VOLUME COMPARISON OF CONFORMALLY COMPACT MANIFOLDS WITH SCALAR CURVATURE $R \geq -n\,(n-1)$

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Abstract. In this paper, we present a global invariant, called renormalized volume, which can be defined for a large class of conformally compact manifolds. We use this definition to show a local volume comparison of conformally compact manifolds with scalar curvature $R \geq -n\,(n-1)$ and also the rigidity result when the renormalized volume is zero.

Dans cet article, nous introduisons un invariant global appelé volume renormalisé et qui peut être défini pour une large classe de variétés conformément compactes. Nous nous servons ensuite de cette définition pour montrer un théorème de comparaison locale pour les variétés conformément compactes dont la courbure scalaire satisfait $R \geq -n(n-1)$. Nous donnons également un résultat de rigidité lorsque le volume renormalisé est nul.

1. Introduction

Volume is one of the natural geometric quantities which is often used to explore geometrical and topological properties of a Riemannian manifold. Classical examples in this direction are various volume comparison theorems which turned out to be fruitful in Riemannian geometry. In order to use those volume comparison theorems efficiently, we have to assume certain lower bound on the Ricci curvature of the manifold. Obviously, we can not expect the same results still to be true if we only assume lower bound on scalar curvature. However, in [11], R.Schoen proposed the following conjecture on the volume functional $V(\cdot)$ on a closed hyperbolic manifold.

Conjecture 1.1. *Let $(M^n, \tilde{g})$ be a closed hyperbolic manifold. Let g be another metric on M with $R(g) \geq R(\tilde{g})$, then $V(g) \geq V(\tilde{g})$.*

Date: September, 2013.

2000 *Mathematics Subject Classification.* Primary 53C25; Secondary 58J05.

Key words and phrases. conformally compact manifold, asymptotically hyperbolic, volume, normalized Ricci flow, normalized Ricci-Deturk flow.

† Research partially supported by China Postdoctoral Science Foundation grant 2012M520414.

‡ Research partially supported by NSF grant of China 10990013.

Recently, inspired by Bray's thesis [1], a notion of renormalized volume for some asymptotically Anti-deSitter-Schwarzschild manifolds was introduced in [2] and an interesting inequality was established when the dimension $n = 3$ and the mass $m > 0$.

With these facts in mind, it may be natural to ask whether or not a version of Schoen's Conjecture which is in terms of renormalized volume introduced in [2] is true for conformally compact manifolds. In order to state our main results, let us recall some basic definitions. Suppose that M^n is a smooth manifold with boundary ∂M. A defining function τ of the boundary in M^n is a smooth function on M^n such that $\tau > 0$ in M^n; $\tau = 0$ on ∂M; $d\tau \neq 0$ on ∂M. A complete non-compact Riemannian metric g on M is said to be conformally compact of regularity $C^{k,\mu}$ if $\tau^2 g$ extends to be a $C^{k,\mu}$ Riemannian metric on $\bar{M}$. The metric $\tau^2 g$ induces a metric $\hat{g}$ on the boundary ∂M, and the metric g induces a conformal class of metric $[\hat{g}]$ on the boundary ∂M when defining functions vary. The conformal manifold $(\partial M, [\hat{g}])$ is called the conformal infinity of the conformally compact manifold (M, g).

Given a $C^{2,\alpha}$ conformally compact manifold (M^n, g) and a representative $\hat{g}$ in $[\hat{g}]$ on the conformal infinity ∂M, there is a uniquely determined defining function τ such that, in a neighborhood of the boundary $[0, \tau_0) \times \partial M \subset M$, g has the form

$$g = \tau^{-2}(d\tau^2 + g_\tau),$$

where g_τ is a $1-$parameter family of metrics on ∂M with $g_\tau|_{\tau=0} = \hat{g}$. We call this τ the special defining function associated with $\hat{g}$.

Let us see some properties of a $C^{2,\alpha}$ conformally compact manifold $(M^n, \tilde{g})$. In order to avoid the complexity of the end structure of conformally compact manifold, we always need the concept of an essential set. Please see Definition 1.1 in [7] for the definition of an essential set. In [6]Lemma 2.5.11 and Corollary 2.5.12 , Gicquaud proved that if a complete non-compact manifold is C^2 conformally compact then it contains essential sets. Once a conformally compact manifold $(M^n, \tilde{g})$ has an essential set $\mathbb{D}$, the volume $vol(B_{\tilde{g}}(x, 1))$ has a lower bound $\Lambda = \Lambda\,(\tilde{g}, n)$ for all $x \in M$ where $B_{\tilde{g}}(x, 1)$ is geodesic ball of radius 1 and center x. Hence if a metric g satisfies $\|g - \tilde{g}\|_{C^0(M^n, \tilde{g})} \leq \epsilon$, it follows that there exists some $v = v\,(\Lambda, \epsilon, n) = v\,(\tilde{g}, \epsilon, n) > 0$ such that $vol(B_g(x, 1)) \geq v$ for all $x \in M$. It follows immediately from the well-known result (see Lemma 3.1 in [10]) that there exists some $i = i\,(k, v, n) = i\,(k, n, \epsilon, \tilde{g}) > 0$ such that the injectivity radius of (M, g) satisfies $inj_{(M,g)} \geq i$ provided $|sec_g| \leq k$.

Now let $\tilde{g}$ be a arbitrary Riemannian metric on M^n. We can define the renormalized volume functional with respect to $\tilde{g}$ (see the same definition in

[2]):

$$\mathcal{V}_{\tilde{g}} : \mathcal{M}^{\infty}(M) \longrightarrow \mathbb{R}$$
$$g \longmapsto \int_M \left(\sqrt{|g|} - \sqrt{|\tilde{g}|} \right) dx$$

where $\mathcal{M}^{\infty}(M)$ denotes the space of smooth Riemannian metrics on M, $|g|$ and $|\tilde{g}|$ denote determinants of metrics g and $\tilde{g}$ under the local coordinates $(x^1, \cdots, x^n)$. Obviously, $\mathcal{V}(g)$ is independent of the choice of $(x^1, \cdots, x^n)$. The well-posedness of the renormalized volume functional will be proved in Lemma 3.1.

Let us recall the non-degeneracy of a conformally compact Einstein metric g, which is defined to be the first L^2-eigenvalue of the linearization of the curvature tensor $Ric + (n-1)\,g$,

$$\lambda = \inf_u \frac{\int_M \langle (\Delta_L + 2(n-1))u, u\rangle_g d\mu_g}{\int_M \|u\|_g^2 d\mu_g}$$

where the infimum is taken among all non zero symmetric 2-tensors u such that

$$\int_M \left(\|u\|_g^2 + \|\nabla u\|_g^2 \right) d\mu_g < \infty$$

and Δ_L is the Lichnerowicz Laplacian with respect to the metric g on symmetric 2-tensor u, i.e.,

$$\Delta_L u_{ij} = -\Delta u_{ij} - 2R_{ipjq}u^{pq} + R_{iq}u_j^q + R_{jq}u_i^q$$

where R_{ipjq}, and $R_{ij} = g^{pq}R_{ipjq}$ are the components of the Riemann curvature tensor and the Ricci curvature tensor of the metric g respectively.

Our first result is:

Theorem 1.2. *Suppose that $(M^n, \tilde{g})$ is a $C^{2,\alpha}$ conformally compact Einstein manifold with non-degeneracy $\lambda > 0$. For $n \geq 4$, and*

$$\delta \in \left(n-1, \frac{(n-1) + \sqrt{(n-1)^2 + 4(n-1)}}{2} \right),$$

there exists some $\epsilon = \epsilon(n, k, \lambda, \tilde{g})$ such that if g is a Riemannian metric on M^n satisfying

$$\|g - \tilde{g}\|_{C^1(M^n, \tilde{g})} \leq \epsilon e^{-\delta\rho},$$
$$\|Rm(g)\| \leq k,$$
$$|R(g) + n(n-1)| \leq Ce^{-\delta\rho}, \quad \textit{for a constant } C > 0$$

and

$$R(g) \geq -n(n-1)$$

where $\rho = d_{\tilde{g}}(\cdot, \mathbb{D})$ *is the distance function to some essential set* $\mathbb{D} \subset M$ *with respect to* $\tilde{g}$*.* $R(g)$ *denotes the scalar curvature of* g*. Then*

$$\mathcal{V}_{\tilde{g}}(g) \geq 0.$$

When $\mathcal{V}_{\tilde{g}}(g) = 0$, we have:

Theorem 1.3. *Under the assumptions of Theorem 1.2 suppose that* $\mathcal{V}_{\tilde{g}}(g) = 0$*. Then there exists a* C^∞ *diffeomorphism* $\Phi : M \longrightarrow M$*, such that* $g = \Phi^* \tilde{g}$*.*

It is known that many conformally compact Einstein manifolds are non-degenerate, for examples, any conformally compact Einstein manifolds with nonpositive sectional curvature and nonnegative Yamabe invariant of its conformal infinity boundaries is non-degenerate (see Theorem A in [8]). In particular, the hyperbolic space $\mathbf{H}^n$ is a non-degenerate conformally compact Einstein manifold, hence as a corollary, we have

Theorem 1.4. *Suppose that* (M, g) *is a* $C^{2,\alpha}$ *conformally compact manifold with topology* $\mathbf{H}^n$ *and satisfies*

$$\|g - h\|_{C^1(\mathbf{H}^n, h)} \leq \epsilon e^{-\delta\rho} \tag{1}$$

where h *is the hyperbolic metric,* ρ *is a distance function to some essential set* $\mathbb{D} \subset M$ *with respect to* h*, and* $\delta \in \left(n-1, \frac{(n-1)+\sqrt{(n-1)^2+4(n-1)}}{2}\right)$*. If* $n \geq 4$*, and* $R \geq -n(n-1)$*, and* $|R(g)+n(n-1)| \leq Ce^{-\delta\rho}$*, for a constant* $C > 0$ *then*

$$\mathcal{V}_h(g) \geq 0,$$

provided that ϵ *is small enough. The equality holds iff* (M, g) *is a standard hyperbolic space.*

