The Limiting Distribution of the q-Derangement Numbers

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Abstract

We prove the normality of the limiting distribution of the coefficients of the q-derangement numbers of type B based on the formula of Foata and Han that contains a parameter z. Setting the parameter z to zero, we are led to the case of ordinary q-derangement numbers. For $z = 1$, we obtain the normality of the distribution of the coefficients of the usual q-derangement numbers of type B.

Keywords: limiting distribution, q-derangement number, flag major index, moment generating function

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1 Introduction

In this paper, we study the limiting distribution of the major index over derangements. Feller [13] proved that the major index is asymptotically normal over the symmetric group. Fulman [15] studied the normal distribution of the major index on conjugacy classes of the symmetric group with large cycles. For the major index over derangements, we prove the normality of the limiting distribution by using Curtiss' theorem. In fact, based on a bivariate generating function of Foata and Han [14] on the q-derangement numbers of type B with a parameter z, we derive the normality of the limiting distribution of the coefficients of these polynomials. Setting $z = 0$ and substituting q^2 by q, we get the normality of the limiting distribution of the major index over derangements. For the case $z = 1$, we get the normality of the limiting distribution of the major index over derangements of type B.

Let $\mathfrak{S}_n$ (resp. $\mathscr{D}_n$) denote the set of permutations (resp. derangements) of $[n] = \{1, 2, \ldots, n\}$. The major index of a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ is defined by

$$\operatorname{maj}\pi = \sum_{\pi_i > \pi_{i+1}} i.$$

The notions of major index and derangements can be extended to signed permutations. A signed permutation or a type B permutation of $[n]$ is a word $\pi_1\pi_2\cdots\pi_n$ such that

$|\pi_1||\pi_2|\cdots|\pi_n|$ is a permutation of $[n]$, and each π_i belongs to the set $\{\pm 1, \pm 2, \ldots, \pm n\}$, see Björner and Brenti [5]. Let $\mathfrak{S}_n^B$ be the set of type B permutations of $[n]$, and let $\mathscr{D}_n^B$ denote the set of type B derangements of $[n]$, namely,

$$\mathscr{D}_n^B = \left\{ \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n^B \,\middle|\, \pi_i \neq i,\ i = 1, 2, \ldots, n \right\}.$$

Adin and Roichman [3] defined the flag major index of a signed permutation π by

$$\operatorname{fmaj}\pi = 2\operatorname{maj}\pi + \operatorname{neg}\pi, \tag{1.1}$$

where $\operatorname{maj}\pi$ (resp. $\operatorname{neg}\pi$) denotes the major index of π (resp. the number of negative elements in π), see also Adin, Brenti and Roichman [2].

Foata and Han [14] obtained the following formula

$$\sum_{\pi\in\mathscr{D}_n^B} q^{\operatorname{fmaj}\pi} z^{\operatorname{neg}\pi} = \sum_{k=0}^{n} (-1)^k q^{k(k-1)} (1+zq)^{n-k} \frac{[n]_{q^2}!}{[k]_{q^2}!}, \tag{1.2}$$

where $[0]_q = [0]_q! = 1$, and for $n \geq 1$, $[n]_q = \sum_{i=0}^{n-1} q^i$ and $[n]_q! = \prod_{i=1}^{n} [i]_q$.

When $z = 1$, (1.2) reduces to the generating function of the q-derangement numbers of type B,

$$\sum_{\pi\in\mathscr{D}_n^B} q^{\operatorname{fmaj}\pi} = \sum_{k=0}^{n} (-1)^k q^{k(k-1)} \frac{[2n]_q!!}{[2k]_q!!}, \tag{1.3}$$

where $[2n]_q!! = [2n]_q[2n-2]_q\cdots[2]_q$. The formula (1.3) is due to Chow [10], see also Chow and Gessel [11]. Setting $z = 0$ and substituting q^2 with q, (1.2) becomes the formula for the q-derangement numbers,

$$\sum_{\pi\in\mathscr{D}_n} q^{\operatorname{maj}\pi} = \sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \frac{[n]_q!}{[k]_q!}. \tag{1.4}$$

This formula was obtained by Gessel [17], see also Gessel and Reutenauer [18]. Combinatorial proofs of (1.4) have been found by Wachs [21], and Chen and Xu [9]. Brown [7] showed that the q-derangement numbers are the multiplicities of the eigenvalues for the q-analogue of the Tsetlin library.

In this paper, we consider the limiting distribution of the coefficients of the polynomial (1.2) in q while z is considered as a parameter. Below is the main result of this paper.

