The Discontinuous Riemann-Hilbert problem for analytic functions in multiply connected domains

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Abstract. In [2], the authors introduce the Keldych-Sedov formula for analytic functions in the upper half-plane, namely the representation of solutions of the mixed boundary value problem for analytic functions, which is a special discontinuous boundary value problem with the integer index. But for many problems in mechanics and physics, for instance some free boundary problems of nonlinear mechanics and the Tricomi problem for some mixed equations (see [4-12]), one needs to apply more general discontinuous boundary value problem for analytic functions and some elliptic equations in the simply and multiply connected domains. Though we have solved the general discontinuous Riemman-Hilbert problem for analytic functions in simply connected domains (see [9]), but the general discontinuous Riemman-Hilbert problem for analytic functions in multiply connected domains has not been completely solved. In this article, we shall handle the general discontinuous Riemann-Hilbert problem for analytic functions by a new method.

Key Words: Discontinuous Riemann-Hilbert problems, analytic functions, A priori estimates and existence of solutions, multiply connected domains.

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1. Formulation and representation of solutions for discontinuous boundary value problems of analytic functions

It is known that an analytic function in a domain D is a continuous solution of the complex equation

$$w_{\bar{z}} = 0,\ z \in D, \tag{1.1}$$

where $z = x+iy$, $w_{\bar{z}} = [w_x + iw_y]/2$. Let D be an $N+1\,(N \geq 1)$-connected bounded domain in $\mathbb{C}$ with the boundary $\partial D = \Gamma = \cup_{j=0}^{N} \Gamma_j \in C^1_\mu\,(0 < \mu < 1)$. Without loss of generality, we assume that D is a circular domain in $|z| < 1$, bounded by the $(N+1)$-circles $\Gamma_j : |z - z_j| = r_j$, $j = 0, 1, ..., N$ and $\Gamma_0 = \Gamma_{N+1} : |z| = 1$, and $z = 0 \in D$. In this article, the notations are the same as in References [4-12]. Now we formulate the discontinuous Riemann-Hilbert problem for equation (1.1) as follows.

Problem A The discontinuous Riemann-Hilbert boundary value problem for analytic functions is to find a continuous solution $w(z)$ of (1.1) in $D^* = \overline{D}\backslash Z$ satisfying the boundary condition:

$$\mathrm{Re}[\overline{\lambda(z)}w(z)] = c(z),\ z \in \Gamma^* = \partial D\backslash Z, \tag{1.2}$$

where $\lambda(z), c(z)$ satisfy the conditions

$$C_\alpha[\lambda(z), \hat{\Gamma}_j] \leq k_0,\ C_\alpha[|z-t_{j-1}|^{\beta_{j-1}}|z-t_j|^{\beta_j}c(z), \hat{\Gamma}_j] \leq k_2,\ j=1,...,m, \tag{1.3}$$

in which $\lambda(z) = a(z) + ib(z), |\lambda(z)| = 1$ on Γ, and $Z = \{t_1, ..., t_m\}$ are the first kind of discontinuous points of $\lambda(z)$ on Γ, $\hat{\Gamma}_j$ is an arc from the point t_{j-1} to t_j on Γ, and does not include the end point $t_j\,(j = 1, 2, ..., m)$, we can assume that $t_j \in \Gamma_0\,(j = 1, ..., m_0), t_j \in \Gamma_1\,(j = m_0+1, ..., m_1), ..., t_j \in \Gamma_N\,(j = m_{N-1}+1..., m)$ are all discontinuous points of $\lambda(z)$ on Γ, (if $\lambda(z)$ on $\Gamma_l\,(0 \leq l \leq N)$ has no discontinuous point, then we can choose a point $t_j \in \Gamma_l\,(0 \leq l \leq N)$ as a discontinuous point of $\lambda(z)$ on $\Gamma_l(0 \leq l \leq N)$, in this case $t_j = t_{j+1}$, and the partial index $K_j = 0$ on $t_j\,(\in \Gamma_l,\ 0 \leq l \leq N)$); there is in no harm assuming that the partial indexes K_l of $\lambda(z)$ on $\Gamma_l\,(l = 0, ..., N_0\,(\leq N))$ are integers, and the partial indexes K_l of $\lambda(z)$ on $\Gamma_l\,(j = N_0+1, ..., N)$ are no integers, (if K_{N+1} of $\lambda(z)$ on Γ_{N+1} is no an integer, then we can similarly discuss), and $t'_l\,(\in \Gamma_l,\ l = N_0+1, ..., N)$ are fixed points, which are not the discontinuous points at Z, set $Z' = \{t'_{N_0+1}, ..., t'_N\}$; and $\alpha\,(1/2 < \alpha < 1), k_0, k_2, \beta_j(0 < \beta_j < 1), \gamma_j\,(j = 1, ..., m)$ are non-negative constants and satisfy the conditions

$$\beta_j + |\gamma_j| < 1,\ j = 1, ..., m, \tag{1.4}$$

where $\gamma_j(j = 1, ..., m)$ are as stated in (1.5) below. Problem A with $A_3(z) = 0$ in D, $c(z) = 0$ on Γ^* is called Problem A_0.

