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COMMUTANTS OF COMPOSITION OPERATORS ON THE HARDY SPACE OF THE DISK

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For my Dad.

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ABSTRACT

James Michael Carter Ph.D., Purdue University, May 2013. Commutants of Composition Operators on the Hardy Space of the Disk. Major Professor: Carl C. Cowen.

The main part of this thesis, Chapter 4, contains results on the commutant of a semigroup of operators defined on the Hardy Space of the disk where the operators have hyperbolic non-automorphic symbols. In particular, we show in Chapter 5 that the commutant of the semigroup of operators is in one-to-one correspondence with a Banach albegra of bounded analytic functions on an open half-plane. This algebra of functions is a subalgebra of the standard Newton space.

Chapter 4 extends previous work done on maps with interior fixed point to the case of the symbol of the composition operator having a boundary fixed point.

1. INTRODUCTION

In this section, we define key terms and provide some background for the motivation of the main result of this work, Theorem 3.0.7, beginning with the case of a compact operator with distinct eigenvalues. Following this is the case of some non-compact operators.

1.1 Background

The basic definition necessary is that of a composition operator

Definition 1.1.1 *For φ a self map of the unit disk, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, the composition operator C_φ is defined for each point f in H, a Hilbert space of analytic functions, by*

$$(C_\varphi f)(z) = f(\varphi(z))$$

for each $z \in \mathbb{D}$ and $f \in H$.

The Hilbert space which the functions belong to will be one of the class of weighted Hilbert spaces $H^2(\beta) = H^2(\mathbb{D}, \beta)$ where $\mathbb{D}$ is the unit disk and $\beta = \beta(n)$ is a sequence of weights in the following sense. Note that the inner product on the weighted Hilbert spaces $H^2(\beta)$ is given by

$$\langle f, g \rangle = \sum_{n=0}^{\infty} a_n \overline{b_n} \beta(n)^2$$

for functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ in $H^2(\beta)$. The Hilbert space of principal interest for this work is the Hardy-Hilbert space of the disk, denoted by H^2 or $H^2(\mathbb{D})$ when necessary with $\beta(n) \equiv 1$. Every function in H^2 satisfies the inequality

$$\|f\|^2 = \sup_{0<r<1} \int_{\partial\mathbb{D}} |f(re^{i\theta})|^2 \, d\theta < \infty$$

where $d\theta$ is normalized arc-length measure on the boundary of the disk. It is not difficult to show that an equivalent definition of $H^2(\beta)$ is given by

$$H^2 = \{f(z) = \sum_{n=0}^{\infty} a_n z^n : \sum_{n=0}^{\infty} |a_n|^2 \beta(n)^2 < \infty\}$$

Therefore, an equivalent formulation of the norm is given by $\|f\|_{H^2} = \sum |a_n|^2$ for $f(z) = \sum a_n z^n$.

One class of functions in the Hardy Hilbert space that is of particular interest is the set of reproducing kernel functions, denoted by K_α for any $\alpha \in \mathbb{D}$, which are defined by

$$K_\alpha(z) = \frac{1}{1 - \overline{\alpha} z} \text{ for } z \in \mathbb{D}$$

It is easy to show that the kernel functions satisfy $f(\alpha) = \langle f, K_\alpha \rangle$ and

$$\|K_\alpha\|^2 = \frac{1}{1 - |\alpha|^2}$$

Another key definition is

Definition 1.1.2 *For any set of operators $\mathcal{A}$, the set of operators B which satisfy $BA = AB$ for every $A \in \mathcal{A}$ is called the commutant of $\mathcal{A}$, and denoted by $\mathcal{A}'$.*

It is worth noting that the class of self-maps $\varphi : \mathbb{D} \to \mathbb{D}$ under investigation all induce bounded composition operators [1] and we will only be interested in bounded operators that satisfy any conditions, including membership in the commutant. Also, we will use the convention that the normalized monomials will constitute the standard basis for H^2, that is $\{\chi_k(z) = z^k : k = 0, 1, 2, \ldots\}$ has dense span in H^2.