The above results can be regarded as a version of Schoen Conjecture in the case of conformally compact manifolds. We suspect that such kind of results might be true without assumption (1). We will use normalized Ricci-DeTurck flow (see (2) below) as a tool to investigate the behavior of the renormalized volume. Indeed, we are able to show that our renormalized volume is non-increasing along the flow (See Proposition 3.3) and will converge to zero as time goes to infinity. In order to get these properties, we need to estimate on the gauge and the scalar curvature (see Lemma 2.7 and Lemma 2.8). We cannot get the estimate like

$$\|g - \tilde{g}\| \leq Ce^{-\sigma t}e^{-\delta\rho}, \quad \delta > n-1,$$

and we believe that such estimate is not true. One possible reason is that the coefficient of the $n-1$ term $g_{(n-1)}$ in the expansion of an conformally compact Einstein manifold at infinity is undetermined. Another renomilized volume for conformally compact manifolds was also introduced in an interesting paper [3], and monotonicity of this quantity along the normalized Ricci flow was also obtained in the paper. However, our renomilized volume is different from that in [3]. We use a non-degenerate conformally compact Einstein metric $\tilde{g}$ as ground state in our definition of $\mathcal{V}_{\tilde{g}}(g)$, and in the case that $R \geq -n(n-1)$, we get a rigidity result when $\mathcal{V}_{\tilde{g}}(g) = 0$. Therefore, in some sense, our renomilized volume $\mathcal{V}_{\tilde{g}}(g)$ can be viewed as a kind of measure of deviation of a conformally compact manifold and the ground state Einstein manifold. Actually, this observation is one of our main motivation of this paper.

This paper is organized as follows. In Sect.2 we will show the long-time existence and convergence of NRDF and prove Theorem 2.1. In Sect.3 we show that the renormalized volume is well-defined under some circumstances first and then give some basic estimates, finally we prove our main results Theorem 1.2 and Theorem 1.3. We fix the notation that the positive constants ϵ with subscripts are always sufficiently small and the positive constants C with subscripts are big and bounded, meanwhile, ϵ_i, $i = 1, 2 \dots$ depend only on n, k, λ, $\tilde{g}$, and sometimes may depend on the existence time. C_j, $j = 1, 2 \dots$ depend only on n, ϵ, k, λ $\tilde{g}$, and sometimes may depend on the decay order with respect to space infinity. In the sequel, we will omit the subscript $\tilde{g}$ of the renormalized volume functional. $\| \cdot \|$, $\langle \cdot, \cdot \rangle$, $d\mu$ and ρ are all with respect to $\tilde{g}$ unless otherwise stated.

Acknowledgements The authors are grateful to Professor Jie Qing and Dr. Romain Gicquaud for their interests in this work and helpful conversations.

2. Long-Time Existence and Convergence of NRDF

In this section, we will frequently use the normalized Ricci-DeTurck flow (NRDF for short in the sequel) with background metric $\tilde{g}$ as well as the normalized Ricci flow (NRF for short in the sequel). We first consider the existence and convergence of NRDF with a non-degenerate conformally compact Einstein metric $\tilde{g}$ as a background metric, and then we show the behavior of the normalized scalar curvature $R + n\,(n-1)$ and the gauge V under NRDF. The NRDF with background metric $\tilde{g}$ is

$$(2) \qquad \begin{cases} \frac{\partial}{\partial t} g_{ij} = -2(R_{ij} + (n-1) g_{ij}) + \nabla_i V_j + \nabla_j V_i \text{ on } M^n \times (0, T), \\ g\,(\cdot, 0) = g. \end{cases}$$

where $V_j = g_{jk} g^{pq} \left(\Gamma^k_{pq} - \tilde{\Gamma}^k_{pq} \right)$ and $\tilde{\Gamma}$ are the Christoffel symbols of the metric $\tilde{g}$.

Assume that $\Phi_t : M^n \longrightarrow M^n$ solves

$$\begin{cases} \frac{\partial}{\partial t} \Phi_t\,(\cdot, t) = -V\,(\Phi_t\,(\cdot, t)\,, t)\,, \\ \Phi_t\,(\cdot, 0) = id, \end{cases}$$

where the components of V are given by $V^i := g^{ij}V_j$, then we obtain a family of smooth diffeomorphisms Φ_t for $t > 0$ such that if $g(t)$, $t \in [0, T)$ is a solution to NRDF (2), $\bar{g}(t) := \Phi_t^* g(t)$, $t \in [0, T)$ is a solution to the following NRF:

$$\begin{cases} \frac{\partial}{\partial t} g_{ij} = -2(R_{ij} + (n-1)g_{ij}) \ on \ M^n \times (0, T). \\ g(\cdot, 0) = g. \end{cases} \tag{3}$$

In [12], Schn$\ddot{u}$rer, Schulze and Simon use the NRDF to get the stability of hyperbolic space under this flow. To be precise, under the assumptions that $\|g - h\|_{C^0(\mathbb{H}^n)} \leq \epsilon$ and $\int_{\mathbb{H}^n} \|g - h\|^2 d\mu_h \leq K$, the NRDF starting from g with background metric being hyperbolic metric h exists globally. Moreover, there exists a constant $C = C(n, K) > 0$ such that

$$\|g(t) - h\| \leq C \cdot e^{-\frac{1}{4(n+2)}t}$$

for all $t \in [0, \infty)$, together with some interior estimates and interpolation, which implies that NRDF converges to h exponentially in C^k as $t \to \infty$ for all $k \in \mathbb{N}$. However, in this paper we need to consider a certain non-degenerate conformally compact Einstein manifold as a background manifold.

It is well known that the curvature tensor of a $n \geq 4$ dimensional Riemannian manifold decomposes into orthogonal parts as follows

$$Rm = W + \frac{1}{n-2} A \odot g + \frac{R}{2n(n-1)} g \odot g,$$

where W, $A := Ric - \frac{R}{n} g$ and R are the Weyl tensor, the traceless Ricci tensor and the scalar curvature respectively; and $\odot$ represents the Kulkarni-Nomizu product, which takes two symmetric $(0, 2)-$tensors h, k and gives a $(0, 4)-$tensor with the same algebraic symmetries of the curvature tensor, defined by

$$\begin{aligned} h \odot k(v_1, v_2, v_3, v_4) =& h(v_1, v_3) \cdot k(v_2, v_4) + h(v_2, v_4) \cdot k(v_1, v_3) \\ & - h(v_1, v_4) \cdot k(v_2, v_3) - h(v_2, v_3) \cdot k(v_1, v_4). \end{aligned}$$

For an Einstein metric $\tilde{g}$, the components of a Riemann curvature tensor is

$$\tilde{R}_{ijkl} = \tilde{W}_{ijkl} - (\tilde{g}_{ik}\tilde{g}_{jl} - \tilde{g}_{il}\tilde{g}_{jk})$$

satisfying $\tilde{g}^{pq}\tilde{W}_{ipjq} = 0$. Hence, the only thing that will cause trouble in generalizing the work of [12] is the Weyl curvature tensor. On the other hand, if we let $u(t) = g(t) - \tilde{g}$ where $g(t)$ is the NRDF with background metric $\tilde{g}$, as $\tilde{R}_{ij}+(n-1)\tilde{g}_{ij}=0$ and $\tilde{V}_i=0$, then $u(t)$ fulfills

$$\begin{aligned}&\frac{\partial}{\partial t}u_{ij}\\&=-2(R_{ij}+(n-1)g_{ij})+\nabla_iV_j+\nabla_jV_i-\left(-2(\tilde{R}_{ij}+(n-1)\tilde{g}_{ij})+\nabla_i\tilde{V}_j+\nabla_j\tilde{V}_i\right)\\&=-\left(\tilde{\Delta}_L+2(n-1)\right)u_{ij}+\mathcal{F}(g,\tilde{g},u)\end{aligned}$$

where $\mathcal{F}$ is the remainder of the linearization of the normalized Ricci-DeTurck operator, see (7) below for the specific expression.

Therefore, if we deal with the Weyl curvature tensor carefully and use the non-degenerate condition instead of the spectrum of the hyperbolic metric to promise the exponential decay with respect to time, we can obtain the following stability of a non-degenerate conformally compact Einstein manifold:

Theorem 2.1. *Let $n\geq 4$. Suppose that $(M^n,\tilde{g})$ is a $C^{2,\alpha}$ conformally compact Einstein manifold with non-degeneracy $\lambda>0$. If*

$$\|g-\tilde{g}\|_{C^1(M^n,\tilde{g})}\leq \epsilon e^{-\delta\rho} \text{ and } \|Rm(g)\|\leq k,$$

where $\delta>\frac{n-1}{2}$, then there exists some $\epsilon=\epsilon(\lambda,k,n,\tilde{g})>0$, such that the normalized Ricci-DeTurck flow (2) with background metric $\tilde{g}$ starting from g exists globally and converges exponentially to $\tilde{g}$.

