MULTIPLE MIXED STATES OF NODAL SOLUTIONS FOR NONLINEAR SCHRÖDINGER SYSTEMS

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ABSTRACT. In this paper we develop a general critical point theory to deal with existence and locations of multiple critical points produced by minimax methods in relation to multiple invariant sets of the associated gradient flow. The motivation is to study non-trivial nodal solutions with each component sign-changing for a class of nonlinear Schrödinger systems which arise from Bose-Einstein condensates theory. Our general method allows us to obtain infinitely many mixed states of nodal solutions for the repulsive case.

1. **Introduction**

The paper is to develop a general critical point theory aiming at applications in the study of existence and multiplicity of mixed states of nodal solutions for nonlinear elliptic systems. The abstract theory is modeled on the classical mountain-pass theorem and the symmetric mountain-pass theorem due to Ambrosetti and Rabinowitz [1, 34]. To motivate our study let us consider the following nonlinear Schrödinger systems of k equations

$$\begin{cases} -\Delta u_j + \lambda_j u_j = \sum_{i=1}^{k} \beta_{ij} u_i^2 u_j, x \in \mathbb{R}^N \\ u_j(x) \to 0, \text{as } |x| \to \infty, j = 1, ..., k \end{cases} \tag{1.1}$$

where $N = 2, 3, k \geq 2, \lambda_j > 0$, for $j = 1, ..., k$, β_{ij} are constants satisfying $\beta_{jj} > 0$ for $j = 1, ..., k$, $\beta_{ij} = \beta_{ji}$ for $1 \leq i < j \leq k$. This class of systems, also known as Gross-Pitaevskii equations, have applications in many physical problems such as in nonlinear optics and in multispecies Bose-Einstein condensates (e.g., [16, 39]). Physically, β_{jj} and β_{ij} $(i \neq j)$ are the intraspecies and interspecies scattering lengths respectively. The sign of the scattering length determines whether the interactions of states are repulsive or attractive. In the repulsive case $(\beta_{ij} < 0)$ the components tend to segregate with each other leading to phase separations. These phenomena have been documented in experiments as well as in numeric simulations (e.g.,[30], [34], [12] and references therein). Mathematical work has been done extensively in recent years and we refer [2, 9, 10, 15, 20, 21, 26, 27, 29, 31, 33, 35, 37, 38, 40, 41] for more references. In particular multiplicity of positive solutions have been established in [4, 15, 32, 37, 38, 41].

There are new challenges in dealing with the existence of multiple solutions, in particular multiple sign-changing solutions. First of all, there are many semi-trivial solutions due to systems collapsing, i.e., solutions with one or more components being zeroes. For example, if $u_1 = \cdots = u_{k-1} = 0$ and u_k satisfies $-\Delta u_k + \lambda_k u_k = \beta_{kk} u_k^3$ then $(0, ..., 0, u_k)$ is a solution of the full k-system. In fact there are infinitely many such semi-trivial solutions. Furthermore there are also infinitely many semi-trivial solutions with two or more components being non-zeroes with the systems collapsing to even lower order ones. Secondly, making the situations more complicated is the fact that there can exist infinitely many positive solutions (solutions with each components being positive). This was established in [15, 41] for the case of 2-systems when $\lambda_1 = \lambda_2 = 1$ and $\mu_1 = \mu_2 = 1$, and extended in [38] to the case of k-equations when $\lambda_j = 1$ and $\beta_{jj} = 1$ for $j = 1, ..., k$ and $\beta_{ij} = \beta$ for $i \neq j$. Thus to obtain sign-changing critical points for the variational formulation one needs to distinguish them from the known existing semi-trivial critical points and the existing positive critical points. It is very difficult to develop a critical theory to accomplish this goal as there are many family of these known existing critical points. Therefore for these nonlinear Schrödinger systems nodal solutions, in particular multiplicity of nodal solutions have not been studied so far due to these difficulties.

On the other hand, over the years there have been systematic studies on nodal solutions for scalar equations by using a combination of minimax methods and the method of invariant sets of gradient flows. Progress have been made along this direction and we refer to [3, 5, 6, 7, 8, 14, 22, 23, 24, 28] and the survey [25] for more references. However most of the methods in treating scalar equations are not applicable directly to systems of equations.

For nonlinear systems using invariant sets of flows we know only two papers. In Liu-Wang [26, 27] a construction of invariant sets has been developed to locate multiple non-trivial solutions but without giving any information about nodal property of the components of solutions. In [36] flow invariance is used to study sign-changing solutions of a nonlinear eigenvalue problem in a bounded domain Ω with Dirichlet boundary condition $-\Delta u_j + a_j u_j^3 + \beta \sum_{i=1}^k u_i^2 u_j = \lambda_{j,\beta} u_j, j = 1, ..., k$ where $a_j \geq 0$ and $\beta > 0$ and infinitely solutions $(u, \vec{\lambda})$ with u sign-changing are given. This is a constrained variational problem for the de-focusing case different from our focusing case.

In this paper we develop a general theory modeled on the classical mountain-pass theorem and the symmetric mountain-pass theorem ([1, 34]) in the presence of a family of invariant sets of the associated pseudo gradient flow. In Section 2 we establish an abstract framework which is new even in applications for scalar equations. The new abstract results are given in Theorems 2.4, 2.5, 2.6. Our new approach allows us to tackle the difficulties in applications to systems of equations as mentioned above. In Section 3 we consider applications of our abstract theory to nonlinear Schrödinger system (1.1). In particular we show that for the repulsive case (i.e., $\beta_{ij} \leq 0$ for $i \neq j$) systems (1.1) has infinitely many mixed states of sign-changing solutions. A mixed state of nodal solution $u = (u_1, ..., u_k)$ is such that some of the k components are sign-changing functions and the rest of the components are one sign functions. Without loss of generality we may take $1 \leq m \leq k$ and we look for solutions $u = (u_1, ..., u_k)$ such that u_j are sign-changing for $j = 1, ..., m$ and u_j are signed functions for $j = m + 1, ..., k$.

Theorem 1.1. *Assume* $N = 2, 3, k \geq m \geq 1, \lambda_j > 0, \beta_{jj} > 0$ *for* $j = 1, ..., k$, $\beta_{ij} = \beta_{ji} \leq 0$ *for* $1 \leq i < j \leq k$. *Then the system (1.1) possesses infinitely many radially symmetric solutions with the first* m *components sign-changing and the last* $k-m$ *components positive.*

The paper is organized as follows. In Section 2 we establish an abstract theory of minimax methods constructing minimax critical points with their locations identified in relation to a family of invariant of sets of the associated gradient flow, aiming toward applications to the nonlinear Schrödinger systems like (1.1). In Section 3 we develop the necessary analytic framework in order to apply the abstract theory from Section 2 and to construct multiple mixed states of nodal solutions, proving Theorem 1.1. We also mention possible extensions and further questions along the line of work.

2. **Intersection property and multiple critical point theorems**

We establish an abstract critical point theory for the existence and multiplicity of critical points with the presence of multiple invariant sets of the associated gradient flow. The theory is modeled on the classical mountain-pass theorem and the symmetric mountain-pass theorem given by Ambrosetti and Rabinowitz ([1, 34]). In the setting of these theorems if we introduce a family of invariant sets of the gradient flow we want to investigate the locations of critical points relative to the invariant sets.

One of the most useful methods in treating sign-changing solutions for scalar equations is by using minimax method in the presence of invariant sets of the gradient flows. Roughly speaking, one shows that the positive and negative cones or their neighborhoods are invariant under the negative gradient flow and builds this information into the construction of minimax critical values so that critical points are obtained outside these invariant neighborhoods. In applications to systems of elliptic equations the situation is that we may construct a finite family of invariant sets and we seek to find critical points outside the union of these invariant sets. The construction of these invariant sets need to have both the semi-trivial critical points and the signed critical points included so critical points found outside these invariant sets are non-trivial sign-changing critical points. The idea here seems quite clear but in order to implement this we would need some new intersection properties. In fact our approach is different from that used for scalar case (i.e., $k = 1$) and is a new and simpler treatment even for scalar case. Though in our applications to nonlinear Schrödinger systems we have smooth functionals, we state our results in quite general settings, for continuous functionals defined on a metric space as the abstract treatments are the same. We refer to [11, 18, 19] for the background of critical point theory of continuous functionals.

Now we introduce some concepts and notations. Let X be a complete metric space with the metric d and f be a continuous functional on X. For $c \in \mathbb{R}$ denote $f^c = \{x \in X \mid f(x) \leq c\}$, $K_c = \{x \in X \mid f(x) = c, f'(x) = 0\}$. We say $G : X \to X$ is an isometric involution if G satisfies $G^2 = Id$ and $d(Gx, Gy) = d(x, y)$ for $x, y \in X$.

Denote the k dimensional simplex Δ in $\mathbb{R}^k$ and its faces $\partial_i\Delta$, $i = 0, 1, ..., k$:

$$\begin{aligned} \Delta &= \{t \in \mathbb{R}^k \mid t = (t_1, ..., t_k), t_i \geq 0, i = 1, ..., k, t_1 + \cdots t_k \leq 1\}, \\ \partial_i\Delta &= \Delta \cap \{t_i = 0\},\ i = 1, ..., k, \\ \partial_0\Delta &= \Delta \cap \{t_1 + \cdots + t_k = 1\}. \end{aligned} \tag{2.1}$$

Lemma 2.1. *Let X be a complete metric space with the metric d. Let P_i, $i = 1, ..., k$ be open subsets of X, and denote*

$$M = \cap_{i=1}^k P_i, \quad \Sigma = \cap_{i=1}^k \partial P_i. \tag{2.2}$$

Assume that a map $\varphi : \Delta \to X$ satisfies $\varphi(\partial_i\Delta) \subset P_i, i = 1, ..., k,$ and $\varphi(\partial_0\Delta) \cap M = \emptyset$. Then $\varphi(\Delta) \cap \Sigma \neq \emptyset$.

