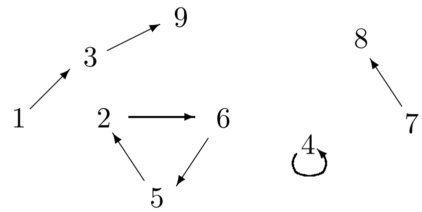

Figure 3: The digraph for a reluctant function

write a cycle $C = (i_1 i_2 \cdots i_r)$ in a form that i_1 is the minimum element of C. Moreover, we may arrange the cycles $C_1, C_2, \ldots, C_k$ in such a way that their minimum elements are in decreasing order. For example $\pi = (6)(38)(274)(15)$ is a canonical cycle representation. Note that the permutation π can be recovered given the sequence obtained from the canonical cycle representation by removing the parentheses.

In fact, we can extend the notion of the canonical cycle representation of a permutation to a reluctant function. Assume f is a reluctant function from A to B and its functional digraph can be decomposed into k cycles $C_1, C_2, \ldots, C_k$, and s directed paths $P_1, P_2, \ldots, P_s$. We first write down the cycles in canonical cycle representation. Then each path P_i is written as $(i_1, i_2, \cdots, i_r)$ with $i_1 \in B$ and $f(i_t) = i_{t-1}$ for $2 \leq t \leq r$, and $P_1, P_2, \ldots, P_s$ are arranged in such a way that their first elements are increasing.

For example, the reluctant function as given in Figure 3, where $A = \{1, 2, 3, 4, 5, 6, 7\}$ and $B = \{8, 9\}$, the canonical cycle representation is

$$(4)(265)(\mathbf{8}7)(\mathbf{9}31).$$

The reluctant function f can be recovered from the sequence 4265**8**7**9**31 according to the elements in B as printed in boldface and the left-to-right minimum elements in A before the first appearance of an element in B.

The following proposition is well known, see, for example, [9, 11, 15, 17]. For completeness, we present a proof based on the notion of the canonical cycle representation of reluctant function.

Proposition 3.1. *Let A and B be two disjoint subsets of $[n]$, and let $|A| = i$ and $|B| = j$. Then the sum of weights of reluctant functions from A to B equals $(a + j)_i$.*

Proof. There is no restriction in supposing that $A = \{1, 2, \cdots, i\}$ and $B = \{i + 1, i + 2, \cdots, i + j\}$. The weight of a reluctant function f from A to B can be visualized as assigning a weight a to each cycle. In the above notation of linear representation of reluctant function f, we see that the weight f is given by a^k, where k is the number of the left-to-right minimum elements in A before the first appearance of the elements in B, equivalently, the left-to-right minimum elements in A before $i + 1$.

On the other hand, the linear representation of f can be generated by the following insertion procedure. Initially, we have the sequence $i + 1, i + 2, \cdots, i + j$ on B. Then we successively insert the elements of $\{1, 2, \cdots, i\}$. Let us first consider how the element 1 can be inserted.

If 1 is inserted at the beginning of the sequence $i{+}1, i{+}2, \cdots, i{+}j$, we obtain a sequence beginning with 1 for which 1 forms a new cycle contributing weight a. Otherwise, there are j possibilities for the insertion of 1 after some element in the sequence $i+1, i+2, \cdots, i+j$. In any case, 1 always joins a path with a terminal in B. Hence the total contribution to the weight sums to j, and we have the factor $(a + j)$.

In general, suppose that $1, 2, \ldots, k - 1$ have already been inserted. We proceed to insert k in an analogous way as we did for the insert of 1. If k is placed at the beginning of the sequence, then k is surely left-to-right minimum, in which case it induces a new cycle with weight a. Otherwise, there are $j + k - 1$ possibilities for the insertion of k after some element in the given sequence. So k will join a cycle or a path, and in either case this insertion does not change the weight of the resulting digraph. Therefore, the total contribution is $j + k - 1$ and we are led to the factor $(a + j + k - 1)$. This completes the proof. ∎

Setting $B = \emptyset$ in Proposition 3.1, a reluctant function from A to B reduces to a permutation of A, and the proposition says that the rising factorial $(x)_n$ equals the generating function of permutations on $[n]$ with respect to the number of cycles.

We now present the combinatorial structures that are needed in our bijection of the Munarini identity.