Since $\|g-\tilde{g}\|_{C^1(M^n,\tilde{g})}\leq \epsilon e^{-\delta\rho}$, due to Theorem 1.1 in [13], there exists some $T=T(n,\tilde{g})$ and a family of metrics $g(t)$ for $t\in[0,T)$ which solves the NRDF (2) starting from $g=g_0$. In addition,

$$\|g(t)-\tilde{g}\|\leq 2\epsilon,\quad t\in[0,T). \tag{4}$$

Since g_0 is Lipschitz, the estimates on the derivatives were improved in [13] Lemma 2.1, hence there exist some constants ϵ_1 and ϵ_2 which only depend on ϵ, n $\tilde{g}$ and T such that

$$\|\tilde{\nabla}g(t)\|\leq\epsilon_1, \tag{5}$$

and

$$\|\tilde{\nabla}^2g(t)\|\leq\frac{\epsilon_2}{\sqrt{t}} \tag{6}$$

for all $t\in(0,T)$. If we assume that $T\leq 2$, then ϵ_1 and ϵ_2 can be independent of T.

In the following we compute some evolution equations under the NRDF which will be used later.

Lemma 2.2. *Let $g(t)$, $t \in [0,T)$ be a solution to NRDF (2). Assume that $\|\tilde{W}\| \leq l$. Then for all $t \in [0,T)$, there exists ϵ_3 and ϵ_4 such that*

$$\begin{aligned}
\frac{\partial}{\partial t}\|g-\tilde{g}\|^2 \leq & g^{ij}\tilde{\nabla}_i\tilde{\nabla}_j\|g-\tilde{g}\|^2 - (2-\epsilon_3)\|\tilde{\nabla}g\|^2 + (4+4l+\epsilon_4)\|g-\tilde{g}\|^2 \\
\frac{\partial}{\partial t}\|\tilde{\nabla}g\|^2 = & g^{ij}\tilde{\nabla}_i\tilde{\nabla}_j\|\tilde{\nabla}g\|^2 - 2g^{ab}\tilde{g}^{mn}\tilde{g}^{ik}\tilde{g}^{jl}\tilde{\nabla}_a\tilde{\nabla}_n g_{ij}\cdot\tilde{\nabla}_b\tilde{\nabla}_m g_{kl} \\
& + \tilde{\nabla}g * \tilde{\nabla}g * \tilde{\nabla}^2 g + \tilde{\nabla}g * \tilde{\nabla}g * \tilde{\nabla}g * \tilde{\nabla}g + \tilde{\nabla}g * \tilde{\nabla}g \\
\frac{\partial}{\partial t}\|\tilde{\nabla}^2 g\|^2 = & g^{ij}\tilde{\nabla}_i\tilde{\nabla}_j\|\tilde{\nabla}^2 g\|^2 - 2g^{ab}\tilde{g}^{mn}\tilde{g}^{ik}\tilde{g}^{jl}\tilde{g}^{pq}\tilde{\nabla}_a\tilde{\nabla}_m\tilde{\nabla}_p g_{ij}\cdot\tilde{\nabla}_b\tilde{\nabla}_n\tilde{\nabla}_q g_{kl} \\
& + \tilde{\nabla}g * \tilde{\nabla}^2 g * \tilde{\nabla}^3 g + \tilde{\nabla}^2 g * \tilde{\nabla}^2 g * \tilde{\nabla}^2 g + \tilde{\nabla}^2 g * \tilde{\nabla}^2 g * \tilde{\nabla}g * \tilde{\nabla}g \\
& + \tilde{\nabla}^2 g * \tilde{\nabla}^2 g + \tilde{\nabla}^2 g * \tilde{\nabla}g * \tilde{\nabla}g * \tilde{\nabla}g * \tilde{\nabla}g + \tilde{\nabla}^2 g * \tilde{\nabla}g * \tilde{\nabla}g
\end{aligned}$$

where $*$ denotes linear combinations with $g(t)$, $\tilde{g}$ and their inverse.

Proof. Since the background metric is Einstein, we can see that

$$\tilde{R}_{ijkl} = \tilde{W}_{ijkl} - (\tilde{g}_{ik}\tilde{g}_{jl} - \tilde{g}_{il}\tilde{g}_{jk})$$

satisfying $\tilde{g}^{pq}\tilde{W}_{ipjq} = 0$. Then a metric g solving the NRDF (2) fulfills

$$\begin{aligned}
&\frac{\partial}{\partial t}g_{ij} = g^{ab}\tilde{\nabla}_a\tilde{\nabla}_b g_{ij} - 2g_{ij}g^{kl}(g_{kl}-\tilde{g}_{kl}) + 2(g_{ij}-\tilde{g}_{ij}) + \frac{1}{2}g^{ab}g^{pq} \\
&\cdot\left(\tilde{\nabla}_i g_{pa}\tilde{\nabla}_j g_{qb} + 2\tilde{\nabla}_a g_{jp}\tilde{\nabla}_q g_{ib} - 2\tilde{\nabla}_a g_{jp}\tilde{\nabla}_b g_{iq} - 2\tilde{\nabla}_j g_{pa}\tilde{\nabla}_b g_{iq} - 2\tilde{\nabla}_i g_{pa}\tilde{\nabla}_b g_{jq}\right) \\
&-\left(g^{kl}-\tilde{g}^{kl}\right)(g_{ip}-\tilde{g}_{ip})\tilde{g}^{pq}\tilde{W}_{jkql} - \left(g^{kl}-\tilde{g}^{kl}\right)(g_{jp}-\tilde{g}_{jp})\tilde{g}^{pq}\tilde{W}_{ikql} \\
&-2\left(g^{kl}-\tilde{g}^{kl}\right)\tilde{W}_{ikjl}
\end{aligned}$$

Comparing with the evolution equations whose background metric is the standard hyperbolic metric h in [12] Lemma 2.1 and Lemma 2.2, the evolution equations with CCE metric $\tilde{g}$ as background metric have some extra terms involving the Weyl tensor, together with the assumption and Lemma 2.2 in [12], we have that

$$\frac{\partial}{\partial t}\|g-\tilde{g}\|^2 \leq g^{ij}\tilde{\nabla}_i\tilde{\nabla}_j\|g-\tilde{g}\|^2 - (2-\epsilon_3)\|\tilde{\nabla}g\|^2 + (4+4l+\epsilon_4)\|g-\tilde{g}\|^2,$$

By direct computation, we can get the other two evolution equations. □

Lemma 2.3. *Let $g(t)$, $t \in [0,T)$ be a solution to the NRDF (2). Then there exists some constants $L(t)$ depends only on t such that*

$$\|g-\tilde{g}\| \leq L(t)e^{-\delta\rho} \text{ and } \|\tilde{\nabla}g\| \leq L(t)e^{-\delta\rho}$$

for all $t \in [0,T)$ and $\delta > \frac{n-1}{2}$.

Proof. Let $\psi = e^{\omega\rho}\left(\|g-\tilde{g}\|^2 + \|\tilde{\nabla} g\|^2\right)$. It follows from Lemma 2.2 and the estimates (4) and (5) that for $t \in [0, T)$,

$$\frac{\partial}{\partial t}\psi \leq g^{ij}\tilde{\nabla}_i\tilde{\nabla}_j\psi + \left(C_1 - \omega^2 g^{ij}\tilde{\nabla}_i\rho \cdot \tilde{\nabla}_j\rho - \omega g^{ij}\tilde{\nabla}_i\tilde{\nabla}_j\rho\right)\psi$$
$$- 2\omega e^{\omega\rho} g^{ij}\tilde{\nabla}_i\rho \cdot \tilde{\nabla}_j\left(\|g-\tilde{g}\|^2 + \|\tilde{\nabla} g\|^2\right) - (2-\epsilon_5)\, e^{\omega\rho}\left(\|\tilde{\nabla} g\|^2 + \|\tilde{\nabla}^2 g\|^2\right)$$

In order to get the decay order of the metric, we modify the essential set to be sufficiently large, together with that $\tilde{g}$ is $C^{2,\alpha}$ conformally compact, by Lemma 2.1 in [7], we have that

$$\tilde{\Delta}\rho = n - 1 + O\left(e^{-a\rho}\right)$$

for some $a > 0$. In the light of the estimates

$$g^{ij}\tilde{\nabla}_i\rho\tilde{\nabla}_j\rho \geq (1-2\epsilon)\,\|\tilde{\nabla}\rho\|^2 = 1 - 2\epsilon,$$
$$g^{ij}\tilde{\nabla}_i\tilde{\nabla}_j\rho \geq (1-2\epsilon)\,\tilde{\Delta}\rho = (1-\epsilon_6)\,(n-1)\,,$$

and

$$|g^{ij}\tilde{\nabla}_i\rho\tilde{\nabla}_j\|\tilde{\nabla}^{k-1}(g-\tilde{g})\|^2|$$
$$\leq (1+2\epsilon)\,\|\tilde{\nabla}\langle\tilde{\nabla}^{(k-1)}(g-\tilde{g})\,,\tilde{\nabla}^{(k-1)}(g-\tilde{g})\rangle_{\tilde{g}}\|$$
$$\leq (1+2\epsilon)\left(b^2\|\tilde{\nabla}^k(g-\tilde{g})\|^2 + \frac{1}{b^2}\|\tilde{\nabla}^{k-1}(g-\tilde{g})\|^2\right),$$

where $b > 0$ is arbitrary, we can choose appropriate b and see that there exists a constant $C_2 > 0$, depending only on ϵ, ω and n, such that

$$\frac{\partial}{\partial t}\psi \leq g^{ij}\tilde{\nabla}_i\tilde{\nabla}_j\psi + C_2\psi$$

Let $\omega = 2\delta$, by using maximum principle outside a sufficiently large essential set (for the proof of this maximum principle, please see the proof of Lemma 4.2 in [10]), we get for $t \in [0, T)$,

$$\psi(\cdot, t) \leq \psi(\cdot, 0)e^{C_2 t} = 2\epsilon^2 e^{C_2 t}.$$

□

Now we move to the long-time existence and convergence of NRDF. In [12] Theorem 3.1, the authors use the first eigenvalue on hyperbolic domains to get a Lyapunov function and finally obtain the convergence. However, as the Weyl curvature will exert a strong influence, the main point to establish the exponential decay of the L^2 norm of $g(t) - \tilde{g}$ with respect to time here is to take advantage of the non-degenerate condition of the background conformally compact Einstein metric $\tilde{g}$, which leads to a linearized stability of

$\tilde{g}$. We now rewrite the evolution equations of $u(t) = g(t) - \tilde{g}$ to which the non-degenerate condition can be related.