Proof. Define $f_i : \Delta \to \mathbb{R}$, $i = 1, ..., k$ by

$$f_i(t) = \begin{cases} d(\varphi(t), \partial P_i), & \text{if } \varphi(t) \notin P_i \\ -\, d(\varphi(t), \partial P_i), & \text{if } \varphi(t) \in P_i. \end{cases} \tag{2.3}$$

Denote $F = (f_1, ..., f_k)$. Then $\varphi(t) \in \Sigma$, if and only if $F(t) = 0$. Define a homotopy $G : [0.1] \times \Delta \to \mathbb{R}^k$ by

$$G(\lambda, t) = \begin{cases} \lambda F(t) + (1-\lambda)t - (\lambda_0 - \lambda)e, & 0 \leq \lambda \leq \lambda_0, t \in \Delta \\ \lambda F(t) + (1-\lambda)t, & \lambda_0 \leq \lambda \leq 1, t \in \Delta \end{cases} \tag{2.4}$$

where $e = (1, ..., 1) \in \mathbb{R}^k$, λ_0 is a constant to be chosen. Assume $F(t) \neq 0$ for $t \notin \partial\Delta$. Otherwise we are done. Denote $G(\lambda, t) = (g_1(\lambda, t), ..., g_n(\lambda, t))$. For $0 < \lambda < 1$ and $t \in \partial_i\Delta$ $i = 1, ..., k$, $t_i = 0$, $\varphi(t) \in P_i$, $f_i(t) = -d(\varphi(t), \partial P_i) < 0$. Hence $g_i(\lambda, t) \leq \lambda f_i(t) + (1-\lambda)t_i < 0$.

For $t \in Int(\partial_0\Delta), t_i > 0, i = 1, ..., k$, and $\varphi(t) \notin M = \cap_{i=1}^k P_i$. There exists an index, say $i = 1$, $\varphi(t) \notin P_1$ and $f_1(t) = d(\varphi(t), \partial P_i) \geq 0$. Hence for $\lambda_0 \leq \lambda < 1$, $t \in Int(\partial_0\Delta)$, $g_1(\lambda, t) = \lambda f_1(t) + (1-\lambda)t_1 > 0$. For $0 \leq \lambda \leq \lambda_0$, $t \in Int(\partial_0\Delta)$

$$\sum_{i=1}^k g_i(\lambda, t) = \lambda \sum_{i=1}^k f_i(t) + (1-\lambda) \sum_{i=1}^k t_i - (\lambda_0 - \lambda)k \geq 1 - \lambda_0(1 + k + a) \geq \frac{1}{2}$$

where $a = \max_{t\in\Delta} \sum_{i=1}^k |f_i(t)|$ and we choose $\lambda_0 = \frac{1}{2(1+k+a)}$. In any case $G(\lambda, t) \neq 0$ for $(\lambda, t) \in [0, 1] \times \partial\Delta$. By homotopy invariance of the Brouwer degree we have $deg(F, Int\Delta, 0) = deg(Id - \lambda_0 e, Int\Delta, 0) = 1$. Hence there exists a point $t \in Int\Delta$ such that $F(t) = 0$, that is, $\varphi(t) \in \Sigma = \cap_{i=1}^k \partial P_i$. ☐

In the following we establish estimates on genus of some symmetric sets for interaction property. We refer to e.g., [34] for the definition and basic properties of the concept of genus.

Lemma 2.2. *Let X be a complete metric space with the metric d. Let $G : X \to X$ be an isometric involution. Assume $k \geq 1$ is an integer. For some $k \geq m \geq 1$ let P_i, $i = 1, ..., m$, be open subsets of X and set $Q_i = GP_i$, $M_i = P_i \cap Q_i$. Let M_j, $j = m + 1, ..., k$, be* *open sets satisfying* $GM_j = M_j$. *Denote* $B^{nk} \subset \mathbb{R}^{nk}$ *the closed unit ball in* $\mathbb{R}^{nk}$ *and* $t = (t_1, ..., t_k) \in B^{nk}$ *with* $t_1, ..., t_k \in \mathbb{R}^n$. *Assume a continuous map* $\varphi : B^{nk} \to X$ *satisfies*

(1) $\varphi(-t) = G\varphi(t), t \in B^{nk}$;

(2) For $i = 1, ..., k$, $\varphi(t) \in M_i$, *if* $t_i = 0$;

(3) $\varphi(t) \notin M := \cap_{i=1}^k M_i$, *if* $t \in \partial B^{nk}$.

Then $\gamma(\varphi(B^{nk} \setminus Y) \cap \Sigma) \geq j - m$, *where* $\Sigma = \cap_{i=1}^m (\partial P_i \cap \partial Q_i) \cap \cap_{j=m+1}^k \partial M_j$, *and open subset* $Y \subset B^{nk}$ *satisfies* $Y = -Y$, $\gamma(\overline{Y}) \leq n - j$, $m + 1 \leq j \leq n$.

Proof. Denote $\mathcal{O} = \{t \in B^{nk} \mid \varphi(t) \in M\}$. Since $\varphi(0) \in M$ and $\varphi(t) \notin M$ for $t \in \partial B^{nk}$, $\mathcal{O}$ is a symmetric open neighborhood of 0 in $\mathbb{R}^{nk}$, and

$$\varphi(\partial\mathcal{O}) \subset \partial M. \tag{2.5}$$

By Borsuk's theorem, $\gamma(\partial\mathcal{O}) = nk$. Decompose ∂M as a disjoint union

$$\partial M = \partial(\cap_{i=1}^k M_i) = C_1 \cup C_2 \cup \cdots \cup C_k \tag{2.6}$$

where

$$C_1 = \cup_{i=1}^k (\partial M_i \cap \cap_{j\neq i} M_j), \; C_k = \cap_{i=1}^k \partial M_i = \cap_{i=1}^m \partial(P_i \cap Q_i) \cap \cap_{j=m+1}^k \partial M_i$$

and in general for $1 \leq p \leq k$

$$C_p = \cup_{s\in S_p} (\cap_{i\in s} \partial M_i \cap \cap_{j\in s^c} M_j) \tag{2.7}$$

where the index set $S_p = \{s = (i_1, ..., i_p) \mid 1 \leq i_1 < i_2 < ... < i_p \leq k\}$, and s^c is an index set of order $k - p$ such that its components have no common with that of s. Let us define for $p = 1, ..., k-1$, $A_p = \{t \in \partial\mathcal{O} \mid \varphi(t) \in C_p\}$. For the set C_k we have

$$\begin{aligned} C_k &= \cap_{i=1}^m \partial(P_i \cap Q_i) \cap \cap_{j=m+1}^k \partial M_i \\ &\subset \cup_{i=1}^m ((\partial P_i \cap Q_i) \cup (P_i \cap \partial Q_i)) \cup \Sigma. \end{aligned} \tag{2.8}$$

Denote

$$B_i = \{t \in \partial\mathcal{O} \mid \varphi(t) \in (\partial P_i \cap Q_i) \cup (P_i \cap \partial Q_i)\}, i = 1, 2, ..., m. \tag{2.9}$$

Let $Z_1 = \{x \in \partial\mathcal{O} \setminus Y | \varphi(x) \in \Sigma\}$ and $Z_2 = \cup_{p=1}^{k-1} A_p \cup \cup_{i=1}^m B_i$. Then $\partial\mathcal{O} \setminus Y \subset Z_1 \cup Z_2$. We have $\gamma(\varphi(B^{nk} \setminus Y) \cap \Sigma) \geq \gamma(Z_1)$. If there exists an odd continuous map $h : \partial\mathcal{O} \to \mathbb{R}^l$ for some integer l such that the restriction on Z_2 satisfies $h : Z_2 \to R^l \setminus \{0\}$ then we we may easily argue that $\gamma(Z_1) \geq \gamma(\partial\mathcal{O} \setminus Y) - l$. Next we show that we may construct such a map with $l = n(k-1) + m$. This will finish the proof by

$$\begin{aligned} &\gamma(Z_1) \\ \geq &\gamma(\partial\mathcal{O} \setminus Y) - (k-1)n - m \\ \geq &nk - (n-j) - (k-1)n - m = j - m. \end{aligned} \tag{2.10}$$

We construct the map as follows. For $p = 1, ..., k-1$, define a map $f_p : B^{nk} \to \mathbb{R}^n$ by

$$f_p(t) = \sum_{s\in S_p} t_{i(s)} d(\varphi(t), \partial(\cap_{j\in s^c} M_j)) \tag{2.11}$$

where $i(s) = i_1$ for $s = (i_1, ..., i_p) \in S_p$. Since M_j, $j = 1, ..., k$, are G-invariant, f is odd in t. In fact,

$$\begin{aligned} f_p(-t) &= \sum_{s\in S_p} (-t)_{i(s)} d(\varphi(-t), \partial(\cap_{j\in s^c} M_j)) \\ &= -\sum_{s\in S_p} (t)_{i(s)} d(G\varphi(t), \partial(\cap_{j\in s^c} M_j)) \\ &= -\sum_{s\in S_p} (t)_{i(s)} d(\varphi(t), G\partial(\cap_{j\in s^c} M_j)) \\ &= -\sum_{s\in S_p} (t)_{i(s)} d(\varphi(t), \partial(\cap_{j\in s^c} M_j)) \\ &= -f_p(t). \end{aligned} \tag{2.12}$$