Denote by $\lambda(t_j - 0)$ and $\lambda(t_j + 0)$ the left limit and right limit of $\lambda(t)$ as $t \to t_j\,(j = 1, 2, ..., m)$ on Γ, and

$$\begin{aligned} &e^{i\phi_j} = \frac{\lambda(t_j-0)}{\lambda(t_j+0)},\ \gamma_j = \frac{1}{\pi i}\ln\frac{\lambda(t_j-0)}{\lambda(t_j+0)} = \frac{\phi_j}{\pi} - K_j, \\ &K_j = \left[\frac{\phi_j}{\pi}\right] + J_j,\quad J_j = 0 \ \text{ or } \ 1,\quad j = 1, ..., m, \end{aligned} \tag{1.5}$$

in which $0 \leq \gamma_j < 1$ when $J_j = 0$, and $-1 < \gamma_j < 0$ when $J_j = 1$, $j = 1, ..., m$. The index K of Problems A and A_0 is defined as follows:

$$K = \frac{1}{2}(K_1 + \cdots + K_m) = \sum_{j=1}^m \left[\frac{\phi_j}{2\pi} - \frac{\gamma_j}{2}\right],$$

where we must give the attention that the boundary circles $\Gamma_j\,(j = 0, 1, ..., N)$ of the domain D are moved round the positive direct. We can see that $\Lambda(z) = \overline{Y(z)}\lambda(z)/|Y(z)|$ on Γ is continuous, it only need to charge the symbol on some arcs on Γ. If $\lambda(t)$ on Γ is continuous, then $K = \Delta_\Gamma \arg\lambda(t)/2\pi$ is a unique integer. Now the function $\lambda(t)$ on Γ is not continuous, we can choose $J_j = 0$ or 1, hence the index K is not unique.

Due to when the index $K < 0$, Problem A may not be solvable, when $K \geq 0$, the solution of Problem A is not necessarily unique. Hence we put forward a well posed-ness of Problem A with modified boundary conditions.

Problem B Find a continuous solution $w(z)$ of the complex equation (1.1) in $\overline{D}$ satisfying the boundary condition

$$\mathrm{Re}[\overline{\lambda(z)}w(z)] = r(z) + h(z)\overline{\lambda(z)}X(z),\ z \in \Gamma^*, \tag{1.6}$$

where

$$h(z)=\begin{cases} 0,\ z\in\Gamma, & \text{if } K\geq N,\\ \left.\begin{array}{l} 0,\ z\in\Gamma_j,\ j=1,...,[K]+1,\\ h_j,\ z\in\Gamma_j, j=[K]+2,...,N-K'+[K]+1,\end{array}\right\} & \text{if } 0\leq K<N\\ \left.\begin{array}{l} h_j+\mathrm{Re}\overline{\lambda(z)}\displaystyle\sum_{m=1}^{[|K|+1/2]-1}(h_m^+ + \mathrm{i}h_m^-)z^m, z\in\Gamma_j, j=1,...,N,\\ [1+(-1)^{2\kappa}]h_0+\mathrm{Re}\overline{\lambda(z)}\displaystyle\sum_{m=1}^{[|K|+1/2]-1}(h_m^+ + \mathrm{i}h_m^-)z^m, z\in\Gamma_0,\end{array}\right\} & \text{if } K<0,\end{cases}$$

in which $K'=[K+1/2]$, $[K]$ is denoted the integer part of K, $h_j\ (j=0,1,...,N)$, $h_m^{\pm}\ (m=1,...,-K-1, K<0)$ are unknown real constants to be determined appropriately. In addition, we may assume that the solution $w(z)$ satisfies the following point conditions

$$\mathrm{Im}[\overline{\lambda(a_j)}w(a_j)]=b_j\overline{\lambda(a_j)}X(a_j), j\in J=\begin{cases}1,...,2K-N+1, & \text{if } K\geq N,\\ 1,...[K]+1, \text{ if } 0\leq K<N,\end{cases} \tag{1.7}$$

where $a_j \in \Gamma_j\ (j=1,...,N_0)$, $a_j\in\Gamma_0\ (j=N_0+1,...,2K-N+1,$ if $K\geq N)$ are distinct points; and when $N-K+1\leq N_0$, $a_j\in\Gamma_j\ (j=N-K'+1,...,N_0)$, $a_j\in\Gamma_0\ (j=N_0+1,...,N-K'+[K]+1,$ if $0\leq K<N)$, otherwise $a_j\in\Gamma_0\ (j=N-K'+1,...,N-K'+[K]+1,$ if $0\leq K<N)$ are distinct points, and

$$Y(z)=\prod_{j=1}^{m_0}(z-t_j)^{\gamma_j}\prod_{l=l}^{N}(z-z_l)^{-[\tilde{K}_l]}\prod_{j=m_0+1}^{m_1}\left(\frac{z-t_j}{z-z_1}\right)^{\gamma_j}\cdots\prod_{j=m_{N_0-1}+1}^{m_{N_0}}\left(\frac{z-t_j}{z-z_{N_0}}\right)^{\gamma_j}$$
$$\times\prod_{j=m_{N_0}+1}^{m_{N_0+1}}\left(\frac{z-t_j}{z-z_{N_0+1}}\right)^{\gamma_j}\left(\frac{z-t'_{N_0+1}}{z-z_{N_0+1}}\right)\cdots\prod_{j=m_{N-1}+1}^{m}\left(\frac{z-t_j}{z-z_N}\right)^{\gamma_j}\left(\frac{z-t'_N}{z-z_N}\right),$$

where $\tilde{K}_l=\sum_{j=m_{l-1}+1}^{m_l}K_j$ are denoted the partial index on $\Gamma_l\ (l=1,...,N)$; and $b_j (j\in J)$ are all real constants satisfying the conditions