Lemma 2.4. *Let $g(t)$, $t \in [0,T)$ be a solution to the NRDF (2). Then*

$$\begin{aligned}\frac{\partial}{\partial t}\|u\|^2 \leq & -2\langle\left(\tilde{\Delta}_L + 2(n-1)\right)u, u\rangle + \left(g^{ab} - \tilde{g}^{ab}\right)\tilde{\nabla}_a\tilde{\nabla}_b\|u\|^2 \\ & + \epsilon_7\left(\|\tilde{\nabla}u\|^2 + \|u\|^2\right).\end{aligned}$$

Proof. The 2-tensor $u(t)$ fulfills

$$\frac{\partial}{\partial t}u_{ij} = -\left(\tilde{\Delta}_L + 2(n-1)\right)u_{ij} + \mathcal{F}(g, \tilde{g}, u)$$

where

(7)

$$\begin{aligned}&\mathcal{F}(g,\tilde{g},u) \\ =& -2\tilde{R}_{ikjl}u^{kl} - g^{kl}g_{ip}\tilde{g}^{pq}\tilde{R}_{jkql} - g^{kl}g_{jp}\tilde{g}^{pq}\tilde{R}_{ikql} + \tilde{R}_{ik}u_j^k + \tilde{R}_{jk}u_i^k + \frac{1}{2}g^{ab}g^{pq} \\ &\cdot\left(\tilde{\nabla}_i g_{pa}\tilde{\nabla}_j g_{qb} + 2\tilde{\nabla}_a g_{jp}\tilde{\nabla}_q g_{ib} - 2\tilde{\nabla}_a g_{jp}\tilde{\nabla}_b g_{iq} - 2\tilde{\nabla}_j g_{pa}\tilde{\nabla}_b g_{iq} - 2\tilde{\nabla}_i g_{pa}\tilde{\nabla}_b g_{jq}\right) \\ &+ \left(g^{ab} - \tilde{g}^{ab}\right)\tilde{\nabla}_a\tilde{\nabla}_b g_{ij} - 2(n-1)\tilde{g}_{ij}.\end{aligned}$$

Hence

$$\mathcal{F}(g,\tilde{g},u) = \left(g^{ab} - \tilde{g}^{ab}\right)\tilde{\nabla}_a\tilde{\nabla}_b g_{ij} + \tilde{\nabla}g * \tilde{\nabla}g + \Theta$$

where

(8)

$$\begin{aligned}\tilde{\nabla}g * \tilde{\nabla}g =& \frac{1}{2}g^{ab}g^{pq}\cdot(\tilde{\nabla}_i g_{pa}\tilde{\nabla}_j g_{qb} + 2\tilde{\nabla}_a g_{jp}\tilde{\nabla}_q g_{ib} - 2\tilde{\nabla}_a g_{jp}\tilde{\nabla}_b g_{iq} - 2\tilde{\nabla}_j g_{pa}\tilde{\nabla}_b g_{iq} \\ & -2\tilde{\nabla}_i g_{pa}\tilde{\nabla}_b g_{jq})\end{aligned}$$

$$\begin{aligned}\Theta =& -2\tilde{R}_{ikjl}u^{kl} - g^{kl}g_{ip}\tilde{g}^{pq}\tilde{R}_{jkql} - g^{kl}g_{jp}\tilde{g}^{pq}\tilde{R}_{ikql} + \tilde{R}_{ik}u_j^k + \tilde{R}_{jk}u_i^k - 2(n-1)\tilde{g}_{ij} \\ =& -4\tilde{R}_{ikjl}u^{kl} + \left(\tilde{g}^{kl} - g^{kl}\right)u_{ip}\tilde{g}^{pq}\tilde{R}_{jkql} + \left(\tilde{g}^{kl} - g^{kl}\right)u_{jp}\tilde{g}^{pq}\tilde{R}_{ikql} + 2\left(g_{ab}\tilde{g}^{ak}\tilde{g}^{bl} - g^{kl}\right)\tilde{R}_{ikjl} \\ =& -4\tilde{W}_{ikjl}u^{kl} + 4\left(\tilde{g}_{ij}\tilde{g}_{kl} - \tilde{g}_{il}\tilde{g}_{kj}\right)u^{kl} + \left(\tilde{g}^{kl} - g^{kl}\right)u_{ip}\tilde{g}^{pq}\tilde{R}_{jkql} + \left(\tilde{g}^{kl} - g^{kl}\right)u_{jp}\tilde{g}^{pq}\tilde{R}_{ikql} \\ &+ 2\left(g_{ab}\tilde{g}^{ak}\tilde{g}^{bl} - g^{kl}\right)\tilde{W}_{ikjl} - 2\left(g_{ab}\tilde{g}^{ak}\tilde{g}^{bl} - g^{kl}\right)\left(\tilde{g}_{ij}\tilde{g}_{kl} - \tilde{g}_{il}\tilde{g}_{kj}\right) \\ =& -4\tilde{W}_{ikjl}u^{kl} + 2\left(g_{ab}\tilde{g}^{ak}\tilde{g}^{bl} - g^{kl}\right)\tilde{W}_{ikjl} + 2\left(g_{ab}\tilde{g}^{ak}\tilde{g}^{bl} + g^{kl} - 2\tilde{g}^{kl}\right)\left(\tilde{g}_{ij}\tilde{g}_{kl} - \tilde{g}_{il}\tilde{g}_{kj}\right) \\ &+ \left(\tilde{g}^{kl} - g^{kl}\right)u_{ip}\tilde{g}^{pq}\tilde{R}_{jkql} + \left(\tilde{g}^{kl} - g^{kl}\right)u_{jp}\tilde{g}^{pq}\tilde{R}_{ikql}.\end{aligned}$$

We decompose Θ into three parts

$$\Theta = \mathbf{P} + \mathbf{Q} + \mathbf{S}$$

where

$$\mathbf{P} = -4\tilde{W}_{ikjl}u^{kl} + 2\left(g_{ab}\tilde{g}^{ak}\tilde{g}^{bl} - g^{kl}\right)\tilde{W}_{ikjl} = -2\left(g_{ab}\tilde{g}^{ak}\tilde{g}^{bl} + g^{kl} - 2\tilde{g}^{kl}\right)\tilde{W}_{ikjl}$$

$$\begin{aligned}\mathbf{Q} =& 2\left(g_{ab}\tilde{g}^{ak}\tilde{g}^{bl} + g^{kl} - 2\tilde{g}^{kl}\right)\left(\tilde{g}_{ij}\tilde{g}_{kl} - \tilde{g}_{il}\tilde{g}_{kj}\right) \\ =& 2\left(g_{ab}\tilde{g}^{ab}\tilde{g}_{ij} - g_{ij} + g^{kl}\tilde{g}_{kl}\tilde{g}_{ij} - g^{kl}\tilde{g}_{il}\tilde{g}_{kj} - 2(n-1)\tilde{g}_{ij}\right)\end{aligned}$$

and

$$\mathbf{S} = \left(\tilde{g}^{kl} - g^{kl}\right)u_{ip}\tilde{g}^{pq}\tilde{R}_{jkql} + \left(\tilde{g}^{kl} - g^{kl}\right)u_{jp}\tilde{g}^{pq}\tilde{R}_{ikql}.$$

Choose a coordinate system $\{x^i\}$ such that at one point, we have $\tilde{g}_{ij} = \delta_{ij}$ and $g_{ij} = \lambda_i\delta_{ij}$ with $|\lambda_i - 1| \leq 2\epsilon$. From the assumption,

$$\begin{aligned}\frac{\partial}{\partial t}\|u\|^2 =& 2\sum_i u_{ii}\frac{\partial}{\partial t}u_{ii} \\ =& -2\langle\left(\tilde{\Delta}_L + 2(n-1)\right)u, u\rangle + 2\langle\left(g^{ab} - \tilde{g}^{ab}\right)\tilde{\nabla}_a\tilde{\nabla}_b u, u\rangle \\ &+ \sum_i\left(\tilde{\nabla}g * \tilde{\nabla}g\right)_{ii}u_{ii} + 2\sum_i\Theta_{ii}u_{ii}.\end{aligned}$$

We have

$$\sum_i\left(\tilde{\nabla}g * \tilde{\nabla}g\right)_{ii}u_{ii} \leq \epsilon_8\|\tilde{\nabla}u\|^2$$

and

$$\begin{aligned}&2\langle\left(g^{ab} - \tilde{g}^{ab}\right)\tilde{\nabla}_a\tilde{\nabla}_b u, u\rangle \\ =& \left(g^{ab} - \tilde{g}^{ab}\right)\tilde{\nabla}_a\tilde{\nabla}_b\|u\|^2 - 2\left(g^{ab} - \tilde{g}^{ab}\right)\langle\tilde{\nabla}_a u, \tilde{\nabla}_b u\rangle \\ \leq& \left(g^{ab} - \tilde{g}^{ab}\right)\tilde{\nabla}_a\tilde{\nabla}_b\|u\|^2 + 4\epsilon\|\tilde{\nabla}u\|^2\end{aligned}$$