We show that $f_p(t) \neq 0$ for $t \in A_p$. Suppose $\varphi(t) \in C_p$. There exists $s_0 \in S_p$ such that $\varphi(t) \in \cap_{i\in s_0} \partial M_i \cap \cap_{j\in s_0^c} M_j$. Then $t_{i(s_0)} \neq 0$ and $d(\varphi(t), \partial(\cap_{j\in s_0^c} M_j)) > 0$. If $s \in S_p$ and $s \neq s_0$, then $s^c \cap s_0 \neq \emptyset$,

$$\varphi(t) \in \cap_{i\in s_0} \partial M_i \cap \cap_{j\in s_0^c} M_j \subset \cap_{i\in s_0\cap s^c} \partial M_i \cap \cap_{j\in s_0^c \cap s^c} M_j \subset \partial(\cap_{j\in s^c} M_j)$$

and $d(\varphi(t), \partial(\cap_{j\in s^c} M_j)) = 0$, for all $s \neq s_0$. We have for $t \in A_p$

$$\begin{aligned} f(t) &= \sum_{s\in S_p} t_{i(s)} d(\varphi(t), \partial(\cap_{j\in s^c} M_j)) \\ &= t_{i(s_0)} d(\varphi(t), \partial(\cap_{j\in s_0^c} M_j)) \neq 0. \end{aligned} \tag{2.13}$$

Next for $i =, 1..., m$ define $g_i : B^{nk} \to \mathbb{R}$ by

$$g_i(t) = d(\varphi(t), \partial P_i) - d(\varphi(t), \partial Q_i). \tag{2.14}$$

Then we have

$$\begin{aligned} g_i(-t) &= d(\varphi(-t), \partial P_i) - d(\varphi(-t), \partial Q_i) \\ &= d(G\varphi(t), \partial P_i) - d(G\varphi(t), \partial Q_i) \\ &= d(\varphi(t), G\partial P_i) - d(\varphi(t), G\partial Q_i) \\ &= d(\varphi(t), \partial Q_i) - d(\varphi(t), \partial P_i) \\ &= -g_i(t). \end{aligned} \tag{2.15}$$

Moreover for $\varphi(t) \in \partial P_i \cap Q_i$, $d(\varphi(t), \partial P_i) = 0$, $d(\varphi(t), \partial Q_i) < 0$ and $g_i(t) = -d(\varphi(t), \partial Q_i) > 0$. Similarly, for $\varphi(t) \in P_i \cap \partial Q_i$, $g_i(t) = d(\varphi(t), \partial P_i) < 0$. Now we may set

$$h = (f_1, ..., f_{k-1}, g_1, ..., g_m) : B^{nk} \to \mathbb{R}^{n(k-1)+m}$$

with the desired property. □

Definition 2.3. Let X be a complete metric space with the metric d and f be a continuous functional on X. Let P_i, $i = 1, ..., k$ be a family of open sets in X. Set $W = \cup_{i=1}^k P_i$.

a) We say $\{P_i\}_1^k$ are an admissible family of invariant sets with respect to f at level c if the following deformation property holds: if $K_c \setminus \bar{W} = \emptyset$ there exists $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$ there exists a continuous map $\eta : X \to X$ satisfying

(1) $\eta(\bar{P}_i) \subset P_i, i = 1, ..., k$.
(2) $\eta|_{f^{c-2\epsilon}} = Id$.
(3) $\eta(J^{c+\epsilon} \setminus W) \subset J^{c-\epsilon}$.

b) Assume further G is an isometric involution of X and f is a G-invariant continuous functional on X. We say $\{P_i\}_1^k$ are a G-admissible family of invariant sets with respect to f at level c if the following deformation property holds: there exists a symmetric closed neighborhood N of $K_c \setminus W$ with $\gamma(N) < \infty$ and $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$ there exists a continuous map $\eta : X \to X$ satisfying

(1) $\eta(\bar{P}_i) \subset P_i, \eta(\bar{Q}_i) \subset Q_i$, $i = 1, ..., k$, here $Q_i = GP_i$, $i = 1, ..., k$.
(2) $\eta \circ G = G \circ \eta$.
(3) $\eta|_{f^{c-2\epsilon}} = Id$.
(4) $\eta(J^{c+\epsilon} \setminus (N \cup W)) \subset J^{c-\epsilon}$.

Theorem 2.4. *Let X be a complete metric space, P_i, $i = 1, ..., k$ be open subsets of X. Denote $M = \cap_{i=1}^k P_i$, $\Sigma = \cap_{i=1}^k \partial P_i$, and $W = \cup_{i=1}^k P_i$. Let f be a continuous functional on X. Assume that $\{P_i\}_1^k$ is an admissible family of invariant sets with respect to f at level c for $c \geq c_* := \inf_{u\in\Sigma} f(u)$. Suppose that there exists a map $\varphi_0 : \Delta \to X$ satisfying*

(1) $\varphi_0(\partial_i\Delta) \subset P_i$, $i = 1, ..., k$.
(2) $\varphi_0(\partial_0\Delta) \cap M = \emptyset$.
(3) $c_0 = \sup_{u\in\varphi_0(\partial_0\Delta)} f(u) < c_*$.

Define

$$c = \inf_{\varphi\in\Gamma} \sup_{u\in\varphi(\Delta)\setminus W} I(u) \tag{2.16}$$

where

$$\Gamma = \{\varphi \in C(\Delta, X) \mid \varphi(\partial_i\Delta) \subset P_i, i = 1, ..., k, \varphi|_{\partial_0\Delta} = \varphi_0\}. \tag{2.17}$$

Then c is a critical value of f and $K_c \setminus W \neq \emptyset$.

Proof. By Lemma 2.2, $\varphi(\Delta) \cap \Sigma \neq \emptyset$ for any $\varphi \in \Gamma$. Hence

$$c = \inf_{\varphi\in\Gamma} \sup_{u\in\varphi(\Delta)\setminus W} f(u) \geq \inf_{u\in\Sigma} f(u) = c_* > c_0 = \sup_{u\in\varphi_0(\partial_0\Delta)} f(u). \tag{2.18}$$

Assume $K_c \setminus W = \emptyset$. Take $0 < \epsilon \leq \epsilon_0$, $c - 2\epsilon \geq c_* - 2\epsilon > c_0$ so that $\varphi_0(\partial_0\Delta) \subset f^{c_0} \subset f^{c-2\epsilon}$. There exists a continuous map $\eta : X \to X$ satisfying (1), (2), and (3) in definition 2.3 a). Then $\eta \circ \varphi \in \Gamma$ for $\varphi \in \Gamma$. In fact, $\eta \circ \varphi(\partial P_i) \subset \eta(P_i) \subset P_i$, $i = 1, ..., k$, and $\eta \circ \varphi|_{\partial_0\Delta} = \eta \circ \varphi_0|_{\partial_0\Delta} = \varphi_0|_{\partial_0\Delta}$. On the other hand, by the definition of c, there exists $\varphi \in \Gamma$ such that $\varphi(\Delta) \setminus W \subset f^{c+\epsilon}$. Hence $\eta \circ \varphi(\Delta) \setminus W \subset (\eta(\varphi(\Delta) \setminus W) \cup \eta(W)) \setminus W \subset \eta(f^{c+\epsilon} \setminus W) \subset f^{c-\epsilon}$, which is a contradiction. □

Theorem 2.5. *Let X be a complete metric space with an isometric involution G, P_i, $i = 1, ..., k$ be open subsets of X. Denote $Q_i = -GP_i, i = 1, ..., k$, $M = \cap_{i=1}^k (P_i \cap Q_i)$, $\Sigma = \cap_{i=1}^k (\partial P_i \cap \partial Q_i)$, and $W = \cup_{i=1}^k (P_i \cup Q_i)$. Let f be a $G-$invariant continuous functional on X. Assume that $\{P_i\}_1^k$ is a G-admissible family of invariant sets with respect to f at level c for $c \geq c_* := \inf_{u\in\Sigma} f(u)$. Suppose that for any $n \in \mathbb{N}$ there exists a continuous map $\varphi^{(n)} : B^{nk} \to X$ satisfying*

(1) Denote $t = (t_1, ..., t_k) \in B^{nk}$, $t_1, ..., t_k \in \mathbb{R}^n$. *Then* $\varphi^{(n)}(t) \in M_i := P_i \cap Q_i$, *if* $t_i = 0, i = 1, ..., k$.

(2) $\varphi^{(n)}(\partial B^{nk}) \cap M = \emptyset$.

(3) $c_0 := \max\{\sup_{u\in F_G} f(u), \sup_{u\in\varphi^{(n)}(\partial B^{nk})} f(u)\} < c_*$.

(4) $\varphi^{(n)}(-t) = G\varphi^{(n)}(t), t \in B^{nk}$,

where $F_G = \{u|Gu = u\}$ *is the set of fixed points of* G. *Define*

$$c_j = \inf_{B\in\Gamma_j} \sup_{u\in B\setminus W} I(u) \tag{2.19}$$

where

$$\Gamma_j = \{B \mid B = \varphi(B^{nk} \setminus Y), \varphi \in G_n, n \geq j, \text{open subset } -Y = Y \subset B^{nk}, \gamma(\overline{Y}) \leq n - j\}, \tag{2.20}$$

and

$$G_n = \{\varphi \mid \varphi \in C(B^{nk}, X), \varphi(-t) = G\varphi(t), t \in B^{nk}; \varphi(t) \in M_i, \text{if } t_i = 0; \varphi|_{\partial B^{nk}} = \varphi^{(n)}\}. \tag{2.21}$$

Then c_j*,* $j \geq k + 1$*, are critical values of* f *with* $c_j \to \infty$ *and* $K_{c_j} \setminus W \neq \emptyset$.

Proof. We first claim $K_{c_j} \setminus W \neq \emptyset$.