Lemma 1.1.3 *For any bounded operator A and any complex number ω, the commutant of $A + \omega I$ is equal to the commutant of A; that is $\{A + \omega I\}' = \{A\}'$.*

Proof Suppose that $AB = BA$, then $B(A + \omega I) = BA + \omega B = AB + \omega B = (A + \omega I)B$ and B commutes with $A + \omega I$. $\blacksquare$

It will be useful to identify certain points of the disk determined by each map for future reference. The first of these are the fixed points of the map.

Definition 1.1.4 *A point b of the closed disk is a fixed point for φ if $\lim_{r\to 1^-}\varphi(rb) = b$, provided this limit exists.*

Since the functions in H^2 are defined on the open disk, we extend the definition to the boundary by using radial limits as in the above definition: for $z \in \partial\mathbb{D}, f(z) = \lim_{r\to 1^-} f(re^{i\theta})$. Temporarily denote the limit function by $f^*(e^{i\theta}) = \lim_{r\to 1^-} f(re^{i\theta})$. It should be observed that this limit exists almost everywhere on the unit circle, $f^* \in L^2(\partial\mathbb{D})$ and the Fourier series for f^* is $\sum a_n e^{in\theta}$ which gives that $\|f^*\|_{L^2(\partial\mathbb{D})} = \|f\|_{H^2(\mathbb{D})}$. These facts can be found in [2, Theorem 2.2]

The first results give sufficient conditions for compactness.

Lemma 1.1.5 *If $\|\varphi\|_\infty < 1$ then C_φ is compact.*

Proof This follows from [2, Proposition 3.11]. ■

Theorem 1.1.6 *If $\varphi(\mathbb{D})$ is contained in a polygon inscribed in the unit circle, then C_φ is compact.*

Proof See [3]. ■

Theorem 1.1.7 *If $\lim_{r\to 1^-}\varphi(rs) = \eta$ where $|\eta| = |s| = 1$ and $\limsup_{r\to 1^-}|\varphi'(rs)| < \infty$, then C_φ is not compact.*

Proof This is Corollary 3.14 in [2]. ■

2. GENERAL RESULTS

In this section we present some results that apply to both the compact and non-compact operators.

Theorem 2.0.8 *The commutant of any set of operators forms an algebra.*

Proof Routine calculation and implementing the definition of commutant provide the proof. ■

Suppose that S is a set of operators. This result provides a "minimal" structure for the commutant of S in that the algebra generated by the operators in S is guaranteed to be a subset of the commutant of S, though this may be a proper subset. For example, the commutant of the identity operator is all operators in the space, but the algebra generated by the identity operator is only the identity operator itself.

Lemma 2.0.9 *If $AB = BA$, then $A^*B^* = B^*A^*$ as well.*

Proof This follows immediately from the fact that $(AB)^* = B^*A^*$. ■

Lemma 2.0.10 *[4] If S is a convex subset of $\mathcal{B}(\mathcal{H})$, then the weak operator topology closure of S equals the strong operator topology closure of S.*

In light of Lemma 2.0.10, the following can be stated unambiguously for either the strong operator topology or the weak operator topology.

Lemma 2.0.11 *If S is a set of operators, then the norm closure of S is contained in the strong operator closure of S, which is contained in the weak operator closure of S.*

Together, these imply that $\{A\}'$ is closed in any of these three topologies. Therefore, it follows from these considerations and Theorem 2.0.8 that the next result is true.

Proposition 2.0.12 *The commutant of any set of operators is a closed subalgebra of $\mathcal{B}(\mathcal{H})$ in the weak operator topology, the strong operator topology, and the norm topology.*

Proposition 2.0.13 *The closure, in the weak operator topology, of the algebra generated by C_φ is contained in the commutant of C_φ.*

Proof Let $B \in \overline{\mathcal{A}(C_\varphi)}^w$ and f, g be in H^2. Then $B = w\text{-lim } B_n$ where $B_n \in \mathcal{A}(C_\varphi)$. So

$$\begin{aligned}
\langle BC_\varphi f, g\rangle &= \lim_{n\to\infty} \langle B_n(C_\varphi f), g\rangle \\
&= \lim_{n\to\infty} \langle C_\varphi(B_n f), g\rangle \\
&= \lim_{n\to\infty} \left\langle B_n f, C_\varphi^* g\right\rangle \\
&= \left\langle Bf, C_\varphi^* g\right\rangle \\
&= \langle C_\varphi Bf, g\rangle
\end{aligned}$$