Next we only need to examine the term $\sum_i\left(\mathbf{P} + \mathbf{Q} + \mathbf{S}\right)_{ii}u_{ii}$. It is obvious that

$$\begin{aligned}\sum_i\mathbf{P}_{ii}u_{ii} =& -2\sum_i\sum_k\left(\frac{1}{\lambda_k} + \lambda_k - 2\right)(\lambda_i - 1)\tilde{W}_{ikik} \\ =& -2\sum_i\sum_k\frac{(\lambda_k - 1)^2}{\lambda_k}(\lambda_i - 1)\tilde{W}_{ikik} \\ \leq& \epsilon_8\|u\|^2,\end{aligned}$$

$$\begin{aligned}\sum_i \mathbf{Q}_{ii} u_{ii} =& 2\sum_i \left(\sum_k \lambda_k - \lambda_i + \sum_k \frac{1}{\lambda_k} - \frac{1}{\lambda_i} - 2(n-1)\right)(\lambda_i - 1)\\ =& 2\sum_i (\lambda_i - 1)\sum_{k\neq i} \frac{(\lambda_k-1)^2}{\lambda_k}\\ \leq& \epsilon_8 \|u\|^2,\end{aligned}$$

and

$$\sum_i \mathbf{S}_{ii} u_{ii} \leq \epsilon_8 \|u\|^2.$$

Hence we finish the proof. □

An immediate consequence is

Theorem 2.5. *Let $n \geq 4$. Suppose that $(M^n, \tilde{g})$ is a $C^{2,\alpha}$ conformally compact Einstein manifold with non-degeneracy $\lambda > 0$ and g is a Riemannian metric on M such that*

$$\|g - \tilde{g}\|_{C^1(M^n,\tilde{g})} \leq \epsilon e^{-\delta\rho}$$

where $\delta > \frac{n-1}{2}$. Let $g(t)$, $t \in [0, T)$ be a solution to the NRDF (2). Then

$$\int_M \|g(t) - \tilde{g}\|^2 d\mu \leq e^{-(2-\epsilon_9)\lambda t} \int_M \|g_0 - \tilde{g}\|^2 d\mu$$

for all $t \in [0, T)$ and some $\epsilon_9 > 0$.

Proof. By the assumption $\delta > \frac{n-1}{2}$ and Lemma 2.3, we have that $\int_M \|u(t)\|^2 d\mu$ and $\int_M \|\tilde{\nabla} u(t)\|^2 d\mu$ are finite for all $t \in [0, T)$. Moreover, $\|u(x,t)\| = 0$ and $\|\tilde{\nabla} u(x,t)\| = 0$ for $(x,t) \in \partial M \times [0,T)$. It immediately follows from Lemma 2.4 that

$$\begin{aligned}&\frac{\partial}{\partial t}\int_M \|u(t)\|^2 d\mu\\ \leq& \int_M -2\langle\left(\tilde{\Delta}_L + 2(n-1)\right)u, u\rangle + \left(g^{ab} - \tilde{g}^{ab}\right)\tilde{\nabla}_a \tilde{\nabla}_b \|u\|^2\\ &+ \epsilon_7\left(\|\tilde{\nabla} u\|^2 + \|u\|^2\right) d\mu.\end{aligned}$$

Using Divergence Theorem and Kato's inequality we have

$$\begin{aligned}
&\int_M \left(g^{ab}-\tilde{g}^{ab}\right)\tilde{\nabla}_a\tilde{\nabla}_b\|u\|^2 d\mu\\
=&-\int_M \tilde{\nabla}_a g^{ab}\tilde{\nabla}_b\|u\|^2 d\mu\\
\leq&\int_M 2\epsilon_1\|u\|\cdot\|\tilde{\nabla}\|u\|\|d\mu\\
\leq&\epsilon_1\int_M\left(\|u\|^2+\|\tilde{\nabla}u\|^2\right)d\mu
\end{aligned}$$

In light of the non-degenerate condition, we have

$$\begin{aligned}
&\int_M -2\langle\Big(\tilde{\Delta}_L+2(n-1)\Big)u,u\rangle d\mu\\
=&-(2-a)\int_M\langle\Big(\tilde{\Delta}_L+2(n-1)\Big)u,u\rangle d\mu-a\int_M\langle\Big(\tilde{\Delta}_L+2(n-1)\Big)u,u\rangle d\mu\\
\leq&-(2-a)\lambda\int_M\|u\|^2 d\mu+a\int_M\langle\tilde{\Delta}u,u\rangle d\mu+2a\int_M\tilde{R}_{ipjq}u^{pq}u^{ij}d\mu\\
\leq&\Big(-(2-a)\lambda+2a\tilde{k}\Big)\int_M\|u\|^2 d\mu-a\int_M\|\tilde{\nabla}u\|^2 d\mu
\end{aligned}$$

here and in the sequel $\tilde{k}$ denotes the upper bound of $\tilde{R}m$ Therefore

$$\begin{aligned}
\frac{\partial}{\partial t}\int_M\|u\|^2 d\mu\leq&\Big(-(2-a)\lambda+2a\tilde{k}+\epsilon_1+\epsilon_7\Big)\int_M\|u\|^2 d\mu\\
&+(-a+\epsilon_1+\epsilon_7)\int_M\|\tilde{\nabla}u\|^2 d\mu
\end{aligned}$$

which implies

$$\int_M\|g(t)-\tilde{g}\|^2 d\mu\leq e^{-(2-\epsilon_9)\lambda t}\int_M\|g_0-\tilde{g}\|^2 d\mu$$

if we choose $a=\epsilon_1+\epsilon_7$. □

Now we prove the long-time existence and convergence.

Proof of Theorem 2.1. Now that we have the exponential convergence of the L^2-norm of $\|g-\tilde{g}\|$, by the same idea as that in Theorem 3.3 in [12], we get exponential convergence in the sup-norm of $\|g-\tilde{g}\|$. For the long-time existence and convergence, we have to check that the injective radius is bounded along NRDF in order that the interior estimates and gradient estimates in Theorem 3.3, Theorem 3.4 and Theorem 3.5 in [12] make sense. Since $\|g(t)-\tilde{g}\|$ and $\|\tilde{\nabla}^2 g(t)\|$ decay exponentially by the same arguments in Theorem 3.4

and Theorem 3.5, together with the analysis in the introduction, the injective radius is always bounded along NRDF. □

Now Let g and $\tilde{g}$ be as stated in Theorem 1.2, and

$$\delta \in \left(n-1, \frac{(n-1)+\sqrt{(n-1)^2+4(n-1)}}{2} \right).$$

For $t \in [1,\infty)$, it follows immediately that there exist some constants $\sigma_i = \sigma_i(n,\lambda,\epsilon,\tilde{g}) > 0$ and $C_3 > 0$ such that

$$\|\tilde{\nabla}^i g(t)\| \le C_3 e^{-\sigma_i t}, \quad i = 1,2,3. \tag{9}$$

In the mean time, there exists some $\tilde{\epsilon} > 0$ which depends only on ϵ, λ, n, k and $\tilde{g}$ such that

$$\|g(t) - \tilde{g}\| \le \tilde{\epsilon} \tag{10}$$

for all $t \in [0,\infty)$.

As a consequence of Lemma 2.2, we can prove the following:

Lemma 2.6. *Let $g(t)$, $t \in [0,\infty)$ be a solution to the NRDF (2). Then there exist constants $\hat{\epsilon}, \bar{\epsilon} > 0$, depending only on ϵ, λ, n, k and $\tilde{g}$ such that*

(1) *for all $t \in [0,1]$, $\|g(t) - \tilde{g}\| \le \hat{\epsilon} e^{-\delta\rho}$, $\|\tilde{\nabla} g(t)\| \le \hat{\epsilon} e^{-\delta\rho}$, where*

$$\delta \in \left(n-1, \frac{(n-1)+\sqrt{(n-1)^2+4(n-1)}}{2} \right);$$

(2) *for all $t \in [1,\infty)$, $\|g(t) - \tilde{g}\| \le \bar{\epsilon} e^{-\sigma_4 t} e^{-\gamma\rho}$, $\|\tilde{\nabla}^k g(t)\| \le \bar{\epsilon} e^{-\sigma_4 t} e^{-\gamma\rho}$, $k = 1,2$, where $\sigma_4 > 0$ is small and*

$$\gamma \in \left(\frac{n-1}{2} - \sqrt{\frac{(n-1)^2}{4} - 2}, \frac{n-1}{2} + \sqrt{\frac{(n-1)^2}{4} - 2} \right).$$

Proof. For $t \in [0,1]$, let $\phi = e^{\mu\rho}\left(\|g-\tilde{g}\|^2 + \|\tilde{\nabla} g\|^2 + t\|\tilde{\nabla}^2 g\|^2 \right)$. It follows from Lemma 2.2 and the estimates (4), (5) and (6) that

$$\begin{aligned}
\frac{\partial}{\partial t}\phi \le\; & g^{ij}\tilde{\nabla}_i\tilde{\nabla}_j\phi + \left(C_4 - \mu^2 g^{ij}\tilde{\nabla}_i\rho\cdot\tilde{\nabla}_j\rho - \mu g^{ij}\tilde{\nabla}_i\tilde{\nabla}_j\rho \right)\phi \\
& - 2\mu e^{\mu\rho} g^{ij}\tilde{\nabla}_i\rho\cdot\tilde{\nabla}_j\left(\|g-\tilde{g}\|^2 + \|\tilde{\nabla} g\|^2 + t\|\tilde{\nabla}^2 g\|^2 \right) \\
& - (2-\epsilon_{10}) e^{\mu\rho}\left(\|\tilde{\nabla} g\|^2 + \|\tilde{\nabla}^2 g\|^2 + t\|\tilde{\nabla}^3 g\|^2 \right) + \left(1 + \epsilon_{10}\sqrt{t} \right)\|\tilde{\nabla}^2 g\|^2
\end{aligned}$$

Using the same way in Lemma 2.3 we see that there exists a constant $C_5 > 0$, depending only on ϵ, λ, n, k and $\tilde{g}$, such that

$$\frac{\partial}{\partial t}\phi \le g^{ij}\tilde{\nabla}_i\tilde{\nabla}_j\phi + C_5\phi$$

Let $\mu = 2\delta$, by maximum principle, we get for $t \in [0,1]$,

$$\phi \le \hat{\epsilon}^2 := \phi(\cdot,0)e^{C_5} = 8\epsilon^2 e^{C_5}.$$

Hence

$$\|\tilde{\nabla}^2 g\| \le \frac{\hat{\epsilon}}{\sqrt{t}}e^{-\delta\rho},$$

for all $t \in (0,1]$.

