Short Papers

Discrete-Time Multivariable Adaptive Control

GRAHAM C. GOODWIN, MEMBER, IEEE, PETER J. RAMADGE, AND PETER E. CAINES, MEMBER, IEEE

Abstract—**This paper establishes global convergence for a class of adaptive control algorithms applied to discrete-time multiinput multioutput deterministic linear systems. It is shown that the algorithms will ensure that the system inputs and outputs remain bounded for all time and that the output tracking error converges to zero.**

I. Introduction

A long-standing problem in control theory has been the question of the existence of simple, globally convergent adaptive control algorithms. By this we mean algorithms which, for all initial system and algorithm states, cause the outputs of a given linear system to asymptotically track a desired output sequence, and achieve this with a bounded-input sequence.

There is a considerable amount of literature on continuous-time deterministic adaptive control algorithms [1]. However, it is only recently that global stability and convergence of these algorithms has been studied under general assumptions. Much interest was generated by the innovative configuration proposed by Monopoli [2] whereby the feedback gains were directly estimated and an augmented error signal and auxiliary input signals were introduced to avoid the use of pure differentiators in the algorithm. Unfortunately, as pointed out in [3] the arguments given in [2] concerning stability are incomplete. New proofs for related algorithms have recently appeared [4], [5]. In [4] Narendra and Valavani treat the case where the difference in orders between the numerator and denominator of the system transfer function (relative degree) is less than or equal to two. In [5], Feuer and Morse propose a solution for general linear systems without constraints on the relative degree. The algorithms in [5] use the augmented error concept and auxiliary inputs as in [2]. The Feuer and Morse result seems to be the most general to date for single-input single-output continuous-time systems. However, these results are technically involved and cannot be directly applied to the discrete-time case.

There has also been interest in discrete-time adaptive control for both the deterministic and stochastic case. This area has particular relevance in view of the increasing use of digital technology in control applications [6], [7].

Ljung [8], [9] has proposed a general technique for analyzing convergence of discrete-time stochastic adaptive algorithms. However, in this analysis a question which is yet to be resolved concerns the boundedness of the system variables. For one particular algorithm [10], it has been argued in [11] that the algorithm possess the property that the sample mean-square output is bounded whenever the sample mean-square of the noise is bounded. However, the general question of stability remains unanswered for stochastic adaptive algorithms.

The study of discrete-time deterministic algorithms is of independent interest but also provides insight into stability questions in the stochastic case [12], [15]. Recent work by Ionescu and Monopoli [13] has been concerned with the extension of the results in [2] to the discrete-time case. As for the continuous case, the augmented error method is used.

In this paper we present new results related to discrete-time deterministic adaptive control. Our approach differs from previous work in several major respects although certain aspects of our approach are inspired by the work of Feuer and Morse [5].

The analysis presented here does not rely upon the use of augmented errors or auxiliary inputs. Moreover, the algorithms have a very simple structure and are applicable to multiple-input multiple-output systems with rather general assumptions.

The paper presents a general method of analysis for discrete-time deterministic adaptive control algorithms. The method is illustrated by establishing global convergence for three simple algorithms. For clarity of presentation, we shall first treat a simple single-input single-output algorithm in detail. The results will then be extended to other single-input single-output algorithms including those based on recursive least squares. Finally, the extension to multiple-input multiple-output systems will be presented.

Since the results in this paper were presented a number of other authors [16]–[18] have presented related results for discrete-time deterministic adaptive control algorithms.

II. Problem Statement

In this paper we shall be concerned with the adaptive control of linear time-invariant finite-dimensional systems having the following state space representation:

$$x(t+1)=Ax(t)+Bu(t); \qquad x(0)=x_0 \tag{2.1}$$

$$y(t)=Cx(t) \tag{2.2}$$

where $x(t)$, $u(t)$, $y(t)$ are the $n\times 1$ state vector, $r\times 1$ input vector, and $m\times 1$ output vector, respectively.

A standard result is that the system (2.1), (2.2) can also be represented in matrix fraction, or ARMA, form as

$$A(q^{-1})y(t)=\begin{bmatrix} q^{-d_{11}}B_{11}(q^{-1}) & \cdots & q^{-d_{1r}}B_{1r}(q^{-1}) \\ \vdots & & \vdots \\ q^{-d_{m1}}B_{m1}(q^{-1}) & \cdots & q^{-d_{mr}}B_{mr}(q^{-1}) \end{bmatrix} u(t) \tag{2.3}$$

with appropriate initial conditions. In (2.3), $A(q^{-1})$, $B_{ij}(q^{-1})$ ($i=1,\cdots,m$; $j=1,\cdots,r$) denote scalar polynomials in the unit delay operator q^{-1} and the factors $q^{-d_{ij}}$ represent pure time delays.

Note that it is not assumed that the system (2.1), (2.2) is completely controllable or completely observable, nor is it assumed that (2.3) is irreducible. The system will be required, however, to satisfy the conditions of Lemma 3.2.

It is assumed that the coefficients in the matrixes A, B, C in (2.1), (2.2) are unknown and that the state $x(t)$ is not directly measurable. A feedback control law is to be designed to stabilize the system and to cause the output, $\{y(t)\}$, to track a given reference sequence $\{y^*(t)\}$.

Manuscript received November 30, 1978; revised June 11, 1979 and November 26, 1979. Paper recommended by K. S. Narendra, Past Chairman of the Adaptive, Learning Systems, Pattern Recognition Committee. This work was supported in part by the Australian Research Grants Committee and the Joint Services Electronics Program under Contract N00014-75-C-0648. The work of G. C. Goodwin was supported in part by a Fulbright Grant and the Division of Applied Sciences, Harvard University, Cambridge, MA.

G. C. Goodwin is with the Department of Electrical Engineering, University of Newcastle, Newcastle, N.S.W., Australia.

P. J. Ramadge is with the Department of Electrical Engineering, University of Toronto, Toronto, Ont., Canada.

P. E. Caines is with the Division of Applied Sciences, Harvard University, Cambridge, MA 02138.

0018-9286/80/0600-0449$00.75 ©1980 IEEE

Specifically, we require $y(t)$ and $u(t)$ to be bounded uniformly in t, and

$$\lim_{t\to\infty} y_i(t)-y_i^*(t)=0 \qquad i=1,\cdots,m. \tag{2.4}$$

III. Key Technical Lemmas

Our analysis of discrete-time multivariable adaptive control algorithms will be based on the following technical results.

Lemma 3.1: If

$$\lim_{t\to\infty} \frac{s(t)^2}{b_1(t)+b_2(t)\sigma(t)^T\sigma(t)}=0 \tag{3.1}$$

where $\{b_1(t)\}$, $\{b_2(t)\}$, and $\{s(t)\}$ are real scalar sequences and $\{\sigma(t)\}$ is a real p-vector sequence; then subject to

1) uniform boundedness condition

$$0<b_1(t)<K<\infty \quad \text{and} \quad 0<b_2(t)<K<\infty \tag{3.2}$$

for all $t>0$ and

2) linear boundedness condition

$$\|\sigma(t)\|<C_1+C_2\max_{0<\tau<t}|s(\tau)| \tag{3.3}$$

where $0<C_1<\infty$, $0<C_2<\infty$, it follows that

$$\lim_{t\to\infty} s(t)=0 \tag{3.4}$$

and $\{\|\sigma(t)\|\}$ is bounded.