Thus

$$BC_\varphi = w\text{- lim } B_n C_\varphi = w\text{- lim}\, C_\varphi B_n = C_\varphi B$$

■

Proposition 2.0.14 *If $B \in \{A\}'$, then for every $v \in \ker(A - \lambda_v I)$, $Bv \in \ker(A - \lambda_v I)$.*

Proof Consider $A(Bv) = (BA)v = B(Av) = B(\lambda_v v) = (B\lambda_v)v = \lambda_v(Bv)$. ■

2.1 The pseudo-iteration semigroup

The pseudo-iteration semigroup is of interest for this work since the symbols of the operators under consideration in Chapter 4 form a semigroup and this provides a characterization of composition operators in the commutant. In order to define the pseudo-iteration semigroup of an analytic function, first we state two theorems which provide a way to express a specific relationship between certain types of functions. Before stating the prerequisite theorems, it will be useful to distinguish a particular fixed point for the symbols of the operators.

Theorem 2.1.1 (Denjoy-Wolff) *Let $f : \mathbb{D} \to \mathbb{D}$ be an analytic non-automorphism. Then there is a unique point a in the closure of $\mathbb{D}$ such that the iterates of f tend to a uniformly on compact subsets of $\mathbb{D}$. The point a in the previous sentence is a fixed point of f and it is the unique fixed point of f for which $|f'(a)| \leq 1$*

Definition 2.1.2 *The unique point a such that the iterates of f converge to a uniformly on compact subsets of $\mathbb{D}$ is called the Denjoy-Wolff point of f.*

In order to provide some categorization of maps based on the Denjoy-Wolff point, it is necessary to define the concept of a fundamental set.

Definition 2.1.3 *If f is a self map of $\mathbb{D}$, then V is a fundamental set for f on $\mathbb{D}$ if V is an open, connected, simply connected subset of $\mathbb{D}$ such that $f(V) \subset V$ and for every compact set $K \subset \mathbb{D}$, there is a positive integer n such that $f_n(K) \subset V$ where f_n is the n^{th} iterate of f.*

A fundamental set is large enough to capture the information about the iterates of f on all of $\mathbb{D}$.

An additional, though more theoretical than practical, tool is that of an interpolating sequence.

Definition 2.1.4 *An interpolating sequence is a sequence of points $\{z_j\}$ in the unit disk such that for any bounded sequence of complex numbers $\{c_j\}$, there is a bounded analytic function f on $\mathbb{D}$ with $f(z_j) = c_j$.*

Theorem 2.1.5 (The Model for Iteration) *Let φ be an analytic mapping of $\mathbb{D}$ into $\mathbb{D}$ with Denjoy-Wolff point a that is non-constant and not an automorphism. If $\varphi'(a) \neq 0$, then there is a fundamental set V for φ on $\mathbb{D}$, a domain Ω, an automorphsim Φ mapping Ω onto Ω and a mapping σ of $\mathbb{D}$ into Ω such that φ and σ are univalent on $V, \sigma(V)$ is a fundamental set for φ on Ω and*

$$\Phi \circ \sigma = \sigma \circ \Phi$$

Moreover, Φ is unique up to conjugation by automorphism of Ω onto Ω, and Φ and σ depend only on φ, not on the particular fundamental set V.

This theorem appears in [2, Theorem 2.53] and provides four categories of maps φ based on the location of the Denjoy-Wolff point and the value of the derivative there:

1. $|a| < 1$ and $\varphi'(a) \neq 0$ (plane/dilation)

2. $|a| = 1$ and $\varphi'(a) < 1$ (half-plane/dilation)

3. $|a| = 1, \varphi'(a) = 1$, and $\Omega = \mathbb{C}$ (plane/translation)

4. $|a| = 1, \varphi'(a) = 1$, and $\Omega = \{z : \mathrm{Im}\,(z) > 0\}$ (half-plane/translation)

The third and fourth cases above are quite similar and not easily distinguished, however, for an operator whose symbol has the property that its iteration sequences are also interpolating, the two cases can be distinguished. In the case that the iteration sequences are also interpolating, then the composition operator is in the half-plane translation case, while if the iteration sequences are not interpolating, the composition operator is in the plane translation case.