Theorem 1.1 *The limiting distribution of the coefficients of the polynomial (1.2) in q is normal for any real number $z \neq -1$.*

To prove Theorem 1.1, we first compute the expectation E_n and the variance σ_n^2 of the coefficients of (1.2). Based on the asymptotic expression of E_n and σ_n^2, we deduce Theorem 1.1 by using Curtiss' theorem [12]. For special values of z, we obtain the normal limiting distributions of coefficients of the ordinary and type B q-derangement numbers.

Corollary 1.2 *Let E_n^B (reps. σ_n^B) be the expectation (resp. standard deviation) of the flag major index of derangements of type B_n. Then the distribution of the random variable $\frac{\mathrm{fmaj}-E_n^B}{\sigma_n^B}$ converges to the standard normal distribution as $n \to \infty$.*

Corollary 1.3 *Let E_n^A (reps. σ_n^A) be the expectation (resp. standard deviation) of the major index of derangements of $[n]$. Then the distribution of the random variable $\frac{\mathrm{maj}-E_n^A}{\sigma_n^A}$ converges to the standard normal distribution as $n \to \infty$.*

2 The expectation and variance

In this section, we compute the expectation and the variance of the coefficients of the polynomial (1.2). Throughout this paper, we assume that $z \neq -1$. Given a polynomial $f(q)$, the expectation E and the variance σ^2 of the coefficients are given by

$$E = \frac{f'(1)}{f(1)}, \quad \sigma^2 = \frac{f''(1)}{f(1)} + E - E^2.$$

See, for example, Harper [16], and Carlitz, Kurtz, Scoville and Stackelberg [8]. On the other hand, the expectation and variance are determined by the moment generating function $M(x) = f(e^x)/f(1)$. In this way, we have

$$E = [x]M(x), \quad \sigma^2 = 2[x^2]M(x) - E^2, \tag{2.1}$$

where $[x^i]M(x)$ denotes the coefficient of x^i in $M(x)$.

Let D_n be the sum of the coefficients of (1.2), that is,

$$D_n = \sum_{k=0}^{n} (-1)^k (1+z)^{n-k} \frac{n!}{k!}. \tag{2.2}$$

In particular, for $z = 0$, D_n becomes the number of derangements of $[n]$. Meanwhile, for $z = 1$, D_n reduces to the number of B_n-derangements. In view of (1.2), the moment generating function equals

$$M_n(x) = \frac{1}{D_n} \sum_{k=0}^{n} (-1)^k e^{k(k-1)x} (1+ze^x)^{n-k} \frac{[n]_{e^{2x}}!}{[k]_{e^{2x}}!}. \tag{2.3}$$

We may assume that

$$M_n(x) = \frac{1}{D_n} \sum_{k=0}^{n} (-1)^k (1+z)^{n-k} \frac{n!}{k!} (1 + c_1 x + c_2 x^2) + \text{higher terms}. \tag{2.4}$$

We find

$$c_1 = \binom{n}{2} + \binom{k}{2} + \frac{z(n-k)}{1+z},$$

$$c_2 = 2\binom{n}{2}\binom{k}{2} + \frac{z(n-k)}{1+z}\left(\binom{n}{2} + \binom{k}{2} + \frac{1+z(n-k)}{2(1+z)}\right) + c_0,$$

where

$$c_0 = \frac{n-k}{72}\left(9n^3 - 14n^2 + 9kn^2 + 15n + 4kn - 9k^2n - 10 - 3k + 22k^2 - 9k^3\right).$$

By (2.1), we obtain the expectation E_n and the variance σ_n^2 of the coefficients of (1.2).

Theorem 2.1 *We have*

$$E_n = \frac{n^2}{2} + \frac{(z-1)n}{2(1+z)} + \frac{1+2z}{2(1+z)^2} + \frac{(-1)^n(n+zn-2z-1)}{2(1+z)^2 D_n}, \tag{2.5}$$

$$\sigma_n^2 = \frac{n^3}{9} + \frac{n^2}{6} - \frac{(5-8z+5z^2)n}{18(1+z)^2} - \frac{8+9z+9z^2}{9(1+z)^3} + \frac{(-1)^n}{D_n}c_3 + \frac{1}{D_n^2}c_4, \tag{2.6}$$

where

$$c_3 = \frac{n^3}{4(1+z)} - \frac{(4+31z)n^2}{36(1+z)^2} - \frac{(46+87z+14z^2)n}{36(1+z)^3} + \frac{41+104z+108z^2+36z^3}{36(1+z)^4},$$