Let $B \in \Gamma_j$, $j \geq k + 1$. Then $B = \varphi(B^{nk} \setminus Y), \varphi \in G_n, n \geq j$, open subset $-Y = Y \subset B^{nk}, \gamma(\overline{Y}) \leq n - j$. By Lemma 2.2, $\gamma(B \cap \Sigma) \geq j - k \geq 1$, for $j \geq k + 1$. Hence

$$c_j = \inf_{B\in\Gamma_j} \sup_{u\in B\setminus W} f(u) \geq \inf_{u\in\Sigma} f(u) = c_* > c_0 = \max\{\sup_{u\in F_G} f(u), \sup_{u\in\varphi^{(n)}(\partial B^{nk})} f(u)\} \tag{2.22}$$

Assume $K_{c_j}\setminus W = \emptyset$. Take $0 < \epsilon \leq \epsilon_0$, $c_j - 2\epsilon \geq c_* - 2\epsilon > c_0$ so that $\varphi^{(n)}(\partial_0\delta) \subset f^{c_0} \subset f^{c_j-2\epsilon}$. Then there exists a continuous map $\eta : X \to X$ satisfying (1), (2), and (3) (4) in definition 2.3 b). Then $A = \eta(B) \in \Gamma_j$ for $B \in \Gamma_j$. On the other hand, by the definition of c_j, there exists $B \in \Gamma_j$ such that $B \setminus W \subset f^{c_j+\epsilon}$. Hence

$$A \setminus W = \eta(B) \setminus W \subset (\eta(B \setminus W) \cup \eta(W)) \setminus W \subset \eta(B \setminus W) \subset f^{c_j-\epsilon}, \tag{2.23}$$

which is a contradiction. The claim is proved.

Next we claim $c_j \to \infty$ as $j \to \infty$.

Since c_j is nondecreasing in j, $c_j \to c$ as $j \to \infty$. If $c < \infty$, then c is a critical value of f and $c \geq c_* > \sup_{u\in F_G} f(u)$. By deformation property there exists a symmetric open neighborhood N of $K_c \setminus W$ with $\gamma(\overline{N}) = m < \infty$. By deformation property there exists a constant $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$ there exists a continuous map $\eta : X \to X$ with the properties in Definition 2.3 b). Assume $c - 2\epsilon \geq c_* - 2\epsilon > c_0$. Choose j large enough such that $c_{m+j} \geq c_j > c - \epsilon/2$. There exists a set $B \in \Gamma_{j+m}$ such that $B \setminus W \subset f^{c_{j+m}+\epsilon} \subset f^{c+\epsilon}$. Then $B = \varphi(B^{nk} \setminus Y), \varphi \in G_n, n \geq m + j$, open subset $-Y = Y \subset B^{nk}, \gamma(\overline{Y}) \leq n - m + j$. We have $\eta \circ \varphi \in G_n$. Let $Y_1 = Y \cup \varphi^{-1}(N)$. Then $\gamma(\overline{Y}_1) \leq \gamma(\overline{N}) + \gamma(\overline{Y}) \leq m + n - (j + m) = n - j$. Let $A = \eta \circ \varphi(B^{nk} \setminus Y)$. It follows

$A \in \Gamma_j$ and

$$\begin{aligned} A \setminus W = \eta \circ \varphi(B^{nk} \setminus Y_1) \setminus W \subset \eta(\varphi(B^{nk} \setminus Y) \setminus N) \setminus W \\ \subset (\eta(f^{c+\epsilon} \setminus (N \cup W)) \cup \eta(W)) \setminus W \subset \eta(f^{c+\epsilon} \setminus (N \cup W)) \subset f^{c-\epsilon} \subset f^{c_j - \frac{\epsilon}{2}} \end{aligned} \tag{2.24}$$

which is a contradiction. □

Theorem 2.6. *Let X be a complete metric space with an isometric involution G, P_i, $i = 1, ..., m$ be open subsets of X. Denote $Q_i = -GP_i$ and $M_i = P_i \cap Q_i$, $i = 1, ..., m$. Let M_j for $j = m+1, ..., k$ be open sets of X satisfying $M_j = GM_j$. Set $M = \cap_{i=1}^k M_j$, $\Sigma = \cap_{i=1}^m (\partial P_i \cap \partial Q_i) \cap \cap_{j=m+1}^k \partial M_j$, and $W = \cup_{i=1}^m (P_i \cup Q_i) \cup \cup_{j=m+1}^k M_j$. Let f be a $G-$invariant continuous functional on X. Assume that $\{\{P_i\}_1^m, \{M_j\}_{i=m+1}^k\}$ are a G-admissible family of invariant sets with respect to f at level c for $c \geq c_* := \inf_{u\in\Sigma} f(u)$. Suppose that for any $n \in \mathbb{N}$ there exists a continuous map $\varphi^{(n)} : B^{nk} \to X$ satisfying*

(1) Denote $t = (t_1, ..., t_k) \in B^{nk}$, $t_1, ..., t_k \in \mathbb{R}^n$. Then $\varphi^{(n)}(t) \in M_i$, if $t_i = 0, i = 1, ..., k$.
(2) $\varphi^{(n)}(\partial B^{nk}) \cap M = \emptyset$.
(3) $c_0 := \sup_{u\in\varphi^{(n)}(\partial B^{nk})} f(u) < c_*$.
(4) $\varphi^{(n)}(-t) = G\varphi^{(n)}(t), t \in B^{nk}$.
Define

$$c_j = \inf_{B\in\Gamma_j} \sup_{u\in B\setminus W} I(u) \tag{2.25}$$

where

$$\Gamma_j = \{B \mid B = \varphi(B^{nk} \setminus Y), \varphi \in G_n, n \geq j, \text{open subset } -Y = Y \subset B^{nk}, \gamma(\overline{Y}) \leq n - j\}, \tag{2.26}$$

and

$$G_n = \{\varphi \mid \varphi \in C(B^{nk}, X), \varphi(-t) = G\varphi(t), t \in B^{nk}; \varphi(t) \in M_i, \text{if } t_i = 0; \varphi|_{\partial B^{nk}} = \varphi^{(n)}\}. \tag{2.27}$$

Assume $(K_{c_j} \setminus W) \cap F_G = \emptyset$ for $j \geq m+1$, where $F_G = \{u|Gu = u\}$ is the set of fixed points of G. Then c_j, $j \geq m+1$, are critical values of f with $c_j \to \infty$ and $K_{c_j} \setminus W \neq \emptyset$.

The proof of this theorem is similar to the proof Theorem 2.5 with some obvious modifications. We omit it here.

Remark 2.7. The theorems are extensions of the symmetric mountain pass theorem due to Ambrosetti and Rabinowitz ([1, 34]). We give the locations of the minimax critical points constructed relevant to a family of invariant sets of the variational flow. These are also considered generalizations of the framework done in [8, 3, 5, 6, 7, 22, 23] where $k = 1$ was treated.

Remark 2.8. In our applications to nonlinear Schrödinger systems the deformation property in the requirements of admissible family are readily satisfied. There are other general sufficient conditions which assure the admissibility of the invariant sets family.

a) Assume X is a Hilbert space, f is a C^1-functional, and the gradient of f is of the form $\nabla f(u) = u - A(u)$ where A is a nonlinear operator satisfying $A(\partial P_i) \subset P_i, A(\partial Q_i) \subset Q_i$, $i = 1, ..., k$. We refer to [6] for more details on this.

b) Assume X is a Banach space, f is a C^1-functional, and there exists a nonlinear compact map $A : X \to X$ satisfying $A(\partial P_i) \subset P_i, A(\partial Q_i) \subset Q_i$, $i = 1, ..., k$. Furthermore assume that there exist $1 < p < \infty$ and positive constants a_1, a_2 such that

$$\langle f'(u), u - A(u)\rangle \geq a_1||u - A(u)||^p,\ ||f'(u)|| \leq a_2||u - A(u)||^{p-1}.$$

Here the constants a_1, a_2 may depend on the value of $f(u)$. More precisely, given $b \in \mathbb{R}$, there exist $a_1 = a_1(b), a_2(b)$ such that the above inequalities hold for $u \in f^b$. We refer to [5, 7, 17] for more details on this. Also in C_0^1 topology some related work in [3, 8, 22, 23].

c) More generally, we may assume

$$\langle f'(u), u - A(u)\rangle \geq ||u - A(u)||g_1(||u - A(u)||),\ ||f'(u)|| \leq g_2(||u - A(u)||),$$

where g_1, g_2 are strictly increasing continuous functions on $[0, \infty)$ satisfying $g_1(0) = g_2(0) = 0$. Again the functions g_1, g_2 may be dependent of $f(u)$. An example of this type is given in next section when we treat the nonlinear Schrödinger system (1.1).

3. Applications to nonlinear Schrödinger systems

We consider the nonlinear Schrödinger system

$$\begin{cases} -\Delta u_j + \lambda_j u_j = \sum_{i=1}^{k} \beta_{ij} u_i^2 u_j, x \in \mathbb{R}^N \\ u_j(x) \to 0, \text{as } |x| \to \infty, j = 1, ..., k \end{cases} \tag{3.1}$$

where $N = 2, 3, k \geq 2, \lambda_j > 0$, for $j = 1, ..., k$, β_{ij} are constants satisfying $\beta_{jj} > 0$ for $j = 1, ..., k$, $\beta_{ij} = \beta_{ji} \leq 0$ for $1 \leq i < j \leq k$.

To make the paper more readable we first prove Theorem 1.1 for a special case $m = k$ whose proof is more straightforward in term of using our abstract results. I.e., we look for solutions $u = (u_1, ..., u_k)$ with each component u_j sign-changing for $j = 1, ..., k$. Later we will point necessary changes for the complications caused in the general case $1 \leq m \leq k-1$. We will also consider the case $m = 0$, i.e., solutions with each component signed. We first prove the following theorem.