Proof: If $\{s(t)\}$ is a bounded sequence, then by (3.3) $\{\|\sigma(t)\|\}$ is a bounded sequence. Then by (3.2) and (3.1) it follows that

$$\lim_{t\to\infty} s(t)=0.$$

Now assume that $\{s(t)\}$ is unbounded. It follows that there exists a subsequence $\{t_n\}$ such that

$$\lim_{t_n\to\infty} |s(t_n)|=\infty$$

and

$$|s(t)|<|s(t_n)| \qquad \text{for } t<t_n. \tag{3.5}$$

Now along the subsequence $\{t_n\}$

$$\left|\frac{s(t_n)}{\left[b_1(t_n)+b_2(t_n)\sigma(t_n)^T\sigma(t_n)\right]^{1/2}}\right| > \frac{|s(t_n)|}{\left[K+K\|\sigma(t_n)\|^2\right]^{1/2}} \qquad \text{using (3.2)}$$

$$> \frac{|s(t_n)|}{K^{1/2}+K^{1/2}\|\sigma(t_n)\|}$$

$$> \frac{|s(t_n)|}{K^{1/2}+K^{1/2}[C_1+C_2|s(t_n)|]} \qquad \text{using (3.3) and (3.5).}$$

Hence,

$$\lim_{t_n\to\infty}\left|\frac{s(t_n)}{\left[b_1(t_n)+b_2(t_n)\sigma(t_n)^T\sigma(t_n)\right]^{1/2}}\right| > \frac{1}{K^{1/2}C_2}>0$$

but this contradicts (3.1) and hence the assumption that $\{s(t)\}$ is unbounded is false and the result follows. □

In order to use this lemma in proving global convergence of adaptive control algorithms it will be necessary to verify (3.1) (with $s(t)$ interpreted as the tracking error) and to check that assumptions (3.2) and (3.3) are satisfied.

The next lemma will be used to verify that the linear boundedness condition (3.3) is satisfied by an important class of linear time-invariant systems. This class corresponds to those linear time-invariant systems for which the control objective (2.4) can be achieved with a bounded-input sequence and for which the tracking error can be reduced to zero if the system parameters are known.

Lemma 3.2: For the system (2.3) *with $r=m$, and subject to*

$$\det\begin{bmatrix} z^{d_{11}-d_1}B_{11}(z) & \cdots & z^{d_{1m}-d_1}B_{1m}(z) \\ \vdots & & \vdots \\ z^{d_{m1}-d_m}B_{m1}(z) & \cdots & z^{d_{mm}-d_m}B_{mm}(z)\end{bmatrix}\neq 0 \tag{3.6}$$

for $|z|<1$ where

$$d_i=\min_{1<j<m} d_{ij} \qquad i=1,\cdots,m,$$

if

$$\max_{\substack{0<t<T\\1<i<m}} |y_i(t+d_i)|=m_2, \tag{3.7}$$

then there exist constants m_3, m_4 which are independent of T with $0<m_3<\infty$, $0<m_4<\infty$ such that

$$|u_i(t)|<m_2m_4+m_3 \qquad 0<t<T,\ i=1,\cdots,m.$$

Proof: The result is standard and simply follows from the fact that (3.6) ensures that the system has a stable inverse. □

In the remainder of the paper these results will be used to prove global convergence of a number of adaptive control algorithms. Sections IV–VII will be concerned with adaptive control of single-input single-output systems. Sections VIII and IX will extend these results to the multiple-input multiple-output case.

IV. Single-Input Single-Output Systems

It is well known that for the single-input single-output (SISO) case, the system output of (2.1), (2.2) can be described by

$$A(q^{-1})y(t)=q^{-d}B(q^{-1})u(t) \tag{4.1}$$

where $\{u(t)\}$, $\{y(t)\}$ denote the input and output sequences, respectively, and $A(q^{-1})$, $B(q^{-1})$ are polynomial functions of the unit delay operator q^{-1},

$$A(q^{-1})=1+a_1q^{-1}+\cdots+a_nq^{-n}$$

$$B(q^{-1})=b_0+b_1q^{-1}+\cdots+b_mq^{-m}; \qquad b_0\neq 0.$$

d represents the system time delay. The initial conditions of (2.1) are replaced by initial values of $y(t)$, $0>t>-n$, and $u(t)$, $-d>t>-d-m$.

The following assumptions will be made about the system.

Assumption Set 4:

a) d is known.

b) An upper bound for n and m is known.

c) $B(z)$ has all zeros strictly outside the closed unit disk. (This is necessary to ensure that the control objective can be achieved with a bounded-input sequence.)

We note that, by successive substitution, (4.1) can be rewritten as

$$y(t+d)=\alpha(q^{-1})y(t)+\beta(q^{-1})u(t) \tag{4.2}$$

where

$$\alpha(q^{-1})=\alpha_0+\alpha_1q^{-1}+\cdots+\alpha_{n-1}q^{-n+1}$$

$$\beta(q^{-1})=\beta_0+\beta_1q^{-1}+\cdots+\beta_{m+d-1}q^{-m-d+1}; \qquad \beta_0\neq 0.$$

As previously stated, the control objective is to achieve

$$\lim_{t\to\infty} [y(t)-y^*(t)]=0 \tag{4.3}$$

where $\{y^*(t)\}$ is a reference sequence. It is assumed that $\{y^*(t)\}$ is known *a priori* and that

$$|y^*(t)|<m_1<\infty \qquad \text{for all } t. \tag{4.4}$$

V. SISO Projection Algorithm I

Let θ_0 be the vector of system parameters (dimension $p=n+m+d$),

$$\theta_0^T=(\alpha_0,\cdots,\alpha_{n-1},\beta_0,\cdots,\beta_{m+d-1}). \tag{5.1}$$

Then (4.2) can be written

$$y(t+d)=\varphi(t)^T\theta_0 \tag{5.2}$$

where

$$\varphi(t)^T=(y(t),\cdots,y(t-n+1),u(t),\cdots,u(t-m-d+1)). \tag{5.3}$$

Now define the output tracking error as

$$\begin{aligned} e(t+d)&=y(t+d)-y^*(t+d)\\ &=\varphi(t)^T\theta_0-y^*(t+d). \end{aligned} \tag{5.4}$$

By choosing $\{u(t)\}$ to satisfy

$$\varphi(t)^T\theta_0=y^*(t+d) \tag{5.5}$$

it is evident that the tracking error is identically zero. However, since θ_0 is unknown, we replace (5.5) by the following adaptive algorithm:

$$\begin{aligned}\hat\theta(t)=\hat\theta(t-1)+a(t)\varphi(t-d)[1+\varphi(t-d)^T\varphi(t-d)]^{-1}\\ \cdot\left[y(t)-\varphi(t-d)^T\hat\theta(t-1)\right]\end{aligned} \tag{5.6}$$

$$\varphi(t)^T\hat\theta(t)=y^*(t+d) \tag{5.7}$$

where $\hat\theta(t)$ is a p-vector of reals depending on an initial vector $\hat\theta(0)$ and on $y(\tau)$, $0<\tau<t$, $u(\tau)$, $0<\tau<t-d$ via (5.6), and where the gain constant $a(t)$ is computed as follows:

$$\begin{aligned} a(t)&=1 \quad \text{if } [(n+1)\text{th component of right-hand side of (5.6) evaluated using } a(t)=1]\neq 0;\\ &=\gamma \quad \text{otherwise where } \gamma \text{ is a constant in the interval } (\epsilon,2-\epsilon),\ \gamma\neq 1 \text{ and } 0<\epsilon<1. \end{aligned} \tag{5.8}$$

This choice of gain constant prevents the computed coefficient of $u(t)$ in (5.7) being zero. We also remark that the purpose of the coefficient 1 in the term $[1+\phi(t-d)^T\phi(t-d)]^{-1}$ of (5.6) is solely to avoid division by zero. Any positive constant could be used in place of the 1.

Apart from the above modification, the algorithm (5.6) is an orthogonal projection of $\hat\theta(t-1)$ onto the hypersurface $y(t)-\varphi(t-d)^T\theta=0$.