$$c_4 = -\frac{(n+zn-2z-1)^2}{4(1+z)^4}.$$

In fact, in order to deduce (2.5) and (2.6), we need recurrence relations concerning the numbers D_n. These relations can be derived from the formulas (6.9) and (6.10) of Foata and Han [14] by setting $q = 1$, that is,

$$\begin{aligned} D_n &= (1+z)nD_{n-1} + (-1)^n, \\ D_n &= (zn+n-1)D_{n-1} + (1+z)(n-1)D_{n-2}, \end{aligned} \tag{2.7}$$

with $D_0 = 1$ and $D_1 = z$. As far as the expectation is concerned, we may rewrite c_1 as

$$c_1 = \frac{1}{2}k(k-1) - \frac{z}{1+z}k + \left(\frac{zn}{1+z} + \frac{n(n-1)}{2}\right).$$

Moreover, we can express $[x]M_n(x)$ in terms of D_n, D_{n-1} and D_{n-2}. With the aid of (2.7), we arrive at (2.5). The expression (2.6) can be derived in the same manner.

We further consider the asymptotic behaviors of E_n and σ_n^2. From (2.2) we see that

$$D_n = n!(1+z)^n\left(e^{-\frac{1}{1+z}} - \sum_{k=n+1}^{\infty}\frac{(-1)^k}{k!\,(1+z)^k}\right),$$

where $z \neq -1$. By the estimate

$$\left| \sum_{k=n+1}^{\infty} \frac{(-1)^k}{k!\,(1+z)^k} \right| \leq \frac{1}{(n+1)!\,(1+z)^{n+1}}$$

of the remainder, we deduce the asymptotic formula

$$D_n \sim n!\,(1+z)^n e^{-\frac{1}{1+z}}. \tag{2.8}$$

This yields the asymptotic expansions of E_n and σ_n^2.

Corollary 2.2 *As $n \to \infty$, we have*

$$E_n \sim \frac{n^2}{2} + \frac{(z-1)n}{2(1+z)} + \frac{1+2z}{2(1+z)^2},$$

$$\sigma_n^2 \sim \frac{n^3}{9} + \frac{n^2}{6} - \frac{(5-8z+5z^2)n}{18(1+z)^2} - \frac{8+9z+9z^2}{9(1+z)^3}.$$

3 The limiting distribution

It is well-known that the moment generating function of a random variable determines its distribution, see Curtiss [12] or Sachkov [20]. In particular, if the moment generating function $M_n(x)$ of a random variable ξ_n has the limit

$$\lim_{n\to\infty} M_n(x) = e^{\frac{x^2}{2}},$$

then ξ_n has as a standard normal distribution as $n \to \infty$.

The moment generating function of the normalized random variable $\frac{\mathrm{fmaj}-E_n}{\sigma_n}$ is

$$e^{\frac{-tE_n}{\sigma_n}} M_n\left(\frac{t}{\sigma_n}\right).$$

So Theorem 1.1 is valid if we can show that

$$\lim_{n\to\infty} e^{\frac{-tE_n}{\sigma_n}} M_n\left(\frac{t}{\sigma_n}\right) = e^{\frac{t^2}{2}}. \tag{3.1}$$

To prove the above relation (3.1), we use an alternative expression for $M_n(x)$ in terms of the Bernoulli numbers. The n-th Bernoulli number, denoted B_n, is defined by

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}. \tag{3.2}$$

Let us recall the following relation, see Mcintosh [19],

$$1 - e^{-x} = x \cdot \exp\left(\sum_{k=1}^{\infty} \frac{B_n\, x^k}{k \cdot k!}\right). \tag{3.3}$$

Lemma 3.1 *We have*

$$M_n(x) = \frac{n!}{D_n} \exp\left(\binom{n}{2}x + \sum_{i=1}^{\infty} \frac{B_{2i}\,(2x)^{2i}}{(2i)\,(2i)!} \sum_{j=1}^{n} \left(j^{2i} - 1\right)\right) \sum_{k=0}^{n} \frac{(-1)^k (1+ze^x)^{n-k}}{[k]_{e^{-2x}}!}.$$

Proof. In view of (3.3), we have for any $j \geq 1$,

$$1 - e^{xj} = -xj \cdot \exp\left(\frac{xj}{2} + \sum_{i=1}^{\infty} \frac{B_{2i}(xj)^{2i}}{(2i)\,(2i)!}\right).$$