1. Suppose (A_1, A_2) is a composition of $[n]$, namely, $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 = [n]$, A_1 and A_2 are allowed to be empty. A pair of permutations (π_1, π_2) on A_1 and A_2 is called a bi-permutation, denoted $(A_1, A_2; \pi_1, \pi_2)$.

2. Suppose (A_1, A_2, A_3) is a composition of $[n]$, defined in the same manner as for the compositions of two components, and suppose $(A_1, A_2; \pi_1, \pi_2)$ is a bi-permutation. If f is a reluctant function from A_3 to $A_1 \cup A_2$ then we say that $(A_1, A_2, A_3; \pi_1, \pi_2; f)$ is a reluctant bi-permutation.

3. Given a 3-coloring of $[n]$, say by the three colors red, black and white, a partisan permutation is defined as a permutation on $[n]$ such that a red element and a black element cannot appear simultaneously in the same cycle. We may regard the red color and black color as banners of two rivalry parties, and consider the white color as a sign for those who are independent. For example,

$$(8, \underline{7}, 9)(\mathbf{2}, 5, \mathbf{4})(\mathbf{10}, 1)(\mathbf{3})(11, \underline{12})(6)$$

   is a partisan permutation, where we represent the red elements as underlined and the black elements in boldface.

To construct the desired combinatorial transformation, we need an extension of Foata's bijection [10, 16] between 'Meixner endofunctions' and 'bicolored permutations'. Recall

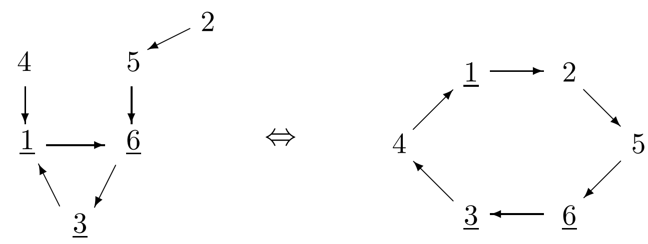

Figure 4: The bijection Φ

that a *Meixner endofunction* on a finite set S is represented by $(A, B; f)$, where (A, B) is a composition of S and f is a map from S to S such that the restriction of f on A is injective and the restriction of f on B is a permutation on B. A *bicolored permutation* on a finite set S is represented by $(A, B; \sigma)$, where (A, B) is a composition of S, and σ is a permutation on S. Note that a composition (A, B) of a set S can be considered as a 2-coloring of S. Foata's bijection can be described as follows.

Proposition 3.2. *There is a bijection between the set of Meixner endofunctions on* $[n]$ *and the set of bicolored permutations on* $[n]$*.*

The following correspondence is essentially Foata's bijection applied to both red cycles and black cycles. Notice that a reluctant bi-permutation $(A_1, A_2, A_3; \pi_1, \pi_2; f)$ becomes a Meixner endofunction when $A_2 = \emptyset$.

Proposition 3.3. *There is a bijection* Φ *between the set of reluctant bi-permutations on* $[n]$ *and the set of partisan permutations on* $[n]$*.*

Proof. Given a reluctant bi-permutation $(A_1, A_2, A_3; \pi_1, \pi_2; f)$, we color the elements in A_1, A_2 and A_3 red, black and white respectively. Consider the cycle decomposition of π_1 and π_2 on A_1 and A_2. We may view a reluctant bi-permutation as a union of disjoint cycles on A_3 and some directed paths on A_3 attached to elements in $A_1 \cup A_2$ together with some permutations on A_1 and A_2. Since f is injective, two directed paths on A_3 cannot be incident to the same element in $A_1 \cup A_2$.

The bijection will be concerned with only the components consisting with cycles on a subset of A_1 or A_2 attached with some paths on A_3. Let C be such a cycle, and P a directed path attached to C. Assume that x is the terminal element of P that is on C. Let y be the element pointing to x on C. Then we can break the arc from y to x, and make y point to the initial element of P. From the color of the elements on the path P, the above operation is reversible. Taking all the paths attached to C into account, we obtain the desired bijection. ∎

For example, as illustrated in Figure 4, the reluctant bi-permutation

$$(\{1, 3, 6\}, \emptyset, \{2, 4, 5\}; (1, 6, 3), \emptyset; (1, 4)(6, 5, 2)),$$

corresponds to the partisan permutation $(\underline{3}, 4, \underline{1}, 2, 5, \underline{6})$, where red elements are underlined.