In the analysis of this algorithm, we will first show that the Euclidean norm of the vector $\tilde\theta(t)=\hat\theta(t)-\theta_0$ is a nonincreasing function along the trajectories of the algorithm. This leads to a characterization of the limiting behavior of the algorithm which will allow us to use Lemma 3.1 to establish global convergence.

Lemma 5.1: Along the solutions of (5.6), (5.7),

$$\|\tilde\theta(t)\|^2-\|\tilde\theta(t-1)\|^2<0, \qquad t>0 \tag{5.9}$$

and

$$\lim_{t\to\infty}\frac{\varphi(t)^T\tilde\theta(t)}{[1+\varphi(t)^T\varphi(t)]^{1/2}}=0. \tag{5.10}$$

Proof: Using the definition of $\tilde\theta(t)$, (5.6) may be rewritten as

$$\begin{aligned}\tilde\theta(t)=\tilde\theta(t-1)-a(t)\varphi(t-d)[1+\varphi(t-d)^T\varphi(t-d)]^{-1}\\ \cdot\varphi(t-d)^T\tilde\theta(t-1).\end{aligned} \tag{5.11}$$

Hence,

$$\begin{aligned}\|\tilde\theta(t)\|^2-\|\tilde\theta(t-1)\|^2=a(t)\left[-2+a(t)\frac{\varphi(t-d)^T\varphi(t-d)}{[1+\varphi(t-d)^T\varphi(t-d)]}\right]\\ \cdot\frac{\tilde\theta(t-1)^T\varphi(t-d)\varphi(t-d)^T\tilde\theta(t-1)}{[1+\varphi(t-d)^T\varphi(t-d)]}<0\end{aligned} \tag{5.12}$$

for all values of $\varphi(t-d)$ provided $0<a(t)<2$. This is satisfied by definition (5.8). Then, since $\|\tilde\theta(t)\|^2$ is a bounded nonincreasing function it converges. Setting

$$\epsilon(t)=-\varphi(t-d)^T\tilde\theta(t-1) \tag{5.13}$$

[and noting that

$$a(t)\left[-2+a(t)\frac{\varphi(t-d)^T\varphi(t-d)}{[1+\varphi(t-d)^T\varphi(t-d)]}\right]$$

is bounded away from zero, with $a(t)$ defined as in (5.8)] we conclude, from (5.12), that

$$\lim_{t\to\infty}\frac{\epsilon^2(t)}{[1+\varphi(t-d)^T\varphi(t-d)]}=0$$

and hence

$$\lim_{t\to\infty}\frac{\epsilon(t)}{[1+\varphi(t-d)^T\varphi(t-d)]^{1/2}}=0. \tag{5.14}$$

Now using (5.13) and (5.11) it follows that

$$\epsilon(t)=-\varphi(t-d)^T\tilde\theta(t-d)-\sum_{i=1}^{d-1}a(t-i)\cdot\frac{\varphi(t-d)^T\varphi(t-d-i)}{[1+\varphi(t-d-i)^T\varphi(t-d-i)]}\epsilon(t-i).$$

Then using (5.4) and (5.7) we have that

$$e(t)=-\varphi(t-d)^T\tilde\theta(t-d). \tag{5.15}$$

Hence,

$$\begin{aligned}\frac{\epsilon(t)}{[1+\varphi(t-d)^T\varphi(t-d)]^{1/2}}&=\frac{e(t)}{[1+\varphi(t-d)^T\varphi(t-d)]^{1/2}}\\ &\quad-\sum_{i=1}^{d-1}a(t-i)\frac{\varphi(t-d)^T}{[1+\varphi(t-d)^T\varphi(t-d)]^{1/2}}\\ &\quad\cdot\frac{\varphi(t-d-i)}{[1+\varphi(t-d-i)^T\varphi(t-d-i)]^{1/2}}\\ &\quad\cdot\frac{\epsilon(t-i)}{[1+\varphi(t-d-i)^T\varphi(t-d-i)]^{1/2}}.\end{aligned} \tag{5.16}$$

Now by the Cauchy–Schwarz inequality and the fact that $|a(t)|<2$

$$\begin{aligned}0&<\left|\frac{a(t-i)\varphi(t-d)^T}{[1+\varphi(t-d)^T\varphi(t-d)]^{1/2}}\frac{\varphi(t-d-i)}{[1+\varphi(t-d-i)^T\varphi(t-d-i)]^{1/2}}\cdot\frac{\epsilon(t-i)}{[1+\varphi(t-d-i)\varphi(t-d-i)]^{1/2}}\right|\\ &<\left|\frac{2\epsilon(t-i)}{[1+\varphi(t-d-i)^T\varphi(t-d-i)]^{1/2}}\right|.\end{aligned}$$

Then using (5.14) it follows that

$$\lim_{t\to\infty}\left|a(t-i)\frac{\varphi(t-d)^T}{[1+\varphi(t-d)^T\varphi(t-d)]^{1/2}}\cdot\frac{\varphi(t-d-i)}{[1+\varphi(t-d-i)^T\varphi(t-d-i)]^{1/2}}\cdot\frac{\epsilon(t-i)}{[1+\varphi(t-d-i)^T\varphi(t-d-i)]^{1/2}}\right|=0 \quad \text{for } i=1,2,\cdots,d-1. \tag{5.17}$$

Hence, using (5.16), (5.17), and (5.14)

$$\lim_{t\to\infty} \frac{e(t)}{[1+\varphi(t-d)^T\varphi(t-d)]^{1/2}}=0. \quad (5.18)$$

This establishes (5.10). □

Note that we do not prove, or claim, that $\hat{\theta}(t)$ converges to θ_0. However, the weaker condition (5.10) will be sufficient to establish convergence of the tracking error to zero and boundedness of the system inputs and outputs. These are the prime properties of concern in adaptive control.

Theorem 5.1: Subject to Assumptions 4a)–c); if the algorithm (5.6), (5.7) is applied to the system (2.1), (2.2) ($r=m=1$), then $\{y(t)\}$ and $\{u(t)\}$ are bounded and

$$\lim_{t\to\infty} [y(t)-y^*(t)]=0. \quad (5.19)$$

Proof: Lemma 5.1 ensures that condition (3.1) of Lemma 3.1 is satisfied, with $s(t)=e(t)$, the tracking error, and $\sigma(t)=\varphi(t-d)$ the vector defined by (5.3). Also $b_1(t)=1$, and $b_2(t)=1$. It follows that the uniform boundedness condition (3.2) is satisfied.

Assumption 4c) and Lemma 3.2 ensure that

$$|u(k-d)|<m_3+m_4 \max_{1<\tau<t} |y(\tau)| \qquad \text{for all } 1<k<t.$$

Therefore, using (5.3)

$$\|\varphi(t-d)\|<p\left\{m_3+[\max(1,m_4)] \max_{1<\tau<t} |y(\tau)|\right\}$$

but

$$|e(t)|>|y(t)|-|y^*(t)|>|y(t)|-m_1.$$

Hence,

$$\|\varphi(t-d)\|<p\left\{m_3+[\max(1,m_4)] \max_{1<\tau<t} [|e(\tau)|+m_1]\right\}$$
$$=C_1+C_2 \max_{1<\tau<t} |e(\tau)|; \qquad 0<C_1<\infty,\ 0<C_2<\infty$$

and it follows that the linear boundedness condition (3.3) is also satisfied.

The result now follows by Lemma 3.1 and by noting that boundedness of $\{\|\varphi(t)\|\}$ ensures boundedness of $\{|y(t)|\}$ and $\{|u(t)|\}$. □

VI. SISO Projection Algorithm II

In this section we present an algorithm differing from that of Section V in that the control law is estimated directly. This approach is adopted in [5], and essentially involves the factorization of β_0 from (4.2). A related procedure is used in the self-tuning regulator [10] where it is assumed that the value of β_0 is known.