$$|\ b_j\ |\leq k_3,\ j\in J, \tag{1.8}$$

herein k_3 is a non-negative constant. We furthermore explain that the above solutions of Problem B are continuous in the domain D, the well posed-ness is firstly proposed in this paper. We can require that the solution $\mathit{\Phi}(z)$ possesses the property

$$\mathit{\Phi}(z)=O(|z-t_j|^{-\delta}),\delta=\begin{cases}\beta_j+\tau,\ \text{for}\ \gamma_j\geq 0,\ \text{and}\gamma_j<0,\ \beta_j\geq|\gamma_j|,\\ |\gamma_j|+\tau,\ \text{for}\ \gamma_j<0,\beta_j<|\gamma_j|,\ j=1,...,m,\end{cases}\tag{1.9}$$

in the neighborhood($\subset D$) of t_j, where $\tau\,(<\alpha)$ is a sufficiently small positive number.

Now we first introduce the solvability result of general discontinuous boundary value problem for analytic functions in a unit-disk $D=\{|z|<1\}$. The general discontinuous Riemann-Hilbert problem (Problem B) for analytic functions in the unit disk, namely $N=0$ with the boundary condition (1.2). We can obtain the explanation formula of solutions of Problem A for analytic functions as follows.

Theorem 1.1 *Problem* A *for analytic functions in $D=\{|z|<1\}$ has the following solvability result.*

(1) *If the index $K\geq 0$, the general solution $\mathit{\Phi}(z)$ of Problem* A *possesses the form*

$$\mathit{\Phi}(z)=\frac{X(z)}{2\pi \mathrm{i}}\left[\int_\Gamma\frac{(t+z)\lambda(t)c(t)}{(t-z)tX(t)}\mathrm{d}t+Q(z)\right],\tag{1.10}$$

if $K\geq 0$, the function $Q(z)$ possesses the form

$$Q(z)=\mathrm{i}\sum_{j=0}^{[K]}(c_jz^j+\overline{c_j}z^{-j})+\begin{cases}0,\ \text{when}\ 2K\ \text{is even},\\ \mathrm{i}c_*\dfrac{1-zt_0}{z-t_0},\ \text{when}\ 2K\ \text{is odd},\end{cases}\tag{1.11}$$

in which $t_0\,(\notin Z)$ is a point on $\Gamma=\{|z|=1\}$, $\sum_{j=0}^{[K]}(c_jz^j+\overline{c_j}z^{-j})$ and $(1-zt_0)/(z-t_0)$ on $\{|z|\}=1$ are real functions, where c_*,c_0 are arbitrary real constants, $c_j\,(j=1,...,[K])$ are arbitrary complex constants, which implies $2K+1$ arbitrary real constants.

(2) *If the index $K<0$, Problem* A *has $-2K-1$ solvability conditions given by*

$$\int_\Gamma\frac{\lambda(t)c(t)}{X(t)\,t^j}\mathrm{d}t=0,j=1,\cdots,-K(=|[K]|),\ \text{when}\ 2K\ \text{is even},$$

$$\left.\begin{aligned}&\int_\Gamma\frac{\lambda(t)c(t)}{X(t)\,t}\mathrm{d}t-\frac{\mathrm{i}c_*}{t_0}=0,\\ &\int_\Gamma\frac{2\lambda(t)c(t)}{X(t)\,t^j}\mathrm{d}t-ic_*(1-t_0^2)t_0^{-j}=0,j=2,...,[-K]+1(=|[K]|),\end{aligned}\right\}\ \text{when}\ 2K\ \text{is odd}.\tag{1.12}$$

When the conditions are satisfied, the solution of Problem B *possesses the form*

$$\mathit{\Phi}(z)=\frac{X(z)z^{|[K]|}}{\pi\mathrm{i}}\left[\int_\Gamma\frac{\lambda(t)c(t)}{(t-z)X(t)t^{|[K]|}}\mathrm{d}t-\frac{\mathrm{i}c_*(1-t_0^2)}{(t_0-z)t_0^{|[K]|}}\right],\quad c_*=0,\ \text{if}\ 2K\ \text{is even}.\tag{1.13}$$

where $t_0(\notin Z)$ is a point on Γ, the constant c_* is determined via (1.12) *as follows*

$$c_*=-\mathrm{i}t_0\int_\Gamma\frac{\lambda(t)c(t)}{X(t)t}\mathrm{d}t,\ \ \text{if}\ 2K\ \text{is odd}.$$

In the above formulas, $X(z)$ is a non-trivial solution of the homogeneous boundary value problem (Problem B_0*) for analytic functions in the form*

$$X(z)=\begin{cases}\mathrm{i}z^K\Pi(z)\mathrm{e}^{\mathrm{i}S(z)}, \text{ when } 2K \text{ is even},\\ \mathrm{i}z^{[K]}(z-t_0)\Pi(z)\mathrm{e}^{\mathrm{i}S(z)}, \text{ when } 2K \text{ is odd},\end{cases}\quad \Pi(z)=\prod_{j=1}^{m}(z-t_j)^{\gamma_j}, \qquad (1.14)$$

in which $S(z)$ is an analytic function in D satisfying the boundary condition

$$\mathrm{Re}[S(z)] = \arg\lambda(z)\arg\overline{z}^{[K]}/\Pi(z)(z-t_0)^{(1-(-1)^{2K})/2}.$$

(see Theorem 4.3, Chapter IV, [5], recently the author gives some improvement).