We now define the pseudo-iteration semigroup of an analytic function.

If $f'(a) \neq 0$, let Ω, V, σ, Φ be as in Theorem 2.1.5. Then an analytic function g that maps $\mathbb{D}$ into $\mathbb{D}$ is in the pseudo-iteration semigroup of f if there exists a linear fractional transformation ψ that commutes with Φ such that $\sigma(g(z)) = \psi(\sigma(z))$.

In 2002, Tami Worner [5] characterized a portion of the commutant in the plane dilation case of the model. Specifically, if φ is in the pseudo-iteration semigroup of $f(z) = \lambda z$ where $0 < |\lambda| < 1$, then the commutant of C_φ is the strong operator closure of the polynomials in C_φ. The maps φ in this case are all compact by Lemma 1.1.5 with fixed point at 0. Another result [5, Theorem 9] by Worner provides examples where the commutant of a composition operator does not coincide with the algebra generated by the composition operator and the Toeplitz operators that commute with the composition operator.

3. COMPACT OPERATORS

Consider the function $\varphi_s(z) = sz$ where $0 < |s| < 1$. This map has fixed point $a = 0$ and the derivative has modulus $|s|$ at the fixed point. In light of the model Theorem 2.1.5, the map φ_s is in the plane/dilation case, the associated composition operator C_{φ_s} has eigenvalues $\{s^n\}_{n\geq 0}$ [2, Theorem 7.30], and is compact. Also, all eigenvalues have multiplicity 1, therefore all eigenspaces have dimension 1. Therefore, C_{φ_s} has $\chi_n(x) = z^n$ as an eigenvector for $n = 0, 1, \ldots$. The commutant of C_{φ_s} is therefore the set of operators A that have all $\chi_n, n = 1, 2, \ldots$ as eigenvectors. Since this is a spanning set of eigenvectors, this completely determines such operators.

If ψ is any automorphism of $\mathbb{D}$ and $f \in H^2$, then $f \circ \psi \in H^2$ as well. This property is referred to as automorphism invariance. Using this, it is not difficult to see that if ψ is an automorphism of $\mathbb{D}$, then for $\varphi_1 = \psi^{-1} \circ \varphi \circ \psi$, the operators C_{φ_1} and C_φ are similar. Since similar operators have the same spectra and and their symbols have fixed point sets that are in one-to-one correspondence, we can assume, without loss of generality that if an analytic function from $\mathbb{D}$ to $\mathbb{D}$ has a single fixed point on the boundary of $\mathbb{D}$, then that point is $z = 1$ using $\psi = \psi_\theta(z) = e^{i\theta}$ where θ is the argument of the boundary fixed point. In addition, we can assume that if the map has an interior fixed point, that point is $z = 0$ using $\psi = \psi_a(z) = \dfrac{a - z}{1 - \bar{a}z}$ where a is the interior fixed point.

Lemma 3.0.6 *If A and B are two operators such that $AB = BA$ and $\ker(A - \lambda I) = \operatorname{span}\{v\}$, then $Bv \in \ker(A - \lambda I)$ and v is an eigenvector for B.*

Proof It follows from Proposition 2.0.14 that $Bv \in \ker(A - \lambda I)$. Since $\dim\ker(A - \lambda I) = 1$, it follows that $Bv = bv$, hence v is an eigenvector of B. ∎

Theorem 3.0.7 *If $\varphi(a) = a$ for some $a \in \mathbb{D}$ and $0 < |\varphi'(a)| < 1$ then $S \in \{C_\varphi\}'$ if and only if for all $v \in \ker(C_\varphi^* - \lambda I), v \in \ker(S^* - \mu_v I)$.*

Proof According to the remarks preceding Lemma 3.0.6, it can be assumed that $a = 0$ by a suitable conjugation with an automorphism.