An advantage of the algorithm is that the precautions required in Section V to avoid division by zero in the calculation of the input are no longer necessary. However, a disadvantage is that additional information is required; specifically, we need to know the sign of β_0 and an upper bound for its magnitude.

Factoring β_0 from (4.2) yields

$$y(t+d)=\beta_0(\alpha_0' y(t)+\cdots+\alpha_{n-1}' y(t-n+1)+u(t) + \beta_1' u(t-1)\cdots+\beta_{m+d-1}' u(t-m-d+1)). \quad (6.1)$$

Let

$$e(t+d)=y(t+d)-y^*(t+d)$$
$$=\beta_0\Big(u(t)+\alpha_0' y(t)\cdots\alpha_{n-1}' y(t-n+1)+\beta_1' u(t-1) \cdots+\beta_{m+d-1}' u(t-m-d+1)-\frac{1}{\beta_0}y^*(t+d)\Big)$$
$$=\beta_0\big(u(t)-\varphi(t)^T\theta_0'\big) \quad (6.2)$$

where

$$\varphi(t)^T=(-y(t)\cdots-y(t-n+1),-u(t-1)\cdots -u(t-m-d+1),y^*(t+d))$$
$$\theta_0'^T=\left(\alpha_0',\cdots,\alpha_{n-1}',\beta_1',\cdots,\beta_{n+d-1}',\frac{1}{\beta_0}\right).$$

It is evident that the tracking error can be made identically zero by choosing $\{u(t)\}$ such that

$$u(t)=\varphi(t)^T\theta_0'. \quad (6.3)$$

However, since θ_0' is unknown, the control law will be recursively estimated. The following adaptive algorithm will be considered:

$$\hat{\theta}(t)=\hat{\theta}(t-d)-\frac{1}{\hat{\beta}_0}\varphi(t-d)[1+\varphi(t-d)^T\varphi(t-d)]^{-1}e(t) \quad (6.4)$$

$$u(t)=\varphi(t)^T\hat{\theta}(t) \quad (6.5)$$

where $\hat{\beta}_0$ is a fixed constant and $\hat{\theta}(t)$ is a p-vector of reals depending on d initial values $\hat{\theta}(0),\cdots,\hat{\theta}(d-1)$ and on $y(\tau)$, $0<\tau<t$, $u(\tau)$, $0<\tau<t-d-1$ via (6.4). Note that (6.4) is actually d separate recursions interlaced. (It has recently been pointed out [18] that it is also possible to analyze a single recursion without interlacing using a different technique but the same general principals.)

The analysis of projection algorithm II has much in common with the analysis for projection algorithm I. We will therefore merely state the analogs of Lemma 5.1 and Theorem 5.1 for the algorithm (6.4), (6.5).

Lemma 6.1: Define

$$\tilde{\theta}(t)=\hat{\theta}(t)-\theta_0'. \quad (6.6)$$

□

Then $\|\tilde{\theta}(t+d)\|^2-\|\tilde{\theta}(t)\|^2\leq 0$ along with the solutions of (6.4) and (6.5) and

$$\lim_{t\to\infty} \frac{\varphi(t)^T\tilde{\theta}(t)}{[1+\varphi(t)^T\varphi(t)]^{1/2}}=0 \quad (6.7)$$

provided

$$0<\frac{\beta_0}{\hat{\beta}_0}<2.$$ □

Lemma 6.1 is used to prove Theorem 6.1 in the same manner that Lemma 5.1 is used to establish Theorem 5.1. We obtain the following theorem in this way.

Theorem 6.1: Subject to Assumptions 4a)–c) and for $0<\beta_0/\hat{\beta}_0<2$; if the algorithm (6.4), (6.5) is applied to the system (2.1), (2.2), then $\{y(t)\}$ and $\{u(t)\}$ are bounded and

$$\lim_{t\to\infty} [y(t)-y^*(t)]=0. \quad (6.6)$$

We note that the condition $0<\beta_0/\hat{\beta}_0<2$ has been previously conjectured [9], [10] in regard to stochastic self-tuning regulators using least squares. The condition can always be satisfied if the sign of β_0 and an upper bound for the magnitude of β_0 are known.

VII. Adaptive Control Using Recursive Least Squares

The wide-spread use of recursive least squares in parameter estimation indicates that it may find application in the adaptive control context. We treat the unit delay case of algorithm I with the projection (5.6) replaced by recursive least squares.

The adaptive control algorithm then becomes

$$\hat{\theta}(t)=\hat{\theta}(t-1)+\frac{a(t)P(t-2)\varphi(t-1)}{[1+a(t)\varphi(t-1)^TP(t-2)\varphi(t-1)]}\left[y(t)-\varphi(t-1)^T\hat{\theta}(t-1)\right] \quad (7.1)$$

$$P(t)=\left[I-\frac{P(t-1)\varphi(t)\varphi(t)^T a(t+1)}{1+\varphi(t)^T P(t-1)\varphi(t)a(t+1)}\right]P(t-1) \tag{7.2}$$

$$\varphi(t)^T\hat{\theta}(t)=y^*(t+1) \tag{7.3}$$

where $P(t)$ is a $p\times p$ matrix and the recursion (7.2) is assumed to be initialized with $P(-1)$ equal to any positive definite matrix.

The scalar $a(t)$ in (7.1), (7.2) plays the same role as in Section V and is required only to avoid the nongeneric possibility of division by zero in (7.3) when evaluating $u(t)$. Hence, $a(t)=1$ will almost always work and for $a(t)=1$ we observe that (7.1) and (7.2) are the standard recursive least squares algorithm. The sequence $\{a(t)\}$ may be chosen as in (5.8).

Lemma 7.1: Along with the solutions of (7.1), (7.2), (7.3) the function $V(t)=\tilde{\theta}(t)^T P(t-1)^{-1}\tilde{\theta}(t)$ *is a bounded, nonnegative, nonincreasing function and*

$$\lim_{t\to\infty}\frac{\varphi(t-1)^T\tilde{\theta}(t-1)}{\left[1+a(t)\varphi(t-1)^T P(t-2)\varphi(t-1)\right]^{1/2}}=0 \tag{7.4}$$

where $\tilde{\theta}(t)=\hat{\theta}(t)-\theta_0$.

Proof: From (7.1), (5.2),

$$\tilde{\theta}(t)=\tilde{\theta}(t-1)-\frac{a(t)P(t-2)\varphi(t-1)\varphi(t-1)^T\tilde{\theta}(t-1)}{\left[1+a(t)\varphi(t-1)^T P(t-2)\varphi(t-1)\right]}. \tag{7.5}$$

Then using (7.2),

$$\tilde{\theta}(t)=P(t-1)P(t-2)^{-1}\tilde{\theta}(t-1).$$

Thus,

$$P(t-1)^{-1}\tilde{\theta}(t)=P(t-2)^{-1}\tilde{\theta}(t-1). \tag{7.6}$$

Now defining $V(t)$ as $\tilde{\theta}(t-1)^T P(t-1)\tilde{\theta}(t-1)$ we have

$$V(t)-V(t-1)=\tilde{\theta}(t)^T P(t-1)^{-1}\tilde{\theta}(t)-\tilde{\theta}(t-1)^T P(t-2)^{-1}\tilde{\theta}(t-1).$$

Using (7.6)

$$\begin{aligned}V(t)-V(t-1)&=\left[\tilde{\theta}(t)-\tilde{\theta}(t-1)\right]^T P(t-2)^{-1}\tilde{\theta}(t-1)\\&=-a(t)\frac{\tilde{\theta}(t-1)^T\varphi(t-1)\varphi(t-1)^T\tilde{\theta}(t-1)}{\left[1+a(t)\varphi(t-1)^T P(t-2)\varphi(t-1)\right]}\end{aligned} \tag{7.7}$$

where we have used (7.5). It is clear from (7.7) that $V(t)$ is a bounded, nonnegative, nonincreasing function and hence converges.