Here we propose that when $2|K|$ is odd, observe that $(z-t_0)/(t-z)(t-t_0) = 1/(t-z)-1/(t-t_0)$, thus the integrals in (1.10), (1.13) should be interpreted as being the difference of two integrals in the sense of the Cauchy principle value. For the case of multiply connected domains, we can use the similar approach.

2. Solvability of discontinuous Riemann-Hilbert problem for analytic functions in multiply connected domains

Now we transform the boundary condition (1.6) into the standard form and first find a solution $S(z)$ of the modified Dirichlet problem with the boundary condition

$$\begin{gathered}\mathrm{Re}S(z)=S_1(z)-\theta(t),\ S_1(z)=\begin{cases}\arg\lambda(z)-[K]\arg z-\arg Y(z),\ z\in\Gamma_0,\\ \arg\lambda(z)-\arg Y(z),\ z\in\Gamma_j, j=1,...,N,\end{cases}\\ \theta(z)=\begin{cases}0,\ z\in\Gamma_0,\\ \theta_j,\ z\in\Gamma_j,\ j=1,...,N,\end{cases}\quad \mathrm{Im}[S(1)]=0,\ \mathrm{Im}[\overline{\lambda(z)}X(z)]=0,\ z\in\Gamma,\end{gathered}\qquad (2.1)$$

in which $\theta_j\ (j=1,...,N)$ are real constants. Denote

$$\Lambda(z)=\begin{cases}z^{[\kappa]},\ z\in\Gamma_0,\\ e^{i\theta_j},\ z\in\Gamma_j, j=1,...,N,\end{cases}\quad X(z)=\begin{cases}z^{[\kappa]}e^{iS(z)}Y(z), z\in\Gamma_0,\\ e^{i\theta_j}e^{iS(z)}Y(z), z\in\Gamma_j, j=1,...,N,\end{cases}\qquad (2.2)$$

where $\kappa = K-(N+1-N_0)/2$. It is not difficult to see that

$$\overline{\lambda(z)}=\begin{cases}\overline{z}^{[\kappa]}|Y(z)|/e^{iS_1(z)}Y(z)=\overline{\Lambda(z)}|Y(z)|/e^{iS_1(z)}Y(z), z\in\Gamma_0,\\ e^{-i\theta_j}|Y(z)|/e^{iS_1(z)}Y(z)=\overline{\Lambda(z)}|Y(z)|/e^{iS_1(z)}Y(z), z\in\Gamma_j, j=1,...,N,\end{cases}$$

and

$$0=\begin{cases}\mathrm{Im}[\overline{\lambda(z)}z^{[\kappa]}e^{iS(z)}Y(z)]=\mathrm{Im}[\overline{\lambda(z)}X(z)],\ z\in\Gamma_0,\\ \mathrm{Im}[\overline{\lambda(z)}e^{i\theta_j}e^{iS(z)}Y(z)]=\mathrm{Im}[\overline{\lambda(z)}X(z)], z\in\Gamma_j, j=1,...,N,\end{cases}\qquad (2.3)$$

namely $\overline{\lambda(z)}X(z)$ on Γ is a real function. Hence we can get the modified boundary condition

$$\frac{\mathrm{Re}[\overline{\lambda(z)}\Phi(z)]}{\overline{\lambda(z)}X(z)}=\mathrm{Re}[\overline{\Lambda(z)}\Psi(z)]=\begin{cases}\mathrm{Re}[\overline{z}^{[\kappa]}\dfrac{\Phi(z)}{e^{iS(z)}Y(z)}]=\dfrac{c(z)\lambda(z)}{X(z)}+h(z), z\in\Gamma_0,\\ \mathrm{Re}[\dfrac{e^{-i\theta_j}\Phi(z)}{e^{iS(z)}Y(z)}]=\dfrac{c(z)\lambda(z)}{X(z)}+h(z), z\in\Gamma_j, 1\le j\le N,\end{cases} \tag{2.4}$$

in which $\Psi(z)=\Phi(z)/e^{iS(z)}Y(z)$, and the point constant (1.7) is transformed to

$$\mathrm{Im}[\overline{\Lambda(a_j)}\Psi(a_j)]=b_j,\ j\in J, \tag{2.5}$$

where $a_j, b_j (j\in J)$ are the same as in (1.7). The boundary value problem (2.4),(2.5) for analytic functions is called Problem B$'$, in which we allow the solution $\Psi(z)$ with simple pole points at $Z'=\{t'_{N_0+1},...,t'_N\}$. Due to the function $R(z)=c(z)\lambda(z)/X(z)$ on Γ in (2.4) possesses some discontinuous points. For the case, we can choose a sequence of functions $\{R_n(t)\}\,(R_n(t)\in C_\alpha(\Gamma),\ n=1,2,...,\ \alpha>0)$ such that $\lim_{n\to\infty}R_n(t)=R(t)=c(t)\lambda(z)/X(z)]$ on Γ^*, hence the modified boundary condition of Problem B$'$ can be proposed by the the corresponding modified boundary condition with the continuous Riemann-Hilbert boundary condition. However by using the method as stated in Theorems 2.2–2.4, we can directly obtain the corresponding results of Problem B$'$ with the discontinuous coefficient $c(z)\lambda(z)/X(z)$ in (2.4).

It is not difficult to see that the equivalence of Problem B with the boundary conditions (1.6), (1.7) and Problem B$'$ with the boundary conditions (2.4), (2.5) for analytic functions, namely

Theorem 2.1 *The function $\Phi(z)$ is a solution of Problem* B *with the boundary conditions* (1.6), (1.7) *for analytic functions in D if and only if $\Psi(z)=\Phi(z)/e^{iS(z)}Y(z)$ is a solution of Problem* B$'$ *with the boundary conditions* (2.4), (2.5).