Theorem 3.1. *Assume $N = 2, 3, k \geq 2, \lambda_j > 0, \beta_{jj} > 0$ for $j = 1, ..., k$, $\beta_{ij} = \beta_{ji} \leq 0$ for $1 \leq i < j \leq k$. Then the system (3.1) possesses infinitely many radially symmetric solutions with each component sign-changing.*

Remark 3.2. We point out that our method does not depend on the radially symmetry of the problems. We work in radially symmetric functions for the compactness of the problems, i.e., the compact embedding from $H^1(\mathbb{R}^N)$ into $L^4(\mathbb{R}^N)$. Thus our result is still valid for other cases as long as the compactness holds. For example, we may consider the systems in an arbitrary bounded domain with Dirichlet boundary condition and we obtain infinitely many nodal solutions by the same methods. In case of $\mathbb{R}^N$ when there are potentials involved, for example, when $\lambda_j = \lambda_j(x)$ for $j = 1, ..., k$ satisfying a compactness condition like

(C) there exists $r > 0$ such that for all $M > 0$, the Lebesgue measure of $\{x \in B_r(y) \mid \lambda_j(x) \leq M\}$ tends to zero as $|y| \to \infty$.

Then we obtain a unbounded sequence of nodal solutions. We leave the precise statements to the readers and the proofs requires little changes.

Let $E = H_r^1(\mathbb{R}^N)$ be the space of radially symmetric functions in $H^1(\mathbb{R}^N)$ in which we shall use the equivalent inner products

$$(u, v)_j = \int_{\mathbb{R}^N} (\nabla u \nabla v + \lambda_j uv)dx, j = 1, ..., k,$$

and the induced norm $|| \cdot ||_j$. The product space $E^k = \overbrace{E \times \cdots \times E}$ is a subspace of $(H^1(\mathbb{R}^N))^k$ endowed with the inner product

$$(u, v) = \sum_{j=1}^{k} (u, v)_j, u = (u_1, ..., u_k), v = (v_1, ..., v_k),$$

which gives rise to a norm on E^k: $|| \cdot ||$. In the following we use $| \cdot |_p$ to denote the L^p norm and constant C may be used from line to line for different constants but independent of the arguments.

Radially symmetric solutions correspond to critical points of the functional

$$J(u) = \frac{1}{2}||u||^2 - \frac{1}{4}\int_{\mathbb{R}^N} \sum_{i,j=1}^{k} \beta_{ij} u_i^2 u_j^2 dx. \tag{3.2}$$

It is easy to check that $J \in C^2(E^k)$ and J satisfies the (PS) condition.

If the matrix $B = (\beta_{ij})$ is positive definite our abstract theory can be applied in a more straight forward way. In the following we consider two cases, one with this assumption and one without this assumption. In the latter case we need to further modify the variational problem and a limiting procedure is used to obtain the desired results.

(B) The matrix (β_{ij}) is positive definite.

3.1 The proof of Theorem 3.1 under the additional condition (B)

We introduce some notations first. Let P be the positive cone in $H_r^1(\mathbb{R}^N)$, $P = \{u \in H_r^1(\mathbb{R}^N) \mid u \geq 0, a.e.\}$. For $\delta > 0$ we define open cones in E^k for $i = 1, ..., k$, by

$$P_i = P_i(\delta) = \{u \in E^k \mid u = (u_1, ..., u_k), d(u_i, -P) < \delta\},$$

$$Q_i = Q_i(\delta) = -P_i = \{u \in E^k \mid u = (u_1, ..., u_k), d(u_i, P) < \delta\}.$$

For a function u, let $u^+ = \max(u, 0)$ and $u^- = \min(u, 0)$ so $u = u^+ + u^-$. Next we define an operator $A : E^k \to E^k$ as follows. Given $u = (u_1, ..., u_k) \in E^k$, define $w = Au = (w_1, ..., w_k)$ by

$$\begin{cases} -\Delta w_j + \lambda_j w_j - \sum_{i \neq j} \beta_{ij} u_i^2 w_j = \beta_{jj} u_j^3, x \in \mathbb{R}^N \\ w_j(x) \to 0, \text{as } |x| \to \infty, j = 1, ..., k. \end{cases} \tag{3.3}$$

Alternatively in the weak form we have

$$\int_{\mathbb{R}^N} (\nabla w_j \nabla \varphi + \lambda_j w_j \varphi) dx - \int_{\mathbb{R}^N} (\sum_{i \neq j} \beta_{ij} u_i^2) w_j \varphi dx = \int_{\mathbb{R}^N} \beta_{jj} u_j^3 \varphi dx, \forall \varphi \in H_r^1(\mathbb{R}^N). \quad (3.4)$$

Then we can show that A is locally Lipschitz continuous. Using the operator A we may construct a negative pseudo gradient flow associated with J as follows. Let $\varphi^t(u)$ with the maximal interval of existence $[0, \sigma(u))$ be the solution of the initial value problem

$$\begin{cases} \dfrac{d}{dt}\varphi^t = -(\varphi^t - A\varphi^t), t \in [0, \sigma(u) \\ \varphi^0 = u. \end{cases} \quad (3.5)$$

Lemma 3.3. *For sufficiently small $\delta > 0$, $A(\partial P_j) \subset P_j$, $A(\partial Q_j) \subset Q_j$, $j = 1, ..., k$. Hence P_j, Q_j, $j = 1, ..., k$ are strictly invariant for the flow φ^t in the sense that $\varphi^t(u) \in intP_j$ (resp. $intQ_j$) for $u \in P_j$ (resp. Q_j) and $t \in (0, \sigma(u))$, $j = 1, ..., k$.*

Proof. Take $\varphi = w_j^-$ in (3.4), we obtain

$$\int_{\mathbb{R}^N} (|\nabla w_j^-|^2 + \lambda_j (w_j^-)^2) dx - \int_{\mathbb{R}^N} (\sum_{i \neq j} \beta_{ij} u_i^2)(w_j^-)^2 dx = \int_{\mathbb{R}^N} \beta_{jj} u_j^3 w_j^- dx. \quad (3.6)$$

Then we have

$$\begin{aligned} d^2(w_j, P) &\leq C \int_{\mathbb{R}^N} (|\nabla w_j^-|^2 + \lambda_j (w_j^-)^2) dx \\ &\leq C \int_{\mathbb{R}^N} \beta_{jj} (u_j^-)^3 w_j^- dx \\ &\leq C (\int_{\mathbb{R}^N} (u_j^-)^4 dx)^{\frac{3}{4}} (\int_{\mathbb{R}^N} (w_j^-)^4 dx)^{\frac{1}{4}} \\ &\leq C d_{L^4}^3(u_j, P) d_{L^4}(w_j, P) \\ &\leq C d^3(u_j, P) d(w_j, P). \end{aligned} \quad (3.7)$$

Choose $C\delta_0^2 = \frac{1}{2}$ and $\delta < \delta_0$. We have

$$d(w_j, P) \leq \frac{1}{2} d(u_j, P).$$

Hence $A(\partial P_j) \subset P_j$. Similarly $A(\partial Q_j) \subset Q_j$, $j = 1, ..., k$. ☐

Lemma 3.4. *It holds*

$$(\nabla J(u), u - Au) = ||u - Au||^2 - \int_{\mathbb{R}^N} \sum_{i \neq j} \beta_{ij} u_i^2 (u_j - w_j)^2 dx. \quad (3.8)$$

Consequently it holds

$$(\nabla J(u), u - Au) \geq ||u - Au||^2, \quad (3.9)$$

Moreover, if we assume that the matrix (β_{ij}) is positive definite, then there exists a constant $C > 0$ such that

$$||\nabla J(u)|| \leq C ||u - Au|| (1 + |J(u)|^{\frac{1}{2}} + ||u - Au||^2). \quad (3.10)$$

Proof. By (3.4) for $\varphi=(\varphi_1,...,\varphi_k)\in E^k$

$$\begin{aligned}&\int_{\mathbb{R}^N}\nabla(u_j-w_j)\nabla\varphi_j dx+\lambda_j(u_j-w_j)\varphi_j dx\\=&\int_{\mathbb{R}^N}\nabla u_j\nabla\varphi_j dx+\lambda_j u_j\varphi_j dx-\int_{\mathbb{R}^N}\beta_{jj}u_j^3\varphi_j dx-\int_{\mathbb{R}^N}(\sum_{i\neq j}\beta_{ij}u_i^2)w_j\varphi_j dx\\=&(\nabla_j J(u),\varphi_j)+\int_{\mathbb{R}^N}(\sum_{i\neq j}\beta_{ij}u_i^2)(u_j-w_j)\varphi_j dx.\end{aligned}\tag{3.11}$$

Thus

$$(\nabla J(u),\varphi)=(u-Au,\varphi)-\int_{\mathbb{R}^N}\sum_{i\neq j}\beta_{ij}u_i^2(u_j-w_j)\varphi_j dx.\tag{3.12}$$

Taking $\varphi=u-Au$, we obtain

$$(\nabla J(u),u-Au)=||u-Au||^2-\int_{\mathbb{R}^N}\sum_{i\neq j}\beta_{ij}u_i^2(u_j-w_j)^2 dx.\tag{3.13}$$

It follows from (3.12) and (3.13)

$$\begin{aligned}&|\int_{\mathbb{R}^N}\sum_{i\neq j}\beta_{ij}u_i^2(u_j-w_j)\varphi_j dx|\\\leq&C\sum_{i\neq j}(\int_{\mathbb{R}^N}(-\beta_{ij}u_i^2(u_j-w_j)^2dx)^{\frac12}(\int_{\mathbb{R}^N}u_i^4dx)^{\frac14}(\int_{\mathbb{R}^N}\varphi_j^4dx)^{\frac14}\\\leq&C(\nabla J(u),u-Au)^{\frac12}|u|_4|\varphi|_4.\end{aligned}\tag{3.14}$$