Observe that for all $j = 0, 1, \ldots$ the subspace $M_j = \text{span}\{1, z, \ldots z^j\}$ is invariant for C_φ^*. It is not difficult to see that C_φ^* restricted to M_j has a representation as an upper triangular $j \times j$ matrix with respect to the basis $\{1, z, \ldots z^j\}$ with diagonal entries are $[C_\varphi^*]_{ii} = \overline{\varphi'(0)}^i$. Therefore $\overline{\varphi'(0)}^i$ for $i = 0, 1, \ldots$ is an eigenvalue of C_φ^*. Now $C_\varphi^* \chi_j \in M_j$ for all $j = 0, 1, \ldots$ since M_j is invariant, hence C_φ^* has a set of eigenvectors with dense span. Moreover, $C_\varphi^* \chi_j$ is a polynomial of degree j.

Now let $v \in \ker(C_\varphi^* - \lambda I)$ and $S \in \{C_\varphi\}'$ so that $S^* C_\varphi^* = C_\varphi^* S^*$, then $S^* C_\varphi^* v = S^* \lambda v = \lambda S^* v = C_\varphi^*(S^* v)$ hence $S^* v \in \ker(C_\varphi^* - \lambda I)$. Since λ is an eigenvalue of C_φ^*, $\lambda = \overline{\varphi'(0)}^{j_v}$ for some minimal j_v and v must be an eigenvector of C_φ^* restricted to M_{j_v}. Thus $S^* v$ is a polynomial of degree j_v and since $0 < |\varphi'(0)| < 1$, $\overline{\varphi'(0)}^i \neq \overline{\varphi'(0)}^j$ for $i \neq j$ each eigenspace is one dimensional, so by Lemma 3.0.6, $S^* v = \mu_v v$.

On the other hand, if $C_\varphi^* v = \lambda v$ implies that $S^* v = \hat{\lambda} v$, then

$$S^* C_\varphi^* v = \lambda S^* v = \lambda \hat{\lambda} v = \hat{\lambda} C_\varphi^* v = C_\varphi^* S^* v \tag{3.1}$$

so $S^* C_\varphi^* = C_\varphi^* S^*$ on the set of eigenvectors of C_φ^*. Since $\{v : v \in \ker(C_\varphi^* - \overline{\varphi'(0)}^k I)\}$ contains a polynomial of degree k for every non-negative integer value of k, then $S^* C_\varphi^* = C_\varphi^* S^*$ on a set of vectors with dense linear span, hence $S^* C_\varphi^* = C_\varphi^* S^*$ on H^2. ∎

4. SOME NON-COMPACT OPERATORS

Previous work done by Worner [5] and Cload [6] provided much insight into the commutant of a composition operator whose symbol has the Denjoy-Wolff point interior to the disk. In this section, a semigroup of operators is studied which all have Denjoy-Wolff point at $z = 1$.

4.1 A semi-group of operators

Consider the family of functions $\varphi_t(z) = e^{-t}z + (1 - e^{-t})$. Each of these is a non-automorphic map of the unit disk to a disk internally tangent to the unit disk at 1, the only fixed point in the closed unit disk is the point 1 and the derivative there has modulus strictly less than 1. We claim that the associated composition operators form a semigroup. For notational convenience, the associated composition operator, C_{φ_t} will be denoted by C_t.

Proposition 4.1.1 *The set of composition operators with symbols* $\varphi_t(z) = e^{-t}z + (1 - e^{-t})$ *forms a strongly continuous semigroup of operators.*

Proof Note that

$$\begin{aligned} \varphi_t(\varphi_s(z)) &= e^{-t}(e^{-s}z + (1 - e^{-s})) + (1 - e^{-t}) \\ &= e^{-(s+t)}z + e^{-t} - e^{-(s+t)} + 1 - e^{-t} \\ &= e^{-(s+t)}z + (1 - e^{-(s+t)}) \\ &= \varphi_{s+t}(z) \end{aligned}$$

This computation shows that $C_tC_s = C_{t+s}$ as well. For a proof of the fact that this semigroup is strongly continuous, see [7]. ■

Recent work by Cowen and Gallardo-Guitérrez provides an interesting idea for constructing invertible operators from non-invertible operators while preserving the commutation relation. Since none of the operators $\{C_t : t > 0\}$ are invertible, it is possible, a priori, to consider a different set of operators that are in some sense easier to work with.