We are now ready to give a combinatorial proof of the identity (3.1) via the road map:

$$\text{reluctant bi-permutations} \Longleftrightarrow \text{partisan permutations} \Longrightarrow \text{bi-permutations.}$$

The $\Longleftrightarrow$ stands for a weight preserving bijection. The $\Longrightarrow$ indicates the equality for the sums of weights, which will be established by an easy binomial summation. Here we note that the equality indicated by $\Longrightarrow$ can be demonstrated combinatorially by a trivial involution, and such formality will be omitted.

We now give the definitions of weights for reluctant bi-permutations and bi-permutations. The weight of a reluctant bi-permutation $(A_1, A_2, A_3; \pi_1, \pi_2; f)$ on $[n]$ is defined as follows. An element in A_1, A_2, A_3 is assigned the weight

$$y + s, \quad x + s, \quad -s,$$

respectively. The weight of a cycle in π_1, π_2 and f is given by

$$\alpha, \quad \beta, \quad \beta + \alpha,$$

respectively. Then the weight of a reluctant bi-permutation is the product of the weights of the elements and the weights of the cycles.

We define the weight of a bi-permutation $(A, B; \pi_1, \pi_2)$. The weight of an element in A is given by x, the weight of an element in B is given by y, the weight of a cycle in π_1 is given by β and the weight of a cycle in π_2 is given by α. Then the weight of a bi-permutation is the product of the weights of the elements and the weights of the cycles. We will see that equation (3.1) expresses the fact that the sum of weights of reluctant bi-permutations on $[n]$ equals the sum of weights of bi-permutations on $[n]$.

Proof. By the definition of the weight of a bi-permutation, it is easily seen that the sum of weights over bi-permutations on $[n]$ equals the the summation on the left hand side of (3.1):

$$\sum_{i+j=n} \binom{n}{i, j} x^i y^j (\beta)_i (\alpha)_j. \tag{3.2}$$

On the other hand, a bi-permutation can be viewed as a permutation on $[n]$ for which each cycle is colored black or white. Hence (3.2) can be expressed in terms of cycle decompositions of permutations on $[n]$, where the weight of a cycle C with cardinality $|C|$ is given by

$$x^{|C|}\beta + y^{|C|}\alpha. \tag{3.3}$$

We proceed to show that the right hand side of (3.1) can also be reduced to a summation over permutations on $[n]$ with each cycle having the above weight (3.3).

First, from Proposition 3.1 we see that the sum of weights of reluctant bi-permutations $(A_1, A_2, A_3; \pi_1, \pi_2; f)$ on $[n]$ equals the summation on the right hand side of (3.1):

$$\sum_{i+j+k=n} \binom{n}{i, j, k} (\alpha)_i (\beta + \alpha + i + k)_j (\beta)_k (y+s)^i (-s)^j (x+s)^k. \tag{3.4}$$

Applying the bijection Φ in Proposition 3.3 between reluctant bi-permutations on $[n]$ and the partisan permutations on $[n]$, (3.1) can be rewritten as a summation of weights of partisan permutations on $[n]$ with the following weight assignments. A red, black, or white element is given the weight

$$y + s, \quad x + s, \quad -s$$

respectively. A cycle containing at least one red element is given the weight α, a cycle containing at least one black element is given the weight β, and a cycle consisting of only white elements is given the weight $\beta + \alpha$.

On the other hand, the total weight of the partisan permutations on $[n]$ can be computed based on the cycle decompositions of permutations on $[n]$. Given a permutation π on $[n]$ and a cycle C in π with $|C| = m$, if C is a cycle consisting of white elements, the weight contribution is

$$(-s)^m (\beta + \alpha); \tag{3.5}$$

if C is used to form a cycle containing at least one red element, the total weight contribution equals

$$\sum_{i=1}^{m} \binom{m}{i} (y+s)^i (-s)^{m-i} \alpha = y^m \alpha - (-s)^m \alpha; \tag{3.6}$$

and if C is used to a cycle containing a black element, the total weight contribution equals

$$\sum_{i=1}^{m} \binom{m}{i} (x+s)^i (-s)^{m-i} \beta = x^m \beta - (-s)^m \beta. \tag{3.7}$$

Summing up (3.5), (3.6) and (3.7), we get the total weight contribution of the cycle C to the summation of weights of partisan permutations on $[n]$:

$$x^m \beta + y^m \alpha. \tag{3.8}$$

Noting that the equalities (3.6) and (3.7) can be easily justified by an involutional argument. Comparing (3.3) and (3.8) completes the combinatorial proof of (3.1). ∎

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