By (3.12) and (3.14)

$$||\nabla J(u)||\leq||u-Au||+C(\nabla J(u),u-Au)^{\frac12}|u|_4.\tag{3.15}$$

Choose $2<s<4$. Then

$$\begin{aligned}&J(u)-\frac1s(u,u-Au)\\=&(\frac12-\frac1s)||u||^2+(\frac1s-\frac14)\int_{\mathbb{R}^N}\sum_{i,j=1}^k\beta_{ij}u_i^2u_j^2dx-\frac1s\int_{\mathbb{R}^N}\sum_{i\neq j}\beta_{ij}u_i^2(u_j-w_j)u_j dx.\end{aligned}\tag{3.16}$$

Now assume that the matrix (β_{ij}) is positive definite. By (3.14) and (3.16)

$$\begin{aligned}||u||^2+|u|_4^4\leq&C(|J(u)|+|(u,u-Au)|+|\int_{\mathbb{R}^N}\sum_{i\neq j}\beta_{ij}u_i^2(u_j-w_j)u_j dx|)\\\leq&C(|J(u)|+||u||||u-Au||+(\nabla J(u),u-Au)^{\frac12}|u|_4^2).\end{aligned}\tag{3.17}$$

Hence

$$||u||^2+|u|_4^4\leq C(|J(u)|+||u-Au||^2+(\nabla J(u),u-Au)),\tag{3.18}$$

and

$$|u|_4 \leq C(|J(u)|^{\frac{1}{4}} + ||u - Au||^{\frac{1}{2}} + (\nabla J(u), u - Au)^{\frac{1}{4}}). \tag{3.19}$$

Substituting (3.19) into (3.15) we obtain

$$||\nabla J(u)|| \leq ||u-Au|| + C(\nabla J(u), u-Au)^{\frac{1}{2}}(|J(u)|^{\frac{1}{4}} + ||u-Au||^{\frac{1}{2}} + (\nabla J(u), u-Au)^{\frac{1}{4}}). \tag{3.20}$$

Finally we have

$$||\nabla J(u)|| \leq C||u - Au||(1 + |J(u)|^{\frac{1}{2}} + ||u - Au||^2). \tag{3.21}$$

□

Next we show that the family $\{P_j\}$ are an admissible family of invariant sets for functional J at level c for any c.

Lemma 3.5. *Let N be a symmetric closed neighborhood of K_c. Then there exists a positive constant $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon' < \epsilon_0$ there exists a continuous map $\sigma : [0,1] \times E^k \to E^k$ satisfying*

(1) $\sigma(0, u) = u$, for all $u \in E^k$.

(2) $\sigma(t, u) = u$, for $t \in [0,1]$, $J(u) \notin [c - \epsilon', c + \epsilon']$.

(3) $\sigma(t, -u) = -\sigma(t, u)$ for all $(t, u) \in [0,1] \times E^k$.

(4) $\sigma(1, J^{c+\epsilon} \setminus N) \subset J^{c-\epsilon}$.

(5) $\sigma(t, \bar{P}_i) \subset P_i$, $\sigma(t, \bar{Q}_i) \subset Q_i$, $i = 1, ..., k$, $t \in [0,1]$.

Proof. For $\delta > 0$ sufficiently small $N(\delta) = \{u \in E^k \mid d(u, K_c) < \delta\} \subset N$. Since J satisfies the (PS) condition, there exist constants $\epsilon_0, b_0 > 0$ such that

$$||\nabla J(u)|| \geq b_0, \text{for } u \in J^{-1}([c - \epsilon_0, c + \epsilon_0]) \setminus N(\frac{1}{2}\delta). \tag{3.22}$$

By Lemma 3.4, there exists a constant $b > 0$ such that $||u - Au|| \geq b$ for $u \in J^{-1}([c - \epsilon_0, c + \epsilon_0]) \setminus N(\frac{1}{2}\delta)$. Decreasing ϵ_0 in necessary we we assume $\epsilon_0 \leq \frac{1}{4}b\delta$.

Define two even continuous functions $g, p : E^k \to [0,1]$ such that

$$g(u) = \begin{cases} 0, u \in N(\frac{1}{4}\delta) \\ 1, u \notin N(\frac{1}{2}\delta), \end{cases} \tag{3.23}$$

$$p(u) = \begin{cases} 0, J(u) \notin [c - \epsilon', c + \epsilon'] \\ 1, J(u) \in [c - \epsilon, c + \epsilon]. \end{cases} \tag{3.24}$$

Let

$$V(u) = \frac{u - Au}{||u - Au||}, u \notin K = \{u \mid J'(u) = 0\}.$$

Consider the initial value problem

$$\begin{cases} \frac{d\tau}{dt} = -g(\tau)p(\tau)V(\tau) \\ \tau(0, u) = u. \end{cases} \tag{3.25}$$

Set $\sigma(t,u) = \tau(\frac{2\epsilon}{b}t, u)$. Then we can verify (1) - (3) as usual. For (4), let $u \in J^{c+\epsilon} \setminus N$. If $J(\tau(t,u)) \geq c - \epsilon$ for $0 \leq t \leq \frac{2\epsilon}{b}$, then $g(\tau(t,u)) = 1$. We also have $p(\tau(t,u)) = 1$ since if for some $0 \leq t \leq \frac{2\epsilon}{b}$, $\tau(t,u) \in N(\frac{1}{2}\delta)$ we have $\frac{1}{2}\delta \leq ||\tau(t,u) - u|| \leq \int_0^t ||\tau'(s,u)||ds \leq t$ a contradiction. Thus we have $J(\tau(\frac{2\epsilon}{b}t, u)) \leq J(u) - \frac{2\epsilon}{b}\int_0^1 ||\tau(s,u) - A\tau(s,u)||ds \leq c + \epsilon - b\frac{2\epsilon}{b} = c - \epsilon$. To verify (5), we need only to notice that $A(\partial P_i) \subset P_i$, $A(\partial Q_i) \subset Q_i$, $i = 1, ..., k$. □

Corollary 3.6. *Let N be a closed symmetric neighborhood of $\tilde{K}_c = K_c \setminus W$. Then there exist a constant $\epsilon_0 > 0$, for $0 < \epsilon < \epsilon_0$ there exists a continuous map $\eta : E^k \to E^k$ such that*

(1) $\eta(-u) = -\eta(u)$ for $u \in E^k$.
(2) $\eta|_{J^{c-2\epsilon}} = Id$.
(3) $\eta(J^{c+\epsilon} \setminus (N \cup W)) \subset J^{c-\epsilon}$.
(4) $\eta(\bar{P}_i) \subset P_i$, $\eta(\bar{Q}_i) \subset Q_i$, $i = 1, ..., k$.

Proof. Note that $N \cup \cup_{i=1}^k (\bar{P}_i \cup Q_i)$ is a closed neighborhood of K_c. According to Lemma 3.5, we can choose $\eta = \sigma(1, \cdot)$. □

Lemma 3.7. *For $\delta > 0$ sufficiently small, it holds $J(u) \geq \delta^2/2$, for $u \in \Sigma = \cap_{i=1}^k (\partial P_i \cap \partial Q_i)$.*

Proof. For $u \in \partial P_j$, we have $||u_j^-|| \geq d(u_j, P_j) = \delta$,

$$\int_{\mathbb{R}^N} (u_j^-)^4 dx = d_{L^4}^4(u_j, P_j) \leq C d^4(u_j, P_j) = C\delta^4.$$

Similar estimates hold for u_j^+. We then have

$$J_\mu(u) = \frac{1}{2}||u||^2 - \frac{1}{4}\int_{\mathbb{R}^N} \sum_{i,j=1}^k \beta_{ij} u_i^2 u_j^2 dx \geq \frac{k}{2}\delta^2 - C(\delta^4) \geq \delta^2/2.$$

□

Let B^{nk} be the unit closed ball of R^{nk}. Denote $t \in \mathbb{R}^{nk}$ by $t = (t_1, ..., t_k)$ with $t_i \in \mathbb{R}^n$ for $i = 1, ..., k$. Define a continuous map $\varphi^{(n)} : B^{nk} \to E^k$ by

$$\varphi^{(n)}(t) = R_n(t_1 \cdot v_1, ..., t_k \cdot v_k) \tag{3.26}$$

where R_n is a large number, $v_j = (v_{1j}, ..., v_{nj}) \in E^n$. For $v = (v_1, ..., v_n) \in E^n$, $s = (s_1, ..., s_n) \in \mathbb{R}^n$, we denote $s \cdot v = s_1 v_1 + \cdots + s_n v_n \in E$. We require v_{ij}, $1 \leq i \leq n, 1 \leq j \leq k$, have disjoint supports. Then

$$\begin{cases} \varphi^{(n)}(-t) = -\varphi^{(n)}(t), t \in B^{nk} \\ \varphi^{(n)}(t) \in M_j = P_j \cap Q_j, \text{if } t_j = 0 \\ \sup\limits_{t \in \partial B^{nk}} J(\varphi(t)) \leq 0 < \inf\limits_{u \in \Sigma} J(u) = c_*. \end{cases} \tag{3.27}$$

Define

$$c_j = \inf_{B\in\Gamma_j} \sup_{u\in B\setminus W} J(u), \tag{3.28}$$

where

$$\Gamma_j = \{B \mid B = \varphi(B^{nk} \setminus Y), \varphi \in G_n, n \geq j, \text{open subset } Y = -Y \subset B^{nk}, \gamma(\overline{Y}) \leq n - j\}, \tag{3.29}$$

and

$$G_n = \{\varphi \in C(B^{nk}, E^k) \mid \varphi(-t) = -\varphi(t); \varphi(t) \in M_j, t_j = 0; \varphi|_{\partial B^{nk}} = \varphi^{(n)}\}. \tag{3.30}$$

By Theorem 2.5 $c_j \geq c_* > 0$, for $j \geq k+1$, are critical values of J, $K_{c_j} \setminus W \neq \emptyset$, and $c_j \to \infty$ as $j \to \infty$.