Thus, from (7.7), and since $a(t)$ is bounded away from zero,

$$\lim_{t\to\infty}\frac{\tilde{\theta}(t-1)^T\varphi(t-1)\varphi(t-1)^T\tilde{\theta}(t-1)}{\left[1+a(t)\varphi(t-1)^T P(t-2)\varphi(t-1)\right]}=0.$$

Hence,

$$\lim_{t\to\infty}\frac{e(t)}{\left[1+a(t)\varphi(t-1)^T P(t-2)\varphi(t-1)\right]^{1/2}}=0 \tag{7.8}$$

where

$$e(t)=-\varphi(t-1)^T\tilde{\theta}(t-1)=y(t)-y^*(t). \tag{7.9}$$

□

Theorem 7.1: Subject to Assumptions 4a)–c) if the algorithm (7.1), (7.2), (7.3) is applied to the system (2.1), (2.2) ($r=m=1$), then $\{y(t)\}$, $\{u(t)\}$ are bounded and

$$\lim_{t\to\infty}\left[y(t)-y^*(t)\right]=0. \tag{7.10}$$

Proof: From Lemma 7.1

$$\lim_{t\to\infty}\frac{e(t)}{\left[1+a(t)\varphi(t-1)^T P(t-2)\varphi(t-1)\right]^{1/2}}=0. \tag{7.11}$$

Now

$$\frac{|e(t)|^2}{\left[1+a(t)\varphi(t-1)^T P(t-2)\varphi(t-1)\right]}>\frac{|e(t)|^2}{\left[1+2\|\varphi(t-1)\|^2(\lambda_{\max}[P(t-2)])\right]}. \tag{7.12}$$

Hence, from (7.11) and (7.12)

$$\lim_{t\to\infty}\frac{|e(t)|^2}{\left[1+2(\lambda_{\max}[P(t-2)])\|\varphi(t-1)\|^2\right]}=0. \tag{7.13}$$

This will be recognized as being condition (3.1) with $s(t)=e(t)$, $b_1(t)=1$, and $b_2(t)=2(\lambda_{\max}[P(t-2)])$.

To establish the uniform boundedness condition (3.2) we proceed as follows. From (7.2) and the matrix inversion lemma,

$$P(t)^{-1}=P(t-1)^{-1}+a(t)\varphi(t)\varphi(t)^T.$$

Hence,

$$\begin{aligned}x^T P(t)^{-1}x&>x^T P(t-1)x\\&>\lambda_{\min}[P(t-1)^{-1}]\|x\|^2 \qquad \text{for each } x\in\mathbb{R}^p.\end{aligned} \tag{7.14}$$

Now choose x as the eigenvector corresponding to the minimum eigenvalue of $[P(t)^{-1}]$.

Then from (7.14)

$$\lambda_{\min}[P(t)^{-1}]>\lambda_{\min}[P(t-1)^{-1}].$$

So $\lambda_{\min}[P(t)^{-1}]$ is a nondecreasing function bounded below by $\lambda_{\min}[P(-1)^{-1}]=K^{-1}>0$.

Hence from (7.13), $0<b_2(t)<2K$. This establishes condition (3.2).

The proof now proceeds as for Theorem 5.1. □

VIII. Multiple-Input Multiple-Output Systems

For the case $m=r>1$, the system (2.1), (2.2) can be represented in the form

$$\begin{bmatrix}A_1(q^{-1}) & & 0\\ & \ddots & \\ 0 & & A_m(q^{-1})\end{bmatrix}y(t)=\begin{bmatrix}q^{-d_{11}}B_{11}(q^{-1}) & \cdots & q^{-d_{1m}}B_{1m}(q^{-1})\\ \vdots & & \vdots\\ q^{-d_{m1}}B_{m1}(q^{-1}) & \cdots & q^{-d_{mm}}B_{mm}(q^{-1})\end{bmatrix}u(t) \tag{8.1}$$

where $A_k(q^{-1})$ and $B_{kl}(q^{-1})$ $1<k<m$, $1<l<m$ are scalar polynomials in the unit delay operator q^{-1} with nonzero constant coefficient.

Using the m identities $1=A_i(q^{-1})F_i(q^{-1})+q^{-d_i}G_i(q^{-1})$ where

$$F_i(q^{-1})=1+f_1^i q^{-1}\cdots+f_{d_i-1}^i q^{-d_i+1}$$

and

$$d_i=\min_{1<j<m}\{d_{ij}\}, \qquad i=1,\cdots,m,$$

(8.1) can be written

$$\begin{bmatrix}y_1(t+d_1)\\ \vdots\\ y_m(t+d_m)\end{bmatrix}=\begin{bmatrix}\alpha_1(q^{-1}) & 0\\ \vdots & \vdots\\ 0 & \alpha_m(q^{-1})\end{bmatrix}y(t)+\begin{bmatrix}\beta_{11}(q^{-1}) & \cdots & \beta_{1m}(q^{-1})\\ \vdots & & \vdots\\ \beta_{m1}(q^{-1}) & \cdots & \beta_{mm}(q^{-1})\end{bmatrix}u(t) \tag{8.2}$$

where

$$\alpha_i(q^{-1}) = G_i(q^{-1})$$

and

$$\beta_{ij}(q^{-1}) = F_i(q^{-1})B_{ij}(q^{-1})q^{d_i-d_{ij}}. \tag{8.3}$$

It can be seen that (8.2) consists of a set of multiple-input single-output (MISO) systems having a common input vector. The following assumptions will be made about the system.

Assumption Set 8:

a) $d_1, \cdots, d_m$ are known.

b) An upper bound for the order of each polynomial in (8.2) is known.

c) The system (8.1) satisfies condition

$$\det\begin{bmatrix} z^{d_{11}-d_1}B_{11}(z) & \cdots & z^{d_{1m}-d_1}B_{1m}(z) \\ z^{d_{m1}-d_m}B_{ml}(z) & \cdots & z^{d_{mm}-d_m}B_{mm}(z)\end{bmatrix} \neq 0 \qquad \text{for } |z| \leq 1.$$

Condition 8c) deserves comment.

First, for any output component y_i, $1 \leq i \leq m$, there exists at least one polynomial $q^{-d_{ij}}B_{ij}(q^{-1})$, $1 \leq j \leq m$, for which the power of q^{-1} associated with its (nonzero) leading coefficient is d_i. For each such polynomial the associated input $u_j(t)$ appears in $y_i(t+d_i)$ with the least possible delay d_i. Evaluated at $z=0$ condition 8c) requires the matrix of these leading coefficients to be nonsingular.

Second, the genericity of condition 8c) [for the model (8.1)] depends upon the initial parameterization of the system from which (8.1) is computed. This dependence is currently under investigation.

The control objective, as before, is to achieve

$$\lim_{t\to\infty} [y_i(t) - y_i^*(t)] = 0 \qquad i = 1, \cdots, m$$

where $y_i^*(t)$ is a reference sequence. It is assumed that each $\{y_i^*(t)\}$ is known *a priori* and that $|y_i^*(t)| \leq m_1 < \infty$ for all t, $i = 1, \cdots, m$.

IX. MIMO Adaptive Control

This section will be concerned with the multivariable versions of the adaptive control algorithms introduced in Sections V and VI. The multivariable version of the algorithm of Section VII also follows analogously.