It follows that

$$[n]_{e^{2x}}! = n! \cdot \exp\left(\binom{n}{2}x + \sum_{i=1}^{\infty} \frac{B_{2i}\,(2x)^{2i}}{(2i)\,(2i)!} \sum_{j=1}^{n} \left(j^{2i} - 1\right)\right). \tag{3.4}$$

Note that

$$\frac{e^{k(k-1)x}}{[k]_{e^{2x}}!} = \frac{1}{[k]_{e^{-2x}}!}. \tag{3.5}$$

Hence the proof is complete by substituting (3.4) and (3.5) into (2.3). ∎

Because of (3.1), it suffices to show that the limit of

$$\frac{n!}{D_n} \exp\left(\binom{n}{2}\frac{t}{\sigma_n} - \frac{tE_n}{\sigma_n} + \sum_{i=1}^{\infty} \frac{B_{2i}\,(2t)^{2i}}{(2i)\,(2i)!\sigma_n^{2i}} \sum_{j=1}^{n} \left(j^{2i} - 1\right)\right) \sum_{k=0}^{n} \frac{(-1)^k (1+ze^{\frac{t}{\sigma_n}})^{n-k}}{[k]_{e^{-\frac{2t}{\sigma_n}}}!} \tag{3.6}$$

as $n \to \infty$, equals $e^{\frac{t^2}{2}}$. We shall deal with the limits of the factors in the above expression. To compute the factor containing the Bernoulli numbers, we need the following asymptotic formula

$$|B_{2n}| \sim 4\sqrt{\pi n}\left(\frac{n}{\pi e}\right)^{2n}, \tag{3.7}$$

see, for example, Abramowitz and Stegun [1, p. 805] and Alzer [4].

Lemma 3.2 *For any real number t that is bounded, we have*

$$\lim_{n\to\infty} \sum_{i=2}^{\infty} \frac{B_{2i}\,t^{2i}}{(2i)\,(2i)!\,\sigma_n^{2i}} \sum_{j=1}^{n} \left(j^{2i} - 1\right) = 0, \tag{3.8}$$

where B_{2i} are the Bernoulli numbers.

Proof. Suppose that t is bounded by $|t| < M$. Let α, β and γ be three constants such that $\alpha > 1$, $\beta > 9$, and $0 < \gamma < 1/2$. Let N be a fixed integer satisfying the following three conditions:

(i) $n+1 < \alpha n$ for any $n > N$;

(ii) $\beta\,\sigma_n^2 > n^3$ for any $n > N$;

(iii) $2\pi N^{\gamma/2} > M\alpha\sqrt{\beta}$.

By Corollary 2.2, the existence of such N is obvious. Let $i \geq 2$ and $n > N$. By (i), we have

$$\sum_{j=1}^{n}\left(j^{2i}-1\right) < \int_{1}^{n+1}\left(t^{2i}-1\right)dt = \frac{(n+1)^{2i+1}-1}{2i+1} - n < \frac{(\alpha n)^{2i+1}}{5}.$$

Using the above inequality and the condition (ii), we find that

$$\frac{1}{\sigma_n^{2i}}\sum_{j=1}^{n}\left(j^{2i}-1\right) < \frac{\alpha}{5}\frac{\left(\alpha\sqrt{\beta}\right)^{2i}}{n^{i-1}}.$$

Moreover, since

$$\frac{1}{n^{i-1}} = \frac{n^{1-i(1-\gamma)}}{n^{\gamma i}} < \frac{n^{2\gamma-1}}{N^{\gamma i}},$$

we see that

$$\left|\lim_{n\to\infty}\sum_{i=2}^{\infty}\frac{B_{2i}\,t^{2i}}{(2i)\,(2i)!\,\sigma_n^{2i}}\sum_{j=1}^{n}\left(j^{2i}-1\right)\right| \leq \frac{\alpha}{5}\sum_{i=2}^{\infty}\frac{|B_{2i}|t^{2i}}{(2i)\,(2i)!}\frac{(\alpha\sqrt{\beta})^{2i}}{N^{\gamma i}}\lim_{n\to\infty}n^{2\gamma-1}. \tag{3.9}$$

The radius of convergence of the series in t on the right hand side of (3.9) equals

$$\lim_{i\to\infty}\left(\frac{|B_{2i}|}{(2i)\,(2i)!}\frac{\left(\alpha\sqrt{\beta}\right)^{2i}}{N^{\gamma i}}\right)^{-\frac{1}{2i}}.$$

Using (3.7), we deduce that the above radius equals

$$\frac{2\pi N^{\gamma/2}}{\alpha\sqrt{\beta}},$$

which is larger than the bound M of t because of the condition (iii). This proves the convergence of the series in t on the right hand side of (3.9). Since $2\gamma-1<0$,

$$\lim_{n\to\infty}n^{2\gamma-1} = 0.$$

Thus (3.8) follows from (3.9). This completes the proof. ∎

In order to evaluate the factor of (3.6) that contains z, that is,

$$\sum_{k=0}^{n}\frac{(-1)^k(1+ze^{\frac{t}{\sigma_n}})^{n-k}}{[k]_{e^{-\frac{2t}{\sigma_n}}}!},$$

we need Tannery's theorem, see, for example, Bromwich [6].