Theorem 4.1.2 *Let* $\hat{\alpha}_t = \|C_t\|, \alpha_t = \widehat{\alpha_t} + 1, B_t = \alpha_t I + C_t$. *Then* $\{C_t : t > 0\}' = \{B_t : t > 0\}', B_t$ *is invertible for all* $t > 0$, *yet* $\{B_t\}$ *is not a semigroup of operators. Moreover, there is no choice of non-zero values* $\{\alpha_t\}$ *such that* $\{B_t : t > 0\}$ *does form a semigroup of operators.*

Proof Note that 0 is not in the spectrum of B_t for any t by construction, hence every operator in $\{B_t : t > 0\}$ is invertible. Also the set of operators $\{C_t : t > 0\}$ is a semigroup. Suppose that $B_t B_s = B_{t+s}$. Then

$$\alpha_t \alpha_s I + \alpha_t C_s + \alpha_s C_t + C_t C_s = \alpha_{t+s} I + C_{t+s}$$

but since $\{C_t : t > 0\}$ is a semigroup,

$$\alpha_t C_s + \alpha_s C_t = (\alpha_{t+s} - \alpha_t \alpha_s) I$$

There are two cases, first consider $\alpha_{t+s} = \alpha_t \alpha_s$. Then

$$\alpha_t C_s + \alpha_s C_t = 0 \tag{4.1}$$

Let $\alpha = \alpha_t, \beta = \alpha_s$, so that $\alpha C_s + \beta C_t = 0$. Further assume that $s \neq t$, recall that $K_0 = 1$, and consider the adjoint equation

$$\begin{aligned} (\bar{\alpha} C_s^*) K_0 + (\bar{\beta} C_t^*) K_0 &= 0 \\ \bar{\alpha}(1 - e^{-s}) + \bar{\beta}(1 - e^{-t}) &= 0 \end{aligned}$$

and the equation $(\alpha C_s) z + (\beta C_t) z = 0$. This gives that $\alpha(e^{-s} z + 1 - e^{-s}) + \beta(e^{-t} z + 1 - e^{-t}) = 0$. Combining this with equation (4.1), $\alpha e^{-s} + \beta e^{-t} = \alpha + \beta = 0$. This implies that $\alpha(e^{-s} + e^{-t}) = 0$ and therefore $\alpha = 0$ or $s = t$. It follows that $\alpha = 0$ and

$\beta = 0$ as well. Now if $s = t, 2\alpha_t C_t = (\alpha_{2t} - \alpha_t^2)I$, however the identity is not a scalar multiple of C_t so this is again a contradiction. In the second case, $\alpha_{t+s} \neq \alpha_t \alpha_s$, then let $\alpha_1 = \frac{\alpha_t}{\alpha_{t+s} - \alpha_t \alpha_s}; \alpha_2 = \alpha_s \alpha_{t+s} - \alpha_t \alpha_s$ and then

$$\alpha_1 C_s + \alpha_2 C_t = I$$

Again, applying the adjoint of both sides to the vector K_0, it is clear that

$$\bar{\alpha_1}(1 - e^{-s}) + \bar{\alpha_2}(1 - e^{-t}) = 1$$

Now using the vector $f(z) = z$ on both sides, $\alpha(e^{-s}z + 1 - e^{-s}) + \alpha_2(e^{-t}z + 1 - e^{-t}) = z$ which gives the system of equations

$$\alpha_1 e^{-s} + \alpha_2 e^{-t} = 1 \tag{4.2}$$

$$\alpha_1(1 - e^{-s}) + \alpha_2(1 - e^{-t}) = 0 \tag{4.3}$$

by equating coefficients on both sides. The equations (4.2) and (4.3) clearly form a contradiction, thus the case when $\alpha_{t+s} \neq \alpha_t \alpha_s$ is also impossible. Therefore no choice of coefficients will make $\{B_t : t > 0\}$ into a semigroup. ■

This result shows that considering invertible operators for the generation of the commutant is possible, however, the structure of the semigroup becomes lost in the transition.