A. MIMO Projection Algorithm I

Let θ_0^i be the vector of parameters in $\alpha_i(q^{-1})$ and $\beta_{i1}(q^{-1}) \cdots \beta_{im}(q^{-1})$. Then (8.2) may be written in the form

$$y_i(t+d_i) = \varphi_i(t)^T\theta_0^i, \qquad 1 \leq i \leq m \tag{9.1}$$

where

$$\varphi_i(t)^T = \left(y_i(t)\cdots y_i(t-n_i+1), u(t)^T, \cdots, u(t-m_i-d_i+1)^T\right).$$

Define

$$\begin{aligned} e_i(t+d_i) &= y_i(t+d_i) - y_i^*(t+d_i) \\ &= \varphi_i(t)^T\theta_0^i - y_i^*(t+d_i). \end{aligned} \tag{9.2}$$

It is evident that the tracking error may be made identically zero if it is possible to choose the vector $u(t)$ to satisfy

$$\phi_i(t)^T\theta_0^i = y_i^*(t+d_i), \qquad 1 \leq i \leq m. \tag{9.3}$$

Obviously (9.3) is a set of simultaneous equations in $u(t)$. Now the matrix multiplying $u(t)$ is nonsingular since in (8.3) $\det(\operatorname{diag} F_i(z)) = 1$ at $z=0$ and Assumption 8c) holds. Hence a unique solution $u(t)$ of (9.3) exists at the true parameter value $\mathring{\theta}$.

Consider the following adaptive algorithm:

$$\begin{aligned} \hat{\theta}_i(t) = \hat{\theta}_i(t-1) + a(t)\varphi_i(t-d_i)\left[1 + \varphi_i(t-d_i)^T\varphi_i(t-d_i)\right]^{-1} \\ \cdot\left(y_i(t) - \varphi_i(t-d_i)^T\hat{\theta}_i(t-1)\right) \end{aligned} \tag{9.4}$$

$$\varphi_i(t)^T\hat{\theta}_i(t) = y_i^*(t+d_i), \qquad 1 \leq i \leq m \tag{9.5}$$

where $\hat{\theta}_i(t)$ is a p_i $(= n_i + m(m_i + d_i))$ vector of reals depending upon an initial vector $\hat{\theta}_i(0)$ and $y_i(\tau)$, $0 \leq \tau \leq t$, $u(\tau)$, $0 \leq \tau \leq t - d_i$ via (9.4).

Clearly, it is critical to ensure that a solution to (9.5) exists for all t. This is guaranteed if the matrix of coefficients of $u(t)$ in (9.5) is nonsingular and this is ensured by the following procedure.

1) At $t=0$ the arbitrary initial value $\hat{\theta}(0)$ of the parameter estimate is chosen so that Assumption Set 8 is satisfied. Hence, the analogous equations to (9.3) evaluated at $\phi(0)$, $y(d_i)$, $1 \leq i \leq m$, $\hat{\theta}(0)$ are solvable for $u(0)$.

2) For $t \geq 1$ the procedure of Lemma 9.1 below guarantees the solvability of the algorithm equations for $u(t)$.

Lemma 9.1: In order that the matrix of coefficients of $u(t)$ in (9.5) is nonsingular for all $t \geq 1$ it is sufficient for $a(t)$ in (9.4) to be chosen as follows:

$$\epsilon < a(t) < 2 - \epsilon$$

where $0 < \epsilon < 1$ and $a(t)^{-1}$ is not an eigenvalue of $-R^{-1}(t-1)V(t)$ with

$$R(t-1) \triangleq [r_1, \cdots, r_m] \tag{9.6}$$

and

$$V(t) \triangleq [v_1, \cdots, v_m]. \tag{9.7}$$

r_i in (9.6) is the vector of coefficients of $u(t)$ in $\hat{\theta}_i(t-1)$, that is,

$$r_i = S_i\hat{\theta}_i(t-1) \tag{9.8}$$

when

$$S_i = [0_{m\times n_i} \quad I_{m\times n} \quad 0_{m\times(m_i+d_i-1)}]. \tag{9.9}$$

v_i in (9.6) is the vector of changes in the coefficient of $u(t)$, that is,

$$\begin{aligned} v_i = S_i\Big[\varphi_i(t-d_i)\left(1 + \varphi_i(t-d_i)^T\varphi_i(t-d_i)\right)^{-1} \\ \cdot\left(y_i(t) - \varphi_i(t-d_i)^T\theta_i(t-1)\right)\Big]. \end{aligned} \tag{9.10}$$

Proof: The proof will be by induction and we first observe that from (9.4) and (9.6)–(9.10)

$$R(t) = R(t-1) + a(t)V(t). \tag{9.11}$$

Then:

i) $R(0)$ is nonsingular by the initial choice of $\hat{\theta}_i(0)$, $i = 1, \cdots, m$.

ii) Assume $R(t-1)$ is nonsingular. Then from (9.11), using $a(t) \neq 0$,

$$\begin{aligned} \det R(t) &= [\det R(t-1)][\det(I + a(t)R(t-1)^{-1}V(t))] \\ &= [\det R(t-1)](a(t))^m\left[\det\left(\frac{1}{a(t)}\right)I + R(t-1)^{-1}V(t)\right] \\ &= 0 \qquad \text{if and only if } \frac{1}{a(t)} \text{ is an eigenvalue of} \\ &\quad -R(t-1)^{-1}V(t). \end{aligned}$$

But the definition of $a(t)$ ensures $a(t)^{-1}$ is not an eigenvalue of $-R(t-1)^{-1}V(t)$, hence $R(t)$ is nonsingular. However, by i) $R(0)$ is nonsingular and it follows by induction $R(t)$, $t \geq 0$ is nonsingular. □

We note that the above choice of $a(t)$ has been included for technical completeness and that $a(t) = 1$ will almost always work since it is a nongeneric occurrence for 1 to be an eigenvalue of $-R(t-1)^{-1}V(t)$. Also since $-R(t-1)^{-1}V(t)$ has only a finite number of eigenvalues it is always possible to find an $a(t)$ to satisfy the lemma by computation of the eigenvalues of $R(t-1)^{-1}V(t)$.

Theorem 9.1: Subject to Assumptions 8a)–c) if the algorithm (9.4), (9.5) is applied to the system (2.1), (2.2) with $r = m$, then $\{y(t)\}$ and $\{u(t)\}$ are bounded and

$$\lim_{t\to\infty} |y_i(t) - y_i^*(t)| = 0; \qquad 1 \leq i \leq m.$$

Proof: Using Lemma (5.1) for each i, we have

$$\lim_{t\to\infty} \frac{e_i(t)}{\left[1 + \varphi_i(t-d_i)^T\varphi_i(t-d_i)\right]^{1/2}} = 0.$$

The proof now follows that of Theorem 5.1, except in the case that the vector $y(t)$ is unbounded. In this case there exists a subsequence $\{t_n\}$ such that

$$\lim_{t_n\to\infty} \|y(t_n)\| = \infty$$

and

$$|y_i(t+d_i)| < |y_{j(t_n)}(t_n+d_{j(t_n)})| \qquad \text{for some } 1<j(t_n)<m$$
$$\text{and for all } i\ 1<i<m \text{ and for all } t<t_n.$$