Proof If $\Psi(z)$ is a solution of of Problem B$'$ with the boundary conditions (2.4), (2.5), we are easy to see that $\Phi(z)=\Psi(z)e^{iS(z)}Y(z)$ is an analytic function in D satisfying the boundary conditions (1.6) and (1.7). However our purpose is to find not only the proper solution $\Psi(z)$ of Problem B$'$, but also the continuous solution $\Phi(z)$ of Problem B in $\overline{D}\backslash Z$, i.e. the solution $\Psi(z)$ may have simple pole points at $Z'=\{t'_{N_0+1},...,t'_N\}$, such that the solution $\Phi(z)=\Psi(z)e^{iS(z)}Y(z)$ has no pole point at Z', because the function $Y(z)$ has simple zero point at S, for this we shall find the solution $\Psi(z)$ of Problem B$'$ with simple pole points at Z'. Because the function $\Lambda(z)$ on Γ is continuous and the index $\kappa=K-(N-N_0)/2$ is an integer, we can find the continuous solution $\psi(z)$ in D satisfying the boundary conditions

$$\begin{aligned}&\mathrm{Re}[\overline{\Lambda(z)}(\psi(z)+\psi_0(z))]=c(z)\lambda(z)/X(z)+h(z)=R(z)+h(z), z\in\Gamma,\\ &\mathrm{Im}[\overline{\Lambda(a_j)}(\psi(a_j)+\psi_0(a_j))]=b_j,\ j\in J,\end{aligned}$$

where $h(t)$ and J are the same as stated in (1.6) and (1.7), $\psi_0(z) = \sum_{j=1}^{N_0} c_j e^{\theta_j}(1-(z-z_j)(t_j-z_j))/(z-t_j)$, the purpose can be realized, which can be verified by the method as stated in proofs of Theorems 2.3$-$2.5 below. Thus the solution $\Psi(z) = \psi(z)+\psi_0(z)$ is a required solution of Problem B$'$ in D, then $\Phi(z) = \Psi(z)e^{iS(z)}Y(z)$ is just the continuous solution of Problem B in $\overline{D}\backslash Z$, which satisfying the boundary conditions

$$\begin{aligned}&\mathrm{Re}[\overline{\lambda(z)}\Phi(z)] = c(z)+h(z)\overline{\lambda(z)}X(z),\ z \in \Gamma,\\ &\mathrm{Im}[\overline{\lambda(a_j)}\Phi(a_j)] = b_j\overline{\lambda(z_j)}X(a_j),\ j \in J.\end{aligned}$$

Theorem 2.2 *Problem* B *for analytic functions in D is unique.*

Proof (1) When $K \geq N-1$, because of Theorem 2.1, we can only discuss Problem B$'$. Let $\Psi_1(z), \Psi_2(z)$ be two solutions of Problem B$'$ for analytic functions. Then the function $\Psi(z) = \Psi_1(z) - \Psi_2(z)$ is a solution of Problem B_0' with the homogeneous boundary conditions

$$\mathrm{Re}[\overline{\Lambda(z)}\Psi(z)]=0 \text{ on } \Gamma,\ \mathrm{Im}[\overline{\Lambda(a_j)}\Psi(a_j)]=0,\ j=1,...,2K-N+1. \tag{2.6}$$

According to the way in [1,5], we see that if $\Psi(z) \not\equiv 0$ in D, then there is

$$N_D + N_\Gamma - P_\Gamma = 2\kappa, \tag{2.7}$$

where N_D, N_Γ are denoted the zero numbers of $\Psi(z)$ in D and Γ respectively, and P_Γ is denoted the pole numbers of $\Psi(z)$ on Γ. If $\Psi(z) \neq 0$, $z \in D$, and we see that $\Psi(z)$ has even zero points on Γ_j $(j = 1, ..., N_0)$, and odd zero points on Γ_j $(j = N_0 + 1, ..., N)$ (see Section 4, Chapter IV, [1] and Section 2, Chapter V, [5]), hence we have

$$2\kappa+1=2\kappa-N+1+2N-N_0-(N-N_0)\leq 2N_D+N_\Gamma-P_\Gamma=2\kappa, \tag{2.8}$$

this contraction verify

$$\Psi(z) \equiv 0 \text{ in } D, \tag{2.9}$$

namely

$$\Psi_1(z) \equiv \Psi_2(z) \text{ in } D,\ i.e.\ \Phi_1(z) \equiv \Phi_2(z) \text{ in } D. \tag{2.10}$$

This completes the proof of the Theorem.