3.2 The proof of Theorem 3.1: the general case

For the general case some of the estimates used in last subsection do not hold anymore, in particular, (3.10) in Lemma 3.4. The idea here is to modify the equation (and therefore the functional) by perturbations so the methods used in last subsection can be accomplished for the modified problems. Then a convergence argument allows us to pass limit to obtain solutions of the original problem with desired properties.

Choose $4 < p < \frac{2N}{N-2}$. For $\mu \in (0, 1]$ consider the functional

$$J_\mu(u) = J(u) - \frac{\mu}{p} \int_{\mathbb{R}^N} \sum_{j=1}^{k} |u_j|^p dx, \ u \in E^k. \tag{3.31}$$

Then it is straightforward to show that $J_\mu \in C^2(E^k)$ and J_μ satisfies the (PS) condition. We define the cones P_j, Q_j, $j = 1, ..., k$ as before and define the operator A_μ, $w = A_\mu u$, $w = (w_1, ..., w_j)$ by

$$\begin{cases} -\Delta w_j + \lambda_j w_j - \sum_{i\neq j} \beta_{ij} u_i^2 w_j = \beta_{jj} u_j^3 + \mu |u_j|^{p-2} u_j, x \in \mathbb{R}^N \\ w_j(x) \to 0, \text{as } |x| \to \infty, j = 1, ..., k. \end{cases} \tag{3.32}$$

In the weak form we have $w_j \in H_r^1(\mathbb{R}^N)$ satisfies $\forall \varphi \in H_r^1(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} (\nabla w_j \nabla \varphi + \lambda_j w_j \varphi) dx - \int_{\mathbb{R}^N} (\sum_{i\neq j} \beta_{ij} u_i^2) w_j \varphi dx = \int_{\mathbb{R}^N} \beta_{jj} u_j^3 \varphi dx + \mu \int_{\mathbb{R}^N} |u_j|^{p-2} u_j \varphi dx. \tag{3.33}$$

Then A_μ is locally Lipschitz continuous. Parallel to Lemmas 3.3, 3.5, 3.7, we have the following results.

Lemma 3.8. *For sufficiently small $\delta > 0$, independent of μ, $A_\mu(\partial P_j) \subset P_j$, $A_\mu(\partial Q_j) \subset Q_j$, $j = 1, ..., k$.*

Proof. Take $\varphi = w_j^-$ in (3.33), we obtain

$$\int_{\mathbb{R}^N}(|\nabla w_j^-|^2+\lambda_j(w_j^-)^2)dx-\int_{\mathbb{R}^N}(\sum_{i\neq j}\beta_{ij}u_i^2)(w_j^-)^2dx \leq \int_{\mathbb{R}^N}\beta_{jj}u_j^3w_j^-dx+\mu\int_{\mathbb{R}^N}(u_j^-)^{p-1}w_j^-dx. \tag{3.34}$$

Then as in the proof of (3.7)

$$d^2(w_j,P_j)\leq C(d^3(u_j,P_j)+d^{p-1}(u_j,P_j))d(w_j,P_j). \tag{3.35}$$

Choose $C(\delta_0^2+\delta_0^{p-2})=\frac{1}{2}$ and $\delta<\delta_0$. We have

$$d(w_j,P_j)\leq \frac{1}{2}d(u_j,P_j).$$

Hence $A_\mu(\partial P_j)\subset P_j$. Similarly $A_\mu(\partial Q_j)\subset Q_j$, $j=1,...,k$. Notice that δ_0 is independent of μ. □

Lemma 3.9. *It holds*

$$(\nabla J_\mu(u),u-A_\mu u)=||u-A_\mu u||^2-\int_{\mathbb{R}^N}\sum_{i\neq j}\beta_{ij}u_i^2(u_j-w_j)^2dx. \tag{3.36}$$

Moreover, there exists $C=C(\mu)>0$ such that

$$||\nabla J_\mu(u)||\leq C(\mu)||u-A_\mu u||(1+|J_\mu(u)|^{\frac{1}{2}}+||u-A_\mu u||^2). \tag{3.37}$$

Proof. First (3.36) can be proved as (3.8). By (3.33) and (3.36) we have

$$||\nabla J_\mu(u)||\leq ||u-A_\mu u||+C(\nabla J_\mu(u),u-A_\mu u)^{\frac{1}{2}}|u|_4. \tag{3.38}$$

Next

$$\begin{aligned}&J_\mu(u)-\frac{1}{4}(u,u-A_\mu u)\\=&\frac{1}{4}||u||^2+(\frac{1}{4}-\frac{1}{p})\mu\int_{\mathbb{R}^N}\sum_{j=1}^k|u_j|^pdx-\frac{1}{4}\int_{\mathbb{R}^N}\sum_{i\neq j}\beta_{ij}u_i^2(u_j-w_j)u_jdx.\end{aligned} \tag{3.39}$$

Then by Hölder inequality we have for some $C>0$

$$\begin{aligned}&||u||^2+\mu|u|_4^4\\\leq&C(\mu)(||u||^2+\mu\int_{\mathbb{R}^N}\sum_{j=1}^k|u_j|^pdx)\\\leq&C(|J(u)|+|(u,u-A_\mu u)|+|\int_{\mathbb{R}^N}\sum_{i\neq j}\beta_{ij}u_i^2(u_j-w_j)u_jdx|)\\\leq&C(|J(u)|+||u||||u-A_\mu u||+(\nabla J_\mu(u),u-A_\mu u)^{\frac{1}{2}}|u|_4^2).\end{aligned} \tag{3.40}$$

From here we have $C=C(\mu)>0$ such that

$$|u|_4\leq C(\mu)(|J(u)|^{\frac{1}{4}}+||u-A_\mu u||^{\frac{1}{2}}+(\nabla J_\mu(u),u-A_\mu u)^{\frac{1}{4}}), \tag{3.41}$$

and

$$||\nabla J_\mu(u)|| \leq C(\mu)||u - A_\mu u||(1 + |J_\mu(u)|^{\frac{1}{2}} + ||u - A_\mu u||^2). \tag{3.42}$$

□

Lemma 3.10. *For $\delta > 0$ sufficiently small, independent of μ, it holds $J_\mu(u) \geq \delta^2/2$, for $u \in \Sigma = \cap_{i=1}^k (\partial P_i \cap \partial Q_i)$.*

This is similar to Lemma 3.7.

Lemma 3.11. *Let $\mu_n \to 0$ and u_n satisfy $\nabla J_{\mu_n}(u_n) = 0$ and $J_{\mu_n}(u_n) \leq C$ for some $C > 0$. Then up to a subsequence, u_n converges to u in E^k, $\nabla J(u) = 0$ and $J(u) = \lim_{n\to\infty} J_{\mu_n}(u_n)$.*

Proof. Using

$$J_{\mu_n}(u_n) - \frac{1}{4}(\nabla J_{\mu_n}(u_n), u_n) = \frac{1}{2}||u_n||^2 + (\frac{1}{4} - \frac{1}{p})\mu_n \int_{\mathbb{R}^N} \sum_{j=1}^k |u_{n,j}|^p dx$$

we obtain that $\{u_n\}$ is bounded in E^k. By the compact imbedding from $H_r^1(\mathbb{R}^N)$ into $L_r^q(\mathbb{R}^N)$, $2 < q < \frac{2N}{N-2}$, a subsequence of $\{u_n\}$ converges to u in $L_r^q(\mathbb{R}^N)$. By a standard argument, $\{u_n\}$ converges to u in E^k and $\nabla J(u) = 0$ and $J(u) = \lim_{n\to\infty} J_{\mu_n}(u_n)$. Since $u_n \notin W$ we have $u \notin W$. □

With these preparations we are ready to finish the proof of Theorem 3.1 for the general case.

Again we define a continuous map $\varphi^{(n)} : B^{nk} \to E^k$ as in (3.25) and (3.48) by choosing R_n large such that

$$\begin{cases} \varphi^{(n)}(t) = R_n(t_1 v_1, \cdots, t_k v_k) \\ \varphi^{(n)}(t) \in M_j = P_j \cap Q_j, \text{if } t_j = 0 \\ \sup_{t\in\partial B^{nk}} J_\mu(\varphi^{(n)}(t)) \leq \sup_{t\in\partial B^{nk}} J(\varphi^{(n)}(t)) < \inf_{u\in\Sigma} J_1(u) = c_*. \end{cases} \tag{3.43}$$

Define

$$c_j(\mu) = \inf_{B\in\Gamma_j} \sup_{u\in B\setminus W} J_\mu(u), \mu \in (0, 1] \tag{3.44}$$

where

$$\Gamma_j = \{B \mid B = \varphi(B^{nk} \setminus Y), \varphi \in G_n, n \geq j, \text{open subset } Y = -Y \subset B^{nk}, \gamma(\overline{Y}) \leq n - j\}, \tag{3.45}$$

and

$$G_n = \{\varphi \in C(B^{nk}, E^k) \mid \varphi(-t) = -\varphi(t); \varphi(t) \in M_j, t_j = 0; \varphi|_{\partial B^{nk}} = \varphi^{(n)}\}. \tag{3.46}$$

By Theorem 2.5 $c_j \geq c_* > 0$, for $j \geq k + 1$, are critical values of J, $\tilde{K}_{c_j} \neq \emptyset$, and $c_j \to \infty$ as $j \to \infty$. We understand $J_0(u) = J(u)$ and $c_j(0) = c_j$, With the help of Lemmas 3.8, 3.9 and 3.10, the deformation property holds for the functionals J_μ, $\mu \in (0, 1]$. By Proposition

2.5, $c_j(\mu), j \geq k+1$ are critical values of J_μ, $K_{c_j(\mu)} \backslash W \neq \emptyset$, and $c_j(\mu) \to \infty$ as $j \to \infty$. We note that $c_j(\mu)$ is nondecreasing as $j \to \infty$ and $\mu \to 0$, and we have the following estimate

$$c^* \leq c_j(\mu) \leq c_j. \tag{3.47}$$

Let $c_j^* = \lim_{\mu \to 0} c_j(\mu)$ (in fact $c_j^* = c_j$). By the following Lemma 3.11, c_j^*, $j \geq k+1$ are critical values of the functional J. Moreover $c_j^* \geq c_j(\mu) \to \infty$ as $j \to \infty$.