It then follows by Lemma 3.2 that there exist constants $0<C_1<\infty$ and $0<C_2<\infty$ such that

$$\|\varphi_i(t_n)\| < C_1 + C_2|y_{j(t_n)}(t_n+d_{j(t_n)})|, \qquad 1<i<m.$$

Since m is finite, there exists a further subsequence $\{t_{n'}\}$ of the subsequence $\{t_n\}$ such that

$$\|\varphi_i(t_{n'})\| < C_1 + C_2|y_i(t_{n'}+d_i)| \qquad \text{for at least one } i,\ 1<i<m$$

and $\{y_i(t_{n'}+d_i)\}$ is unbounded. The remainder of the proof then follows that of Theorem 5.1 where we note that

$$|e_i(t)| = |y_i(t) - y_i^*(t)|. \qquad \square$$

B. MIMO Projection Algorithm II

From (8.2), (8.3) the fact that $F_i(z)=1$ for $z=0$, $i=1,\cdots,m$ and by Assumption 8c) we can factor out the nonsingular matrix Γ_0 $(=[\beta_{ij}(0)])$ giving

$$\begin{bmatrix} y_1(t+d_1) \\ \vdots \\ y_m(t+d_m) \end{bmatrix} = \Gamma_0\{u(t) + C(q^{-1})y(t) + D(q^{-1})u(t-1)\} \qquad (9.12)$$

where

$$C(q^{-1}) = \Gamma_0^{-1}\begin{bmatrix} \alpha_1(q^{-1}) & & 0 \\ & \ddots & \\ 0 & & \alpha_m(q^{-1}) \end{bmatrix}$$

and

$$D(q^{-1}) = \Gamma_0^{-1}\left\{\begin{bmatrix} \beta_{11}(q^{-1}) & \cdots & \beta_{1m}(q^{-1}) \\ \vdots & & \vdots \\ \beta_{m1}(q^{-1}) & \cdots & \beta_{mm}(q^{-1}) \end{bmatrix} - \Gamma_0\right\}.$$

Define

$$\begin{bmatrix} e_1(t+d_1) \\ \vdots \\ e_m(t+d_m) \end{bmatrix} = \begin{bmatrix} y_1(t+d_1) \\ \vdots \\ y_m(t+d_m) \end{bmatrix} - \begin{bmatrix} y_1^*(t+d_1) \\ \vdots \\ y_m^*(t+d_m) \end{bmatrix}$$
$$= \Gamma_0\left[u(t) + C(q^{-1})y(t) + D(q^{-1})u(t-1) - \Gamma_0^{-1}\begin{bmatrix} y_1^*(t+d_1) \\ \vdots \\ y_m^*(t+d_m) \end{bmatrix}\right]$$
$$= \Gamma_0\left(u(t) - \theta_0'^T\varphi(t)\right) \qquad (9.13)$$

where $\theta_0'^T$ is an $m\times n'$ matrix whose ith row contains the parameters from the ith rows of $C(q^{-1})$, $D(q^{-1})$, and $E=\Gamma_0^{-1}$. $\varphi(t)$ is an $n'\times 1$ vector containing the appropriate delayed versions of $y(t)$, $u(t-1)$, and $y^*(t)$:

$$\varphi(t)^T = \left(-y(t)^T, -y(t-1)^T, \cdots, -u(t-1)^T, -u(t-2)^T, \cdots, y_1^*(t+d_1), \cdots, y_m^*(t+d_m)\right).$$

Analogously to Section VI we introduce the following adaptive control algorithm,

$$\hat\theta(t+d)^T = \hat\theta(t)^T - P\begin{bmatrix} e_1(t+d_1) \\ \vdots \\ e_m(t+d_m) \end{bmatrix}[1+\varphi(t)^T\varphi(t)]^{-1}\varphi(t)^T \qquad (9.14)$$

$$u(t) = \hat\theta(t)^T\varphi(t) \qquad (9.15)$$

where $d=\max(d_1,\cdots,d_m)$. $\hat\theta(t)^T$ is an $m\times n'$ matrix of reals depending on d initial matrices $\hat\theta(i)^T$, $1<i<d$ and past data from the system. P is a matrix of constants specified *a priori*. As in Section VI, (9.14) represents d interlaced recursions.

Once the recursions (9.14), (9.15) are initialized in a manner that leads to a unique solution for $u(0)$ it is sufficient to ensure that all successive recursions lead to equations solvable for $u(t)$, $t>1$. In analogy to Lemma 9.1 we have the following.

Lemma 9.2: Define $\tilde\theta(t+d)^T = \hat\theta(t+d)^T - \theta_0'^T$ and let $K=P\Gamma_0$. If K^T+K-K^TK is positive definite, then, along the trajectories of (9.14), (9.15):

a) $\operatorname{trace}\left[\tilde\theta(t+d)^T\tilde\theta(t+d)\right] - \operatorname{trace}\left[\tilde\theta(t)^T\tilde\theta(t)\right] < 0.$

b) $$\lim_{t\to\infty}\frac{e_i(t+d_i)}{[1+\varphi(t)^T\varphi(t)]^{1/2}} = 0, \qquad 1<i<m.$$

Proof:

a) We can rewrite (9.13) using (9.15) as

$$\begin{bmatrix} e_1(t+d_1) \\ \vdots \\ e_m(t+d_m) \end{bmatrix} = \Gamma_0\tilde\theta(t)^T\varphi(t).$$

Hence, from (9.14)

$$\tilde\theta(t+d)^T = \tilde\theta(t)^T - K\tilde\theta(t)^T\varphi(t)[1+\varphi(t)^T\varphi(t)]^{-1}\varphi(t)^T \qquad (9.18)$$

and

$$\operatorname{trace}\left(\tilde\theta(t+d)^T\tilde\theta(t+d)\right) - \operatorname{trace}\left(\tilde\theta(t)^T\tilde\theta(t)\right)$$
$$= -\operatorname{trace}\left[\left(K^T+K-K^TK\frac{\varphi(t)^T\varphi(t)}{[1+\varphi(t)^T\varphi(t)]}\right)\cdot\left(\tilde\theta(t)^T\varphi(t)[1+\varphi(t)^T\varphi(t)]^{-1}\varphi(t)^T\tilde\theta(t)\right)\right]$$
$$<0 \qquad \text{if } K^T+K-K^TK \text{ is positive definite.}$$

b) As in the proof of Lemma (7.1) it follows that

$$\lim_{t\to\infty}\operatorname{trace}\left(K^T+K-K^TK\frac{\varphi(t)^T\varphi(t)}{[1+\varphi(t)^T\varphi(t)]}\right)\cdot\left(\tilde\theta(t)^T\varphi(t)[1+\varphi(t)^T\varphi(t)]^{-1}\varphi(t)^T\tilde\theta(t)\right) = 0.$$

Since K^T+K-K^TK is positive definite then

$$\lim_{t\to\infty}\left[\tilde\theta(t)^T\varphi(t)[1+\varphi(t)^T\varphi(t)]^{-1}\varphi(t)^T\tilde\theta(t)\right] = 0$$

or

$$\lim_{t\to\infty}\Gamma_0^{-1}\begin{bmatrix} e_1(t+d_1) \\ \vdots \\ e_m(t+d_m) \end{bmatrix}[1+\varphi(t)^T\varphi(t)]^{-1}[e_1(t+d_1)\cdots e_m(t+d_m)]\Gamma_0^{-1} = 0.$$

This implies that (9.17) holds. $\square$

Using Lemma 9.2 and following the proof of Theorem 9.1 we have the following.

Theorem 9.2: Subject to Assumptions 8a)–c), and K^T+K-K^TK positive definite, if the algorithm (9.14), (9.15) is applied to the system (2.1), (2.2) with $r=m$, then the vectors $y(t)$ and $u(t)$ are bounded and

$$\lim_{t\to\infty}|y_i(t)-y_i^*(t)| = 0, \qquad 1<i<m. \qquad \square$$

X. Nonlinear Systems

Although the analysis in the paper has been carried out for deterministic linear systems, it is clear that it could be readily extended to certain classes of nonlinear systems of known form. The essential points are the form of (5.2) or (6.2), and the linear bound condition (3.3). The latter point would indicate that systems with cone bounded nonlinearities would satisfy the conditions.