(2) Next we consider $0 \leq K < N-1$, for verifying the uniqueness of solutions of Problem B$'$, let $\mathit{\Psi}_1(z)$, $\mathit{\Psi}_2(z)$ be two solutions of Problem B$'$ for analytic functions. Then the function $\mathit{\Psi}(z) = \mathit{\Psi}_1(z) - \mathit{\Psi}_2(z)$ is a solution of homogeneous Problem B_0' with the boundary conditions

$$\mathrm{Re}[\overline{\Lambda(z)}\,\mathit{\Psi}(z)]=h(z) \text{ on } \mathit{\Gamma},\ \mathrm{Im}[\overline{\lambda(a_j)}\,\mathit{\Psi}(a_j)]=0,\ j=1,...,[K]+1. \tag{2.11}$$

According to the way in [1,5], we see that if $\mathit{\Psi}(z) \not\equiv 0$ in D, then there are

$$\begin{cases} 2\kappa + 1 \leq [\kappa] + [\kappa + 1/2] + 1 \leq [\kappa] + 1 + N_0 - (N - [\kappa + 1/2]) + N - N_0 \\ \leq 2N_D + N_\Gamma - P_\Gamma \leq 2\kappa, \text{ if } N - N_0 < [\kappa + 1/2], \text{ i.e. } N - [\kappa + 1/2] < N_0, \\ 2\kappa+1\leq[\kappa]+[\kappa+1/2]+1\leq[\kappa]+1+N-(N-[\kappa+1/2])+N-N_0-(N-N_0) \\ \leq 2N_D+N_\Gamma-P_\Gamma\leq 2\kappa, \text{ if } 0 \leq [\kappa+1/2] \leq N-N_0, \text{ i.e. } N_0\leq N-[\kappa+1/2], \end{cases} \tag{2.12}$$

where N_D, N_Γ are denoted the zero numbers of $\Psi(z)$ in D and Γ respectively and P_Γ is denoted the pole numbers of $\Psi(z)$ on Γ, this contradiction prove that $\Psi(z)\equiv 0$, i.e. $\Psi_1(z)\equiv\Psi_2(z)$ in D.

(3) When $K<0$, it is clear that the solution of Problem B is unique, if there exists a solution of Problem B. In particular when $K=-1/2$ or -1, and $\Psi(z)(\not\equiv 0)$ is the solution of the homogeneous Problem B_0' in D, then we can derive a contradictious inequality

$$0=(N-N_0)-(N-N_0)\le 2N_D+N_\Gamma-P_\Gamma=2\kappa<0.$$

Hence $\Psi(z)\equiv 0$ in D. For general case $K<0$ with continuous solutions, we can use the following method.

Theorem 2.3 *Under the above conditions, Problem* B *with the index* $K>N-1$ *for analytic functions has a solution.*

Proof Firstly we discuss the case of Problem B′ with the index $\kappa\ge N-1$ and find a solution $\Psi_0(z)$ of Problem B′ with the index κ satisfying the boundary conditions

$$\begin{aligned}&\mathrm{Re}[\overline{\Lambda(t)}\Psi_0(t)]=c(t)\lambda(t)/X(t),\ t\in\Gamma,\\&\mathrm{Im}[\overline{\Lambda(a_j)}\Psi_0(a_j)]=b_j,\ j=1,...,2\kappa-N+1,\end{aligned}\tag{2.13}$$

which is continuous at Z', and find the similar solutions $\psi_{N_0+1}(z),...,\psi_N$ of Problem B′ in D with the boundary conditions

$$\begin{aligned}&\mathrm{Re}[\overline{\Lambda(t)}(\psi_l(t)+\tilde\psi_l(t))]=0,\ t\in\Gamma^*,\\&\mathrm{Im}[\overline{\Lambda(a_j)}(\psi_l(a_j)+\tilde\psi_l(a_j))]=0,\ j=1,...,s,\end{aligned}\quad l=N_0+1,...N,\tag{2.14}$$

where $s=2\kappa-N+1$, $\tilde\psi_l(z)=c_le^{-i\theta_l}(1-(z-z_l)(t_j'-z_l))/(z-t_l')\,(l=N_0+1,...,N)$, herein $c_l\,(l=N_0+1,...,N)$ are real constants. It is easy to see that $\Psi_l(z)=\psi_l(z)+\tilde\psi_l(z)\,(l=N_0+1,...,N)$ are linearly independent, and denote

$$\mathrm{Im}[\overline{\Lambda(a_j)}\Psi_0(a_j)]=b_j^*,\ j\in S'=\{s+1,...,s+s'\},\tag{2.15}$$

in which $s'=N-N_0$. If $b_j^*=b_j$, $j\in S'$, then $\Phi(z)=Y(z)e^{iS(z)}\Psi_0(z)$ is just the required solution of Problem B. Otherwise, we can verify

$$I=\begin{vmatrix}\Psi_{N_0+1}(a_{s+1})&\cdots&\Psi_{N_0+1}(a_{s+N-N_0})\\\Psi_{N_0+2}(a_{s+1})&\cdots&\Psi_{N_0+2}(a_{s+N-N_0})\\\vdots&\ddots&\vdots\\\Psi_{N-1}(a_{s+1})&\cdots&\Psi_{N-1}(a_{s+N-N_0})\\\Psi_N(a_{s+1})&\cdots&\Psi_N(a_{s+N-N_0})\end{vmatrix}\ne 0.\tag{2.16}$$

In fact if $I=0$, then there exits $N-N_0$ real constants $d_l\,(l=N_0+1,...,N)$, such that

$$\sum_{l=N_0+1}^{N}\Psi_l(a_j)=0,\ j\in S'.\tag{2.17}$$

Due to $\tilde{\Psi}(z) = \sum_{l=N_0+1}^{N} \Psi_l(z) \neq 0$ in D, we see that $\tilde{\Psi}(z) = \sum_{l=N_0+1}^{N+1} d_l\Psi_l(z)$ have even zero points on Γ_j $(j = 1, ..., N_0)$, and at least one zero points on Γ_j $(j = N_0 + 1, ..., N)$, hence we have

$$2\kappa+1 = 2\kappa - N + 1 + N - N_0 + 2N_0 - N_0 \leq 2N_D + N_\Gamma - P_\Gamma = 2\kappa, \tag{2.18}$$