Theorem 3.3 (Tannery) *Let $\{v_k(n)\}_{k\geq 0}$ be an infinite series satisfying the following two conditions:*

(i) For any fixed k, there holds $\lim_{n\to\infty} v_k(n) = w_k$;

(ii) For any non-negative integer k, $|v_k(n)| \leq M_k$, where M_k is independent of n and the series $\sum_{k\geq 0} M_k$ is convergent.

Then

$$\lim_{n\to\infty} \sum_{k=0}^{m(n)} v_k(n) = \sum_{k=0}^{\infty} w_k,$$

where $m(n)$ is an increasing integer-valued function which trends steadily to infinity as n does.

The limit of the factor containing z can be determined by the following lemma.

Lemma 3.4 *For any real number t that is bounded by $|t| \leq M$, we have*

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{(-1)^k}{[k]_{e^{-2t/\sigma_n}}!\,(1+ze^{\frac{t}{\sigma_n}})^k} = e^{-\frac{1}{1+z}}. \tag{3.10}$$

Proof. We apply Tannery's theorem. Set $m(n) = n$ and

$$v_k(n) = \frac{(-1)^k}{[k]_{e^{-2t/\sigma_n}}!\,(1+ze^{\frac{t}{\sigma_n}})^k}.$$

It is clear from Corollary 2.2 that

$$w_k = \lim_{n\to\infty} v_k(n) = \frac{(-1)^k}{k!\,(1+z)^k},$$

and therefore the right hand side of (3.10) coincides with $\sum_{k=0}^{\infty} w_k$. By virtue of Tannery's theorem, it suffices to find an upper bound M_k for

$$|v_k(n)| = \left| \frac{1}{[k]_{e^{-2t/\sigma_n}}!\,(1+ze^{\frac{t}{\sigma_n}})^k} \right|,$$

such that M_k is independent of n and $\sum_{k=0}^{\infty} M_k$ converges. Note that we can always assume that n is sufficiently large. Since $1+ze^{\frac{t}{\sigma_n}} \to 1+z$ as $n\to\infty$, we see that there exists a constant $c \neq 0$, say, $c = \frac{|1+z|}{2}$, satisfying

$$|1+ze^{\frac{t}{\sigma_n}}| \geq c. \tag{3.11}$$

For $t \leq 0$, we have $e^{-\frac{t}{\sigma_n}} \geq 1$ and thus $[k]_{e^{-2t/\sigma_n}}! \geq k!$. Using (3.11), we get the following upper bound for $|v_k(n)|$

$$M_k = \frac{1}{k!\, c^k},$$

which is independent of n. Clearly, M_k satisfies the convergence condition.

For $t \geq 0$, Corollary 2.2 implies that σ_n has a positive lower bound as n runs over all positive integers and so does $e^{-\frac{2t}{\sigma_n}}$. Suppose that $e^{-\frac{2t}{\sigma_n}} \geq c_t \in (0, 1]$. Since the function $e^{-\frac{2t}{\sigma_n}}$ is continuous in t and t is bounded, there exists a constant $c' \in (0, 1]$ independent of t such that $e^{-\frac{2t}{\sigma_n}} \geq c'$ for all $|t| \leq M$. Hence for any $k \geq 1$,

$$[k]_{e^{-2t/\sigma_n}}! = \prod_{j=2}^{k} (1 + e^{-2t/\sigma_n} + \cdots + e^{-2(j-1)t/\sigma_n}) \geq (1 + c')^{k-1}.$$

Again, it follows from (3.11) that $|v_k(n)|$ has an upper bound

$$M'_k = \frac{1}{c^k (1 + c')^{k-1}}.$$

It is easy to check that M'_k satisfies the convergence condition. This completes the proof. ∎

Combining (2.8), Corollary 2.2, Lemma 3.2, and Lemma 3.4, we deduce that the limit of the sum (3.6) equals $e^{\frac{t^2}{2}}$ as $n \to \infty$. So the proof of Theorem 1.1 is complete.

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