XI. Conclusion

The paper has analyzed a general class of discrete-time adaptive control algorithms and has shown that, under suitable conditions, they will be globally convergent. The algorithms have a very simple structure and are applicable to both single-input single-output and multiple-input multiple-output systems with arbitrary time delays provided only that a stable control law exists to achieve zero tracking error. The results resolve a long standing question in adaptive control regarding the existence of simple, globally convergent adaptive algorithms.

References

[1] I. D. Landau, "A survey of model reference adaptive techniques—Theory and applications," *Automatica*, vol. 10, pp. 353–379, 1974.

[2] R. V. Monopoli, "Model reference adaptive control with an augmented error signal," *IEEE Trans. Automat. Contr.*, vol. AC-19, pp. 474–485, Oct. 1974.

[3] A. Feuer, B. R. Barmish, and A. S. Morse, "An unstable dynamical system associated with model reference adaptive control," *IEEE Trans. Automat. Contr.*, vol. AC-23, pp. 499–500, June 1978.

[4] K. S. Narendra and L. S. Valavani, "Stable adaptive controller designs—Direct control," *IEEE Trans. Automat. Contr.*, vol. AC-23, pp. 570–583, Aug. 1978.

[5] A. Feuer and S. Morse, "Adaptive control of single-input single-output linear systems," *IEEE Trans. Automat. Contr.*, vol. AC-23, pp. 557–570, Aug. 1978.

[6] G. A. Dumont and P. R. Bélanger, "Self-tuning control of a titanium dioxide kiln," *IEEE Trans. Automat. Contr.*, vol. AC-23, pp. 532–538, Aug. 1978.

[7] K. J. Åström, U. Borisson, L. Ljung, and B. Wittenmark, "Theory and applications of self tuning regulators," *Automatica*, 19, pp. 457–476, 1977.

[8] L. Ljung, "Analysis of recursive stochastic algorithms," *IEEE Trans. Automat. Contr.*, vol. AC-22, pp. 551–575, Aug. 1977.

[9] ——, "On positive real transfer functions and the convergence of some recursive schemes," *IEEE Trans. Automat. Contr.*, vol. AC-22, pp. 539–551, Aug. 1977.

[10] K. J. Åström and B. Wittenmark, "On self-tuning regulators," *Automatica*, vol. 9, pp. 195–199, 1973.

[11] L. Ljung and B. Wittenmark, "On a stabilizing property of adaptive regulators," *Preprint, IFAC Symp. on Identification*, Tbilisi, U.S.S.R., 1976.

[12] B. Egardt, "A unified approach to model reference adaptive systems and self tuning regulators," Dep. Automat. Contr., Lund Inst. Technol. Tech. Rep., Dec. 1977.

[13] T. Ionescu and R. V. Monopoli, "Discrete model reference adaptive control with an augmented error signal," *Automatica*, vol. 13, pp. 507–517, Sept. 1977.

[14] J. L. Willems, *Stability Theory of Dynamical Systems*. New York: 1970.

[15] G. C. Goodwin, P. J. Ramadge, and P. E. Caines, "Discrete time stochastic adaptive control," *SIAM J. Contr. Optimiz.*, to be published.

[16] A. S. Morse, "Global stability of parameter adaptive control systems," Yale Univ., S & IS Rep. 7902R, Mar. 1979.

[17] K. S. Narendra and Y-H. Lin, "Stable discrete adaptive control," Yale Univ., S & IS Rep. 7901, Mar. 1979.

[18] B. Egardt, "Stability of model reference adaptive and self tuning regulators," Dep. Automat. Contr., Lund Inst. Technol., Tech. Rep., Dec. 1978.

Stable Discrete Adaptive Control

KUMPATI S. NARENDRA, FELLOW, IEEE, AND YUAN-HAO LIN

***Abstract*—The paper presents a proof of stability of the model reference adaptive control problem for the discrete case.**

I. Introduction

At present there is widespread interest in the stable adaptive control of unknown linear time-invariant plants using input-output data. Schemes have been suggested for both direct [1]–[3] and indirect [4], [5] control of continuous as well as discrete [6], [7] systems and the equivalence of the two schemes in some cases has also been demonstrated [4], [5]. Probably the single most important problem to arise in the course of these investigations concerns the proof of stability of the overall adaptive control loop.

Manuscript received March 9, 1979; revised November 19, 1979. Paper recommended by J. L. Speyer, Past Chairman of the Stochastic Control Committee. This work was supported by the Office of Naval Research under Contract N00014-76-C-0017.

The authors are with the Department of Engineering and Applied Science, Yale University, New Haven, CT 06520.

Monopoli [1] proposed a scheme for continuous systems involving an auxiliary signal fed into the reference model and a corresponding augmented error between model and plant outputs. Narendra and Valavani [2], using positive real operators, suggested a similar approach and clarified the resulting stability problem when the relative degree of the plant is greater than or equal to three. They offered a conjecture that the adaptive loop would also be stable for the general case. Feuer and Morse [3] proposed a stable solution to the adaptive control problem but the resulting controller is much too complex for use in practical applications. Thus, the search has continued for a controller with a simple structure which will assure the global asymptotic stability of the adaptive loop. The results presented in this paper demonstrate the desired stability behavior for discrete versions of the simple controllers suggested in [1] and [2]. Similar results have also been reported recently in [9] and [10] for the discrete adaptive control problem and in [11] and [12] for the continuous case.

This paper examines the discrete version of the problem considered in [2] recapitulating the basic philosophy as well as the specific technique used for the design of the adaptive controller in that paper. Hence the first few sections of this paper have been considerably condensed and the interested reader is referred to the earlier work for all details. The principal contribution made here is the verification of the conjecture made in [2] regarding the stability of the adaptive loop, for the discrete problem when an additional feedback signal suggested in [8] is used. Accordingly most of the paper is devoted to the proof of stability. While the proof given in [11] for continuous systems can be directly extended to the discrete case, we present here a simpler proof which is valid for discrete systems.

II. Statement of the Problem

A single-input single-output discrete linear time-invariant plant P is described by the input–output pair $\{u(k), y_p(k)\}$ and can be represented by the transfer function

$$W_p(z) = k_p \frac{Z_p(z)}{R_p(z)} \tag{1}$$

where $W_p(z)$ is proper, with $R_p(z)$ a monic polynomial of degree n, $Z_p(z)$ a monic stable[1] polynomial of degree $m < n$, and k_p a constant gain parameter. The integer $n-m$ is called the relative degree of the plant. We assume that only m, n and the sign of k_p as well as an upper bound on $|k_p|$ are known, while the coefficients of Z_p and R_p are unknown.

A reference model M whose output $y_M(k)$ represents the behavior desired from the plant when augmented by a suitable controller can be represented by the transfer function

$$W_M(z) \triangleq k_M \frac{Z_M(z)}{R_M(z)} \tag{2}$$

where $R_M(z)$ and $Z_M(z)$ are monic stable polynomials of degrees n and $r < m$ respectively and k_M is a constant. Hence the relative degree of the model is assumed to be greater than or equal to that of the plant. The reference input $r(k)$ to the model is specified and is assumed to be uniformly bounded.

The adaptive control problem is to determine a suitable control function $u(k)$ such that

$$y_p(k) - y_M(k) \to 0 \qquad \text{as } k \to \infty. \tag{3}$$

For the sake of simplicity we shall assume that $r = m$. As in the continuous case, the solution to the above problem may be divided into two parts. The first part which is algebraic in nature addresses itself to the realizability of a suitable controller structure. It can be shown exactly as in the continuous case [2] that a controller can be found which can achieve (3) with a fixed set of parameters. In the following section the

[1] With all zeros inside the unit circle.