3.3 The proof of Theorem 1.1, the general case: $1 \leq m < k$

We shall use Theorem 2.6 here. We consider the case when (B) holds first. The general case can be treated similarly with what we discuss here and the approximation scheme in section 3.2.

Using the notations in section 3.1 we set $X = \cap_{j=m+1}^k \bar{P}_j$ which is a complete metric space. Since P_j are invariant sets the negative gradient flow is still well defined on X. In the setting of Theorem 2.6 we have $P_1, .., P_m$ and take $M_i = P_i \cap Q_i$ for $i = 1, ..., m$. Take $M_j = Q_j \cap X$ for $j = m+1, ..., k$. Define $G : X \to X$ by

$$G(u_1, ..., u_m, u_{m+1}, ..., u_k) = (-u_1, ..., -u_m, u_{m+1}, ..., u_k).$$

Then G is an isometric involution on X and the functional J is G-invariant. Set

$$M = \cap_{i=1}^k M_j, \ \Sigma = \cap_{i=1}^m (\partial P_i \cap \partial Q_i) \cap \cap_{j=m+1}^k \partial M_j,$$

and

$$W = \cup_{i=1}^m (P_i \cup Q_i) \cup \cup_{j=m+1}^k M_j.$$

Lemmas 3.3 and 3.4 are still valid. For Lemma 3.5 replacing E^k by X we may obtain the result. To construct the proper maps $\varphi^{(n)}$ we need to modify the proof as follows.

Let B^{nk} be the unit closed ball of R^{nk}. Denote $t \in \mathbb{R}^{nk}$ by $t = (t_1, ..., t_k)$ with $t_i \in \mathbb{R}^n$ for $i = 1, ..., k$. Define a continuous map $\varphi^{(n)} : B^{nk} \to X$ by

$$\varphi^{(n)}(t) = R_n(t_1 \cdot v_1, ..., t_m \cdot v_m, \bar{t}_{m+1} v_{m+1}, ..., \bar{t}_k \cdot v_k) \tag{3.48}$$

where $v_j = (v_{1j}, ..., v_{nj}) \in E^n$, and R_n is a large number such that $\varphi^{(n)}(\partial B^{nk}) \cap M = \emptyset$. For $v = (v_1, ..., v_n) \in E^n$, $s = (s_1, ..., s_n) \in \mathbb{R}^n$, we denote $s \cdot v = s_1 v_1 + \cdots + s_n v_n \in E$ and $\bar{s} \cdot v = |s_1| v_1 + ... + |s_n| v_n$. We require v_{ij}, $1 \leq i \leq n, 1 \leq j \leq k$, have disjoint supports. Then

$$\begin{cases} \varphi^{(n)}(-t) = G\varphi^{(n)}(t), t \in B^{nk} \\ \varphi^{(n)}(t) \in M_j, \text{if } t_j = 0, j = 1, ..., k. \\ \sup\limits_{t \in \partial B^{nk}} J(\varphi(t)) \leq 0 < \inf\limits_{u \in \Sigma} J(u) = c_*. \end{cases} \tag{3.49}$$

Define

$$c_j = \inf_{B \in \Gamma_j} \sup_{u \in B \backslash W} J(u), \tag{3.50}$$

where

$$\Gamma_j = \{B \mid B = \varphi(B^{nk} \setminus Y), \varphi \in G_n, n \geq j, \text{open subset } Y = -Y \subset B^{nk}, \gamma(\overline{Y}) \leq n - j\}, \tag{3.51}$$

and

$$G_n = \{\varphi \in C(B^{nk}, E^k) \mid \varphi(-t) = -\varphi(t); \varphi(t) \in M_j, t_j = 0; \varphi|_{\partial B^{nk}} = \varphi^{(n)}\}. \tag{3.52}$$

Note that if $u \in F_G$ we have $u = (0, ..., 0, u_{m+1}, ..., u_k) \in \cap_{i=1}^k (P_i \cap Q_i) \subset W$. Thus $(K_{c_j} \setminus W) \cap F_G = \emptyset$. By Proposition 2.6, $c_j \geq c_* > 0$, for $j \geq m+1$, are critical values of J, $K_{c_j} \setminus W \neq \emptyset$, and $c_j \to \infty$ as $j \to \infty$. Any critical point u in $K_{c_j} \setminus W$ is a mixed state with the first m components sign-changing and the last $k - m$ components positive.

3.4 Further extensions and remarks

First we remark that we may use the invariant sets constructed to obtain positive solutions.

Theorem 3.12. *Under the conditions of Theorem 3.1, system (3.1) possesses a positive solution.*

We use Theorem 2.4 to the problem. Set $X = \cap_{i=1}^k \bar{P}_i$, $R_i = Q_i \cap X$ for $i = 1, ..., k$. Set $M = \cap_{i=1}^k R_i$, $\Sigma = \cap_{i=1}^k \partial R_i$, $W = \cup_{i=1}^k R_i$. Define for $t \in \mathbb{R}^k$,

$$\varphi_0(t) = R(t_1 v_1, t_2 v_2, ..., t_k v_k)$$

where $v_i \in R_i$ such that they have mutually disjoint supports, R is large such that $\varphi_0(\partial\Delta) \cap M = \emptyset$. Define

$$c = \inf_{\varphi \in \Gamma} \sup_{u \in \varphi(\Delta) \setminus W} J(u)$$

where

$$\Gamma = \{\varphi \in C(\Delta, X) \mid \varphi(\partial_i \Delta) \subset P_i, i = 1, ..., k, \varphi|_{\partial_0 \Delta} = \varphi_0\}.$$

By Theorem 2.4, there is a critical point $u \in K_c \setminus W$ which has every component positive. Theorem 3.12 is proved.

Next we remark that we do not need the full even symmetry of the functional to obtain infinitely many mixed states of nodal solutions, only on the first m components for which we expect to have sign-changing components. To obtain the existence of at least one sign-changing solution we do not need to assume the evenness of the functional at all. Instead stating more general results we are content by giving an example here. We replace u_i^3 by $f_i(u_i)$ in the systems

$$\begin{cases} -\Delta u_j + \lambda_j u_j = f_j(u_j) + \sum_{i \neq j} \beta_{ij} u_i^2 u_j, x \in \mathbb{R}^N \\ u_j(x) \to 0, \text{as } |x| \to \infty, j = 1, ..., k. \end{cases} \tag{3.53}$$

Assume that $f_j \in C^1(\mathbb{R})$ satisfies $f_j'(0) = 0$, that

there exists $C_j > 0, |f_j(t)| \leq C_j(|t| + |t|^{\alpha_j - 1})$ for $t \in \mathbb{R}$, $2 < \alpha_j < \frac{2N}{N-2}$, and

there exists $\mu_j > 2$ such that $0 < \mu_j F_j(t) \leq t f_j(t)$ for $t \neq 0$.

Theorem 3.13. *Assume $N = 2, 3, k \geq 2, \lambda_j > 0$, $\beta_{ij} = \beta_{ji} \leq 0$ for $1 \leq i < j \leq k$. Then under the above conditions on f_j $j = 1, ..., k$ the system (3.53) possesses a solution with each component sign-changing.*

To prove this result we use Theorem 2.4 again. Set $X = E^k$. We may prove by similar arguments that P_i is an invariant set for each $i = 1, ..., k$. Similarly this is also for $Q_i = -P_i$. We choose P_i and Q_i as $2k$ invariant sets in the setting of Theorem 2.4. Set $M = \cap_{i=1}^k (P_i \cap Q_i)$, $\Sigma = \cap_{i=1}^k (\partial P_i \cap \partial Q_i)$, $W = \cup_{i=1}^k (P_i \cup Q_i)$. Define for $t \in \mathbb{R}^{2k}$,

$$\varphi_0(t) = R(t_1 v_1, t_2 v_2, ..., t_{2k} v_{2k})$$

where $v_i \in P_i$ for $i = 1, ..., k$ and $v_{k+i} \in Q_i$ for $i = 1, ..., k$ such that they have mutually disjoint supports, R is large such that $\varphi_0(\partial\Delta) \cap M = \emptyset$. Define

$$c = \inf_{\varphi \in \Gamma} \sup_{u \in \varphi(\Delta) \setminus W} J(u)$$

where

$$\Gamma = \{\varphi \in C(\Delta, X) \mid \varphi(\partial_i \Delta) \subset P_i, \varphi(\partial_{k+i}\Delta) \subset Q_i, i = 1, ..., k, \varphi|_{\partial_0 \Delta} = \varphi_0\}.$$

By Theorem 2.4, there is a critical point $u \in K_c \setminus W$ which has every component sign-changing. Theorem 3.12 is proved.

Finally from the proof we see our method applies to situations of variable coeeficients functions. Assume $\lambda_j = \lambda_j(|x|)$ and $\beta_{ij} = \beta_{ij}(|x|)$ are continuous radial functions such that $\lambda_j \geq \lambda_0 > 0$ for some constant $\lambda_0 > 0$, $j = 1, ..., k$, $\beta_{ij} \in L^\infty(\mathbb{R}^N)$ and $\beta_{jj} \geq \beta_0 > 0$ for some $\beta_0 > 0$, and $\beta_{ij} \leq 0$ for $i \neq j$.

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