where N_D, N_Γ are denoted the zero numbers of $\tilde{\Psi}(z) = \sum_{l=N_0+1}^{N} d_l\Psi_l(z)$ in D and Γ respectively, and P_Γ is denoted the pole number of $\tilde{\Psi}(z) = \sum_{l=N_0+1}^{N} d_l\Psi_l(z)$ on Γ. This contraction prove that there exist $N - N_0$ real constants d_l $(l = N_0 + 1, ..., N)$, which are not all equal to 0, such that

$$\mathrm{Im}[\overline{\Lambda(a_j)} \sum_{l=N_0+1}^{N} d_l\Psi_l(a_j)] = b_j^* - b_j,\ j \in S', \tag{2.19}$$

and then

$$\mathrm{Im}[\overline{\Lambda(a_j)}(\Psi_0(a_j) - \sum_{l=N_0+1}^{N} d_l\Phi_l(a_j)] = b_j,\ j \in S'.$$

Thus

$$\Phi(z) = \Psi(z)Y(z)e^{iS(z)} = [\Psi_0(z) - \sum_{l=N_0+1}^{N} d_l\Psi_l(z)]Y(z)e^{iS(z)} \tag{2.20}$$

is just a solution of Problem B for analytic functions with the index $K > N - 1$.

Theorem 2.4 *Under the above conditions, Problem* B *with the index* $-1 \leq K \leq N - 1$ *for analytic functions has a unique solution.*

Proof First of all, we discuss the case: $K = N - 1$, and find a solution $\Psi_0(z)$ of Problem $\tilde{\mathrm{B}}$ with the index $\tilde{\kappa} = N$ satisfying the boundary conditions

$$\begin{aligned} &\mathrm{Re}[\overline{\Lambda_1(t)}\Psi_0(t)] = [c(t)\lambda(t)/X(t) + h(t)]/|t|,\ t \in \Gamma^*, \\ &\mathrm{Im}[\overline{\Lambda_1(a_j)}\Psi_0(a_j)] = b_j/|a_j|,\ j = 1, ..., N+1, \end{aligned} \tag{2.21}$$

for analytic functions, where the index of $\Lambda_1(t) = \Lambda(t)/\bar{t}$ is $\tilde{\kappa} = N$, in this case $h(t) = 0$ and $N - [\tilde{\kappa}] = 0$. Due to the above boundary value problem belongs to Problem $\tilde{\mathrm{B}}$ with $\tilde{\kappa} = N$, by the result in Theorem 2.3, there exists a solution $\Psi_0(z)$ of the Problem $\tilde{\mathrm{B}}$, if $\Psi_0(0) = 0$, then $\Psi(z) = \Psi_0(z)/z$ is just a solution of Problem B′ with $\kappa = N - 1$. If $\Psi_0(0) \neq 0$, we can find two solutions $\Psi_1(z), \Psi_2(z)$ satisfying the boundary conditions

$$\mathrm{Re}[\overline{\Lambda_1(t)}\Psi_1(t)] = 0,\ t \in \Gamma^*,\ \mathrm{Im}[\overline{\Lambda_1(a_j)}\Psi_1(a_j)] = \begin{cases} 0,\ j = 1, ..., N, \\ 1/|a_j|, j = N + 1, \end{cases} \tag{2.22}$$

$$\mathrm{Re}[\overline{\Lambda_1(t)}\Psi_2(t)] = \begin{cases} 1/|t|, t \in \Gamma_1, \\ 0,\ z \in \Gamma\backslash\Gamma_1, \end{cases}\ \mathrm{Im}[\overline{\Lambda_1(a_j)}\Psi_2(a_j)] = 0,\ j = 1, ..., N+1, \tag{2.23}$$

It is clear that $\Psi_1(z), \Psi_2(z)$ are linearly independent, and we can verify

$$I = \begin{vmatrix} \mathrm{Re}\Psi_1(0) & \mathrm{Re}\Psi_2(0) \\ \mathrm{Im}\Psi_1(0) & \mathrm{Im}\Psi_2(0) \end{vmatrix} \neq 0. \tag{2.24}$$

Otherwise, if $I = 0$, then there exits two real constants c_1, c_2 $(|c_1| + |c_2| \neq 0)$, such that $c_1\Psi_1(0) + c_2\Psi_2(0) = 0$, and we have

$$2N+1 = N - N_0 + 2N_0 + N + 1 - N_0 \leq 2N_D + N_\Gamma - P_\Gamma = 2K = 2N. \tag{2.25}$$

This contraction prove that there exist two real constants c_1, c_2 such that

$$\begin{cases} c_1\mathrm{Re}\Psi_1(0) + c_2\mathrm{Re}\Psi_2(0) = \mathrm{Re}\Psi_0(0), \\ c_1\mathrm{Im}\Psi_1(0) + c_2\mathrm{Im}\Psi_2(0) = \mathrm{Im}\Psi_0(0), \end{cases} \tag{2.26}$$

thus $\tilde{\Psi}(z) = \Psi_0(z) - c_1\Psi_1(z) - c_2\Psi_2(z)$ has a zero point at $z = 0$, and then $\Psi(z) = \tilde{\Psi}(z)/z$ is a solution of Problem B$'$ with $K = N - 1$. According to the way, we can derive the existence of solutions of Problem B$'$ with the indexes $K = N - 2, ..., 0, -1$, but when the index $K = -1$, the corresponding boundary conditions should be replaced by

$$\mathrm{Re}[\overline{\Lambda_1(t)}\Psi_1(t)] = 0,\ t \in \Gamma^*,\ \mathrm{Im}[\overline{\Lambda_1(a_j)}\Psi_1(a_j)] = 1/|a_j|,\ j = N+1,$$

$$\mathrm{Re}[\overline{\Lambda_1(t)}\Psi_2(t)] = \begin{cases} 1/|t|, t \in \Gamma_0, \\ 0,\ t \in \Gamma\backslash\Gamma_0, \end{cases} \mathrm{Im}[\overline{\Lambda_1(a_j)}\Psi_2(a_j)] = 0,\ j = N+1. \tag{2.27}$$