In particular, at $t = 1$,

$$\|g-\tilde{g}\|(\cdot,1), \|\tilde{\nabla}g\|(\cdot,1), \|\tilde{\nabla}^2 g\|(\cdot,1) \le \hat{\epsilon}e^{-\delta\rho}. \tag{11}$$

For $t \in [1,\infty)$, let $\varphi = e^{\sigma_4 t}e^{\nu\rho}\left(\|g-\tilde{g}\|^2 + a\|\tilde{\nabla}g\|^2 + b\|\tilde{\nabla}^2 g\|^2\right)$. Due to (9) and (10), we can choose a and b such that

$$\begin{aligned}\frac{\partial}{\partial t}\varphi \le\; & g^{ij}\tilde{\nabla}_i\tilde{\nabla}_j\varphi + \left(4 + 4l + \tilde{\epsilon} + \sigma_4 - \nu^2 g^{ij}\tilde{\nabla}_i\rho\cdot\tilde{\nabla}_j\rho - \nu g^{ij}\tilde{\nabla}_i\tilde{\nabla}_j\rho\right)\varphi \\ & - (2-\epsilon_{11})\, e^{\sigma_4 t}e^{\nu\rho}\left(\|\tilde{\nabla}g\|^2 + a\|\tilde{\nabla}^2 g\|^2 + b\|\tilde{\nabla}^3 g\|^2\right) \\ & - 2\nu e^{\sigma_4 t}e^{\nu\rho}g^{ij}\tilde{\nabla}_i\rho\cdot\tilde{\nabla}_j\left(\|g-\tilde{g}\|^2 + a\|\tilde{\nabla}g\|^2 + b\|\tilde{\nabla}^2 g\|^2\right)\end{aligned}$$

Let $c + d = -2\nu$, then

$$\begin{aligned}& - 2\nu e^{\nu\rho}g^{ij}\tilde{\nabla}_i\rho\cdot\tilde{\nabla}_j\|\tilde{\nabla}^k(g-\tilde{g})\|^2 \\ =& (c+d)\, e^{\nu\rho}g^{ij}\tilde{\nabla}_i\rho\cdot\tilde{\nabla}_j\|\tilde{\nabla}^k(g-\tilde{g})\|^2 \\ =& ce^{\nu\rho}g^{ij}\tilde{\nabla}_i\rho\cdot\tilde{\nabla}_j\|\tilde{\nabla}^k(g-\tilde{g})\|^2 + de^{\nu\rho}g^{ij}\tilde{\nabla}_i\rho\cdot\tilde{\nabla}_j\|\tilde{\nabla}^k(g-\tilde{g})\|^2 \\ \le& |c|\,(1+\tilde{\epsilon})\left(m^2 e^{\nu\rho}\|\tilde{\nabla}^{k+1}(g-\tilde{g})\|^2 + \frac{1}{m^2}e^{\nu\rho}\|\tilde{\nabla}^k(g-\tilde{g})\|^2\right) \\ & + \left(-b\nu g^{ij}\tilde{\nabla}_i\rho\cdot\tilde{\nabla}_j\rho\left(e^{\nu\rho}\|\tilde{\nabla}^k(g-\tilde{g})\|^2\right) + dg^{ij}\tilde{\nabla}_i\rho\cdot\tilde{\nabla}_j\left(e^{\nu\rho}\|\tilde{\nabla}^k(g-\tilde{g})\|^2\right)\right) \\ =& |c|\,(1+\tilde{\epsilon})\, m^2 e^{\nu\rho}\|\tilde{\nabla}^{k+1}(g-\tilde{g})\|^2 + dg^{ij}\tilde{\nabla}_i\rho\cdot\tilde{\nabla}_j\left(e^{\nu\rho}\|\tilde{\nabla}^k(g-\tilde{g})\|^2\right) \\ & + \left(\frac{|c|\,(1+\tilde{\epsilon})}{m^2} - d\nu g^{ij}\tilde{\nabla}_i\rho\cdot\tilde{\nabla}_j\rho\right)\left(e^{\nu\rho}\|\tilde{\nabla}^k(g-\tilde{g})\|^2\right).\end{aligned}$$

Let $|c|\left(1+\tilde{\epsilon}\right)m^2=2-\epsilon_{11}$, then we have

(12)

$$\begin{aligned}\frac{\partial}{\partial t}\varphi &\leq g^{ij}\tilde{\nabla}_i\tilde{\nabla}_j\varphi+\left(-2\nu-c\right)g^{ij}\tilde{\nabla}_i\rho\cdot\tilde{\nabla}_j\varphi\\ &\left(4+4l+\tilde{\epsilon}+\sigma_4+\frac{\left(1+\tilde{\epsilon}\right)^2c^2}{2-\epsilon_{11}}+\left(\nu^2+c\nu\right)g^{ij}\tilde{\nabla}_i\rho\cdot\tilde{\nabla}_j\rho-\nu g^{ij}\tilde{\nabla}_i\tilde{\nabla}_j\rho\right)\varphi\\ &\leq g^{ij}\tilde{\nabla}_i\tilde{\nabla}_j\varphi+\left(-2\nu-c\right)g^{ij}\tilde{\nabla}_i\rho\cdot\tilde{\nabla}_j\varphi+C_6\varphi\end{aligned}$$

where

$$C_6=4+4l+\tilde{\epsilon}+\sigma_4+\frac{\left(1+\tilde{\epsilon}\right)^2c^2}{2-\epsilon_{11}}+\left(\nu^2+c\nu\right)\left(1+\tilde{\epsilon}\right)-\nu\left(n-1\right)\left(1-\tilde{\epsilon}\right)\leq 0$$

$\tilde{\epsilon}$ and σ_4 are sufficiently small, l is also sufficiently small as the the essential set can be chosen sufficiently large, $c=-\nu$ and $\nu=2\gamma$ for a given

$$\gamma\in\left(\frac{n-1}{2}-\sqrt{\frac{\left(n-1\right)^2}{4}-2},\frac{n-1}{2}+\sqrt{\frac{\left(n-1\right)^2}{4}-2}\right).$$

Since we have estimates (11) at $t=1$,

$$\varphi(\cdot,1)=e^{\sigma_4}e^{2\gamma\rho}\left(\|g-\tilde{g}\|^2+a\|\tilde{\nabla}g\|^2+b\|\tilde{\nabla}^2g\|^2\right)\leq e^{\sigma_4}\hat{\epsilon}^2,$$

it follows from the evolution equation (12) and maximum principle that

$$\varphi\left(\cdot,t\right)\leq\bar{\epsilon}^2=\frac{e^{\sigma_4}\hat{\epsilon}^2}{\min\{a,b,1\}}$$

for $t\in[1,\infty)$. Note that for our purpose here we need a variant of the maximum principle in Theorem 4.2 in [10], where Qing, Shi and Wu proved a variant of the maximum principle in Theorem 4.3 in [5], originally from [9]. The proof goes the same as that in [5] Theorem 4.3 and [10] Theorem 4.2 if we change the time-dependent laplacian Δ_{g_t} there to be $g^{ij}\tilde{\nabla}_i\tilde{\nabla}_j$ in our case. □

Lemma 2.7. *Under the NRDF, for all* $t\in[0,\infty)$, *the* 1*-form* V *satisfies*

$$\|V\left(\cdot,t\right)\|\leq\tilde{C}e^{-\tilde{\sigma}t}e^{-\delta\rho},$$

where $\delta\in\left(n-1,\frac{(n-1)+\sqrt{(n-1)^2+4(n-1)}}{2}\right)$, $\tilde{C}$ *depends only on* ϵ, λ, n, k *and* $\tilde{g}$, *and arbitrary* $\tilde{\sigma}>0$.

Proof. Let $E = Ric + (n-1)\,g$, we consider the evolution equation of V under NRDF,

$$\begin{aligned}\frac{\partial}{\partial t}V_j &= \frac{\partial}{\partial t}[g_{jk}g^{pq}\left(\Gamma^k_{pq} - \tilde{\Gamma}^k_{pq}\right)] \\ &= g_{jk}g^{pq}\frac{\partial}{\partial t}\Gamma^k_{pq} + \frac{\partial}{\partial t}\left(g_{jk}g^{pq}\right)\left(\Gamma^k_{pq} - \tilde{\Gamma}^k_{pq}\right) \\ &= \mathbf{I} + \mathbf{J}\end{aligned}$$

where

$$\begin{aligned}\mathbf{I} =& \frac{1}{2}g^{kl}g_{jk}g^{pq}\left(\nabla_p\left(-2E_{lq} + \nabla_l V_q + \nabla_q V_l\right) + \nabla_q\left(-2E_{lp} + \nabla_l V_p + \nabla_p V_l\right)\right) \\ & - \frac{1}{2}g^{kl}g_{jk}g^{pq}\left(\nabla_l\left(-2E_{pq} + \nabla_q V_p + \nabla_p V_q\right)\right) \\ =& - g^{pq}\left(\nabla_p R_{jq} + \nabla_q R_{jp} - \nabla_j R_{pq}\right) + \Delta V_j \\ & + \frac{1}{2}g^{pq}\left(\nabla_p\nabla_j V_q - \nabla_j\nabla_p V_q + \nabla_q\nabla_j V_p - \nabla_j\nabla_q V_p\right) \\ =& - \left(2\left(div_g Ric\right)_j - \nabla_j R\right) + \Delta V_j + R^k_j V_k \\ =& \Delta V_j + R^k_j V_k\end{aligned}$$

and

$$\begin{aligned}\mathbf{J} =& \frac{\partial}{\partial t}\left(g_{jk}g^{pq}\right)\left(\Gamma^k_{pq} - \tilde{\Gamma}^k_{pq}\right) \\ =& \left(g^{pq}\left(-2E_{jk} + \nabla_j V_k + \nabla_k V_j\right) - g_{jk}g^{ap}g^{bq}\left(-2E_{ab} + \nabla_a V_b + \nabla_b V_a\right)\right)\left(\Gamma^k_{pq} - \tilde{\Gamma}^k_{pq}\right) \\ \leq& C_7\|g - \tilde{g}\|_{C^2}\cdot\|g - \tilde{g}\|_{C^1}.\end{aligned}$$

From the above discussion, it follows that for $t \in [1, \infty)$,

$$\|R^k_j + (n-1)\,\delta^k_j\| \leq C_8\|g - \tilde{g}\|_{C^2}$$

and

$$\|V\| \leq C_9\|g - \tilde{g}\|_{C^1},$$

then we have

$$\frac{\partial}{\partial t}V_j \leq \Delta V_j - (n-1)\,V_j + C_{10}\|g - \tilde{g}\|_{C^2}\cdot\|g - \tilde{g}\|_{C^1},$$

where Δ and ∇ are with respect to g, which implies that

$$\begin{aligned}\frac{\partial}{\partial t}\|V\| \leq& \Delta\|V\| - (n-1)\,\|V\| + C_{10}\|g - \tilde{g}\|_{C^2}\cdot\|g - \tilde{g}\|_{C^1} \\ \leq& \Delta\|V\| - (n-1)\,\|V\| + C_{10}\bar{\epsilon}^2 e^{-2\sigma_4 t}e^{-2\gamma\rho}\end{aligned}$$

At $t=1$, due to Lemma 2.6, we have

$$\|V(\cdot,1)\| \le \epsilon_{13} e^{-\delta\rho}.$$

Let $2\gamma=\delta$ and $v=e^{\delta\rho}\|V\|$, we can see that v satisfies

$$\begin{aligned}\frac{\partial}{\partial t}v \le& \Delta v - 2\delta\nabla\rho\cdot\nabla v - \left(\delta\Delta\rho + n - 1 - \delta^2\|\nabla\rho\|^2\right)v + C_{10}\bar{\epsilon}^2 e^{-2\sigma_4 t}\\ \le& \Delta v - 2\delta\nabla\rho\cdot\nabla v - Bv + C_{10}\bar{\epsilon}^2 e^{-2\sigma_4 t}\end{aligned} \tag{13}$$

where B is a positive constant when $\delta \in \left(n-1, \frac{(n-1)+\sqrt{(n-1)^2+4(n-1)}}{2}\right)$ and we choose $B \ne 2\sigma_4$ and the essential set to be sufficiently large.

Consider ODE

$$\begin{cases} \frac{du}{dt} = -Bu + C_{10}\bar{\epsilon}^2 e^{-2\sigma_4 t}, t\in[1,\infty),\\ u(1) = \epsilon_{13}.\end{cases} \tag{14}$$

the solution

$$u(t) = \epsilon_{13} e^{B} e^{-Bt} + \frac{C_{10}\bar{\epsilon}^2}{B-2\sigma_4}\left(e^{-2\sigma_4 t} - e^{B-2\sigma_4} e^{-Bt}\right).$$

Since u is a subsolution to the equation (13) with $v(\cdot,1)\le u(1)$, due to Theorem 4.2 in [10], we have $v(\cdot,t)\le u(t)$ for all $t\in[1,\infty)$. Hence

$$\|V(\cdot,t)\| \le C_{11} e^{-\tilde{\sigma}t} e^{-\delta\rho}$$

for all $t\in[1,\infty)$, where $\tilde{\sigma} = \min\{2\sigma_4, B\}$ and C_{11} depends only on ϵ, λ, n, k and $\tilde{g}$. Together with Lemma 2.6, we conclude that

$$\|V(\cdot,t)\| \le \tilde{C} e^{-\tilde{\sigma}t} e^{-\delta\rho}$$

for all $t\in[0,\infty)$. □

Lemma 2.8. *Under the NRDF, the scalar curvature R satisfies*

$$R(\cdot,t) \ge -n(n-1)$$

for all $t\in[0,\infty)$, and

$$|R(\cdot,t) + n(n-1)| \le \bar{C} e^{-\bar{\sigma}t} e^{-\delta\rho}$$

for all $t\in[1,\infty)$ where $\delta \in \left(n-1, \frac{(n-1)+\sqrt{(n-1)^2+4(n-1)}}{2}\right)$, $\bar{C}$ depends only on ϵ, λ, n, k and $\tilde{g}$, and arbitrary $\bar{\sigma}>0$.

Proof. By a direct computation, we see that under the NRF of $\bar{g}$, the scalar curvature $\bar{R} = R(\bar{g}(t))$ satisfies the following evolution equation

$$\frac{\partial}{\partial t}\bar{R} = \Delta\bar{R} + 2\|\bar{R}ic + (n-1)\bar{g}\|_{\bar{g}}^2 - 2(n-1)(\bar{R} + n(n-1)),$$

where Δ and $\|\cdot\|_{\bar{g}}$ are with respect to the solution of NRF $\bar{g}(t)$. Let $\bar{S} = \bar{R} + n(n-1)$, under NRF, it satisfies

$$\frac{\partial}{\partial t}\bar{S} = \Delta\bar{S} + 2\|\bar{R}ic + (n-1)\bar{g}\|_{\bar{g}}^2 - 2(n-1)\bar{S},$$

with $\bar{S}(0) \geq 0$. Then by maximum principle due to Karp and Li (See Theorem 7.39 in [4]), we see that

$$\bar{S}(t) \geq 0,$$

which means

$$R(\bar{g}(t)) \geq -n(n-1).$$

Because of the diffeomorphism invariance, we have

$$R(g(t)) = R(\Phi_t^* g(t)) = R(\bar{g}(t)) \geq -n(n-1).$$

Moreover, as

$$\begin{aligned} &\|\bar{R}ic + (n-1)\bar{g}\|_{\bar{g}} \\ =&\|Ric(\Phi_t^* g) + (n-1)\Phi_t^* g\|_{\Phi_t^* g} \\ =&\|Ric(g(t)) + (n-1)g(t)\|_{g(t)} \\ \leq& C_{12}\|g(t) - \tilde{g}\|_{C^2}, \end{aligned}$$

by the same arguments as those in the proof of Lemma 2.7 and note that the constant $-2(n-1)$ before the zero order term S will make B in ODE more positive hence will bring no trouble to the order, we have

$$|\bar{S}| \leq C_{13} e^{-\sigma_5 t} e^{-\delta\rho} \tag{15}$$

where C_{13} depends only on ϵ, λ, n, k and $\tilde{g}$.

For any $x \in M$ and $s \in (0,t)$, let $\gamma(s) = \Phi_s(x,s)$ be an integral curve of V, together with Lemma 2.7, we see that

$$\begin{aligned} d_{\tilde{g}}(\Phi_t(x,t), x) &\leq \int_0^t \|\dot{\gamma}(s)\| ds \\ &\leq C_{14}\int_0^t \|V(x,s)\| ds \\ &\leq C_{15} e^{-\delta\rho} \end{aligned} \tag{16}$$

where $d_{\tilde{g}}(,)$ denotes the distance function in M with respect to metric $\tilde{g}$. Hence, we get

$$\begin{aligned}|R(g(t))(x)+n(n-1)| &\le |\bar{S}(x)|+|R(g(t))(x)-R(\bar{g}(t))(x)|\\ &\le |\bar{S}(x)|+|R(g(t))(x)-R(g(t))(\Phi_t(x,t))|\\ &\le |\bar{S}(x)|+C_{16}\|\tilde{\nabla}R\|d_{\tilde{g}}(\Phi_t(x,t),x)\\ &\le C_{17}e^{-\sigma_6 t}e^{-\delta\rho}\end{aligned}$$

where we used (15), (16) and $\|\tilde{\nabla}R\|\le C_{18}\|\tilde{\nabla}g\|_{C^3}\le C_{19}e^{-\sigma_7 t}$ for $t\in[1,\infty)$ in above inequality. □

3. Some Basic Estimates and Proof of Main Results

In this section, we first show that the renormalized volume is well-defined under some conditions and then give some basic estimates, and finally we prove our main results Theorem 1.2 and Theorem 1.3.