As for the case $K = N - 3/2$, by using the same way, we can consider the boundary conditions

$$\begin{aligned} &\mathrm{Re}[\overline{\Lambda_1(t)}\Psi_0(t)] = [c(t)\overline{\lambda(t)}X(t) + h(t)]/|t|,\ t \in \Gamma^*, \\ &\mathrm{Im}[\overline{\Lambda_1(a_j)}\Psi_0(a_j)] = b_j/|a_j|,\ j = 1, ..., N, \end{aligned} \tag{2.28}$$

for analytic functions, where $\lambda_1(t) = \lambda(t)/\bar{t}$ with the index $\tilde{\kappa} = N - 1/2$, and derive the solvability of Problem B$'$ with $K = N - 3/2$. Moreover we also obtain the solvability of Problem B$'$ with $K = N - 5/2, ..., 1/2, -1/2$.

Theorem 2.5 *Under the above conditions, Problem* B *with the index* $K < -1$ *for analytic functions has a unique solution.*

Proof We first discuss the index $K = -3/2$ (or $K = -2$) of problem $\tilde{\mathrm{B}}$ with the boundary condition

$$\mathrm{Re}[\overline{\Lambda_1(t)}\Psi_0(t)] = [c(t)\lambda(t)/|X(t)| + h(z)]/|t|,\ z \in \Gamma^*, \tag{2.29}$$

in which $\Lambda_1(z) = \Lambda(z)/|z|$, and its index is $\tilde{\kappa} = -1/2$ (or -1). If $\Psi_0(0) = 0$, then $\Psi(z) = \Psi_0(z)/z$ is just a solution of Problem B$'$ with the index $K = -3/2$ (or -2). Otherwise, we find two solutions of Problem $\tilde{\mathrm{B}}$ for analytic functions with the boundary conditions

$$\begin{aligned}
&\mathrm{Re}[\overline{\Lambda_1(t)}\Psi_1(t)] = \begin{cases} (h_j + \mathrm{Re}\overline{\Lambda_1(t)}t)/|t|, \ t \in \Gamma_j, \ j = 1, ..., N, \\ \mathrm{Re}\overline{\Lambda_1(t)}t/|t|, \ t \in \Gamma_0, \end{cases} \\
&(\text{or } \mathrm{Re}[\overline{\Lambda_1(t)}\Psi_1(t)] = (h_j + \mathrm{Re}\overline{\Lambda_1(t)}t)/|t|, \ t \in \Gamma_j, \ j = 0, 1, ..., N, \\
&\mathrm{Re}[\overline{\Lambda_1(t)}\Psi_2(t)] = \begin{cases} (h_j + \mathrm{Re}\overline{\Lambda_1(t)}it)/|t|, \ t \in \Gamma_j, j = 1, ..., N,) \\ \mathrm{Re}\overline{\Lambda_1(t)}it/|t|, \ t \in \Gamma_0, \end{cases} \\
&(\text{or } \mathrm{Re}[\overline{\Lambda_1(t)}\Psi_2(t)] = (h_j + \mathrm{Re}\overline{\Lambda_1(t)}it)/|t|, \ t \in \Gamma_j, \ j = 0, 1, ..., N.)
\end{aligned} \tag{2.30}$$

It is clear that $\Psi_1(z), \Psi_2(z)$ are linearly independent, and we can verify

$$I = \begin{vmatrix} \mathrm{Re}\Psi_1(0) & \mathrm{Re}\Psi_2(0) \\ \mathrm{Im}\Psi_1(0) & \mathrm{Im}\Psi_2(0) \end{vmatrix} \neq 0. \tag{2.31}$$

Otherwise, if $I = 0$, then there exits two real constants c_1, c_2 $(|c_1| + |c_2| \neq 0)$, such that $c_1\Psi_1(0) + c_2\Psi_2(0) = 0$, and we have

$$0 = (N - N_0) - (N - N_0) \leq 2N_D + N_\Gamma - P_\Gamma = 2\kappa < 0. \tag{2.32}$$

This contraction prove that there exist two real constants c_1, c_2 such that

$$\begin{cases} c_1\mathrm{Re}\Psi_1(0) + c_2\mathrm{Re}\Psi_2(0) = \mathrm{Re}\Psi_0(0), \\ c_1\mathrm{Im}\Psi_1(0) + c_2\mathrm{Im}\Psi_2(0) = \mathrm{Im}\Psi_0(0), \end{cases} \tag{2.33}$$

thus $\Psi(z) = \Psi_0(z) - c_1\Psi_1(z) - c_2\Psi_2(z)$ has a zero point at $z = 0$, and then $\Phi(z) = \Psi(z)/z$ is a solution of Problem B$'$ with $K = -3/2$ (or -2). According to the way, we can derive the existence of solutions of Problem B with the indexes $K = -5/2, (-3), -7/2, (-4), ...$

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