Lemma 3.1. *Suppose that $(M,\tilde{g})$ is $C^{2,\alpha}$ conformally compact manifold. If g is a metric on M satisfying $\|g-\tilde{g}\|_{C^0(M,\tilde{g})}\le Ke^{-\delta\rho}$, where $\delta>n-1$ and ρ is the distance function to some essential set in M with respect to $\tilde{g}$. Then $\mathcal{V}_{\tilde{g}}(g)$ is well-defined, i.e. for any exhausting domains $\{\Omega_i\}$ of M with $\Omega_i\subset\Omega_{i+1}$,*

$$\mathcal{V}_{\tilde{g}}(g)=\lim_{i\to\infty}I_i\triangleq\lim_{i\to\infty}\int_{\Omega_i}\left(\sqrt{|g|}-\sqrt{|\tilde{g}|}\right)dx,$$

Proof. It suffices to show that $\lim_{i\to\infty}\int_{\Omega_i}\left(\sqrt{|g|}-\sqrt{|\tilde{g}|}\right)dx$ exists and is finite for any exhausting domains $\{\Omega_i\}$ of M with $\Omega_i\subset\Omega_{i+1}$. Since $(M^n,\tilde{g})$ is $C^{2,\alpha}$ conformally compact, $\tilde{g}$ has the form

$$\tilde{g}=\tau^{-2}(d\tau^2+\tilde{g}_\tau),$$

where τ is the special defining function for some representative in the conformal infinity of $(M,\tilde{g})$. Without loss of generality, we may assume

$$\{x\in M:\tau(x)\ge\tau_i\}\subset\Omega_i\subset\{x\in M:\tau(x)\ge\tau_{i+1}\},$$

where $\{\tau_i\}$ is a decreasing sequence approaching to 0.

By a direct computation together with the fact that $\frac{1}{A}e^{-\rho}\le\tau\le Ae^{-\rho}$ for some constant $A>0$, we see that for any $i>j$ we have

$$
\begin{aligned}
|I_i - I_j| &\le \int_{\{x\in M:\tau_{i+1}\le\tau(x)\le\tau_j\}} |\sqrt{|g|}-\sqrt{|\tilde{g}|}|dx \\
&\le \int_{\{x\in M:\tau_{i+1}\le\tau(x)\le\tau_j\}} |1-\sqrt{\frac{|g|}{|\tilde{g}|}}|\sqrt{|\tilde{g}|}dx \\
&\le C\left(K,\tilde{g},n\right)\int_0^{\tau_j}\tau^{\delta-n}d\tau \\
&\le C\left(K,\tilde{g},n,\delta\right)\tau_j^{\delta-n+1},
\end{aligned}
$$

which implies that $\{I_i\}$ is a convergence sequence and thus we finish to prove the lemma.

□

Now let g and $\tilde{g}$ be as in Theorem 1.2.

Remark 3.2. *Due to Lemma 3.1 and Lemma 2.3, we see that for each t, $\mathcal{V}(g(t))$ is well-defined.*

Now, we can show that

Proposition 3.3. *Let $g\left(t\right)$ be a solution to the NRDF (2), then we have*

$$
\mathcal{V}\left(g\left(t\right)\right)=\mathcal{V}\left(g\left(0\right)\right)-\int_0^t\int_M\left(R\left(g\left(s\right)\right)+n\left(n-1\right)\right)d\mu_g ds,
$$

where $d\mu_g$ is volume element with respect to metric $g(s)$. Moreover, $\mathcal{V}\left(g\left(t\right)\right)$ is non-increasing in t and

$$
\lim_{t\to\infty}\mathcal{V}\left(g\left(t\right)\right)=0. \tag{17}
$$

Proof. Let Ω be any compact domain in M with smooth boundary, then by a direct computation, under NRDF (2), we have

$$
\frac{d}{dt}\int_\Omega\left(\sqrt{|g|}-\sqrt{|\tilde{g}|}\right)dx=-\int_\Omega\left(R+n(n-1)\right)d\mu_g+\int_{\partial\Omega}\langle V,\nu\rangle_g d\sigma,
$$

where ν is the outward unit normal vector of $\partial\Omega$, hence, we obtain

$$
\begin{aligned}
&\int_\Omega\left(\sqrt{|g(t)|}-\sqrt{|\tilde{g}|}\right)dx-\int_\Omega\left(\sqrt{|g(t_0)|}-\sqrt{|\tilde{g}|}\right)dx \\
&=-\int_{t_0}^t\int_\Omega\left(R\left(g(s)\right)+n(n-1)\right)d\mu_g ds+\int_{t_0}^t\int_{\partial\Omega}\langle V,\nu\rangle_g d\sigma ds
\end{aligned} \tag{18}
$$

Combine (18) with Lemma 2.7 and Lemma 2.8, and let Ω exhaust the whole manifold M we get

$$\mathcal{V}(g(t)) = \mathcal{V}(g(t_0)) - \int_{t_0}^{t} \int_M (R(g(s)) + n(n-1))\, d\mu_g ds. \tag{19}$$

Subtract (18) from (19), we get

$$\begin{aligned} \int_{M\backslash\Omega} \left(\sqrt{|g(t)|} - \sqrt{|\tilde{g}|}\right) dx = & \int_{M\backslash\Omega} \left(\sqrt{|g(t_0)|} - \sqrt{|\tilde{g}|}\right) dx \\ & - \int_{t_0}^{t} \int_{M\backslash\Omega} (R(g(s)) + n(n-1))\, d\mu_g ds \\ & - \int_{t_0}^{t} \int_{\partial\Omega} \langle V, \nu\rangle_g d\sigma ds \end{aligned} \tag{20}$$

Let $t_0 = 1$ and $\Omega_\tau = \{x \in M : \tau(x) \geq \tau\}$, we have

$$\begin{aligned} |\int_{M\backslash\Omega_\tau} \left(\sqrt{|g(t)|} - \sqrt{|\tilde{g}|}\right) dx| \leq & \int_{M\backslash\Omega_\tau} |\sqrt{|g(1)|} - \sqrt{|\tilde{g}|}| dx \\ & + \int_1^t \int_{M\backslash\Omega_\tau} |R(g(s)) + n(n-1)|\, d\mu_g ds \\ & + \int_1^t \int_{\partial\Omega_\tau} \|V\|_g d\sigma ds \\ & \leq C_{20} \tau^{\delta+1-n}, \end{aligned} \tag{21}$$

where C_{20} depends only on ϵ, n, k, λ and $\tilde{g}$, and we have already used Lemma 2.3, Lemma 2.8, Lemma 2.7 in the last inequality. On the other hand, due to (2) in Lemma 2.6, for any fixed $\tau > 0$, and any small $\eta > 0$, there is a large $T_0 \geq 1$ which depends only on τ and η so that for any $t \geq T_0$ we have

$$|\int_{\Omega_\tau} (\sqrt{|g(t)|} - \sqrt{|\tilde{g}|}) dx| \leq \frac{\eta}{2}.$$

Combine this with (21) we see that for any small $\eta > 0$ there is a large T_0 which depends only on η so that for any $t \geq T_0$

$$|\mathcal{V}(g(t))| \leq \eta,$$

which implies that

$$\lim_{t\to\infty} \mathcal{V}(g(t)) = 0.$$

Thus we finish to prove the proposition. □

Now, we can prove our main results.

Proof of Theorem 1.2. Consider NRDF (2) and NRF (3) starting from $g = g_0$, and let $g(t)$ and $\bar{g}(t)$ be the solution to (2) and (3) respectively. By Proposition 3.3, we obtain

$$\mathcal{V}(g) = \mathcal{V}(g_0) \geq \mathcal{V}(g(t)) \geq 0,$$

that is,

$$\mathcal{V}(g) \geq 0.$$

□

Proof of Theorem 1.3. If equality holds, we have

$$\mathcal{V}(g) = \mathcal{V}(g(t)) = 0, \text{ for } t \in [0, \infty),$$

which implies

$$\int_{M^n} (R(g(t)) + n(n-1)) d\mu_g = 0,$$

together with the fact that $R(g(t)) \geq -n(n-1)$, we get that on M and for all $t \in [0, \infty)$

$$R(g(t)) = -n(n-1),$$

which means

$$R(\bar{g}(t)) = -n(n-1).$$

By the evolution equation of R under NRF (3), we see that

$$Ric(\bar{g}(t)) = (1-n)\bar{g}(t)$$

for all $t \in [0, \infty)$. Thus the initial metric g is an Einstein metric, which means that the NRDF (2) is just acting by diffeomorphisms. According to the NRDF

$$\begin{cases} \frac{\partial}{\partial t} g_{ij} = \nabla_i V_j + \nabla_j V_i, \\ g(\cdot, 0) = g = g_0, \\ V_j = g_{jk} g^{pq} \left(\Gamma^k_{pq} - \tilde{\Gamma}^k_{pq} \right). \end{cases}$$

and the NRF

$$\begin{cases} \frac{\partial}{\partial t} \bar{g}_{ij} = 0, \\ \bar{g}(\cdot, 0) = g = g_0. \end{cases}$$

Hence

$$g = g_0 = \bar{g}(t) = \Phi_t^* g(t).$$

By the same arguments in [12] Theorem 4.1 we get that if $g(t)$, $t \in [0, \infty)$ is a solution to NRDF (2), then $\bar{g}(t) := \Phi_t^* g(t)$, $t \in [0, \infty)$ is a solution to the NRF which satisfies

$$\bar{g}(t) \to \Phi_\infty^* \tilde{g} \text{ in } \mathcal{M}^\infty(M) \text{ as } t \to \infty$$

for some smooth diffeomorphism Φ_∞ of M^n satisfying $\Phi_t \to \Phi_\infty$ in $C^\infty(M^n, M^n)$ as $t \to \infty$. Therefore

$$g = \Phi_\infty^* \tilde{g}.$$

and thus we finish to prove Theorem 1.3. □

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