The logarithm will be defined on $\mathbb{C} \setminus (-\infty, 0]$ such that $\ln(1) = 0$.

The inner products of the vectors from the set $\{v_n(z) = (1 - z)^n : n = 0, 1, 2, \dots\}$ is related to the Gamma function by the following result.

Proposition 4.1.3 *For* $w_1, w_2 \in \{0, 1, 2, \dots\}$,

$$\langle v_{w_1}, v_{w_2} \rangle = \frac{\Gamma(w_1 + w_2 + 1)}{\Gamma(w_1 + 1)\Gamma(w_2 + 1)}$$

Proof See [8, Theorem 2.6, Theorem 4.1] ■

Corollary 4.1.4 *The functions* $(1 - z)^k$ *are in* H^2 *if and only if* $\mathrm{Re}\,(k) > -1/2$.

Proof This is a consequence of Lemma 7.3 in [2]

Corollary 4.1.5 *The set of vectors* $\{(1-z)^k : \mathrm{Re}\,(k) > -1/2\}$ *has dense span in* H^2.

Proof Let $v_k(z) = (1-z)^k$ and consider the case where k is a non-negative integer. Then using the binomial theorem,

$$\begin{aligned} v_k(z) &= (1-z)^k \\ &= \sum_{j=0}^{k} \binom{k}{j} (-1)^j z^j \end{aligned}$$

And solving for z^k gives

$$z^k = \sum_{j=0}^{k} \binom{k}{j} (-1)^j v_j(z)$$

Therefore $\{\chi_n(z) : n = 0, 1, \ldots\} \subset \{v_k(z) : k = 0, 1, \ldots\} \subset \{v_m(z) : \mathrm{Re}\,(m) > -1/2\}$ and since $\{\chi_n(z) : n = 0, 1, 2, \ldots\}$ has dense linear span, the result follows. ∎

According to [2, Lemma 7.24], the operators C_t have eigenvalues of infinite multiplicity and therefore the eigenvalue equation has a non-trivial solution. Now let $f(z) \in H^2$ be fixed and consider the eigenvalue equation

$$C_t f = \lambda(t) f \tag{4.4}$$

By Proposition 4.1.1, $C_t^2 f = C_{2t} f$ and then the spectral mapping theorem implies that $C_t^2 f = \lambda(t)^2 f$. Hence for any non-negative, rational value of the parameter q, $C_q f = \lambda(1)^q f$. The mapping $t \mapsto C_t$ is strongly continuous [7, Theorem 5.1, 5.2] so that $\lambda(t)$ is a continuous function of t. Therefore $\lambda(t) = \lambda(1)^t$ for $t > 0$. By computing

$$\left.\frac{d}{dt}\right|_{t=0} [(C_t f)(z)] = \left.\frac{d}{dt}\right|_{t=0} \lambda(t) f(z) \tag{4.5}$$

the eigenfunctions f can be determined. Note that the left hand side of equation (4.5) is $\left.\frac{d}{dt}\right|_{t=0} [(C_t f)(z)] = f'(\varphi_0(z))(z-1)(-1) = f'(z)(1-z)$. Then since the right hand side is equal to $\lambda'(0)f(z)$, the functions f are found to satisfy $\frac{f'}{f} = \lambda'(0)\frac{1}{(1-z)}$ or $f(z) = b(1-z)^{-\ln(\lambda(1))}$ for any complex number b.

Lemma 4.1.6 *For every k with real part greater than -1/2,* $\dim(\bigcap_{t>0} \ker(C_t - e^{-tk}I))) = 1.$

Proof Let $f(z) \in \bigcap_{t>0} \ker(C_t - e^{-tk}I)$. Then $f(z)$ must satisfy (4.5). Since the equation 4.5 has a one dimensional solution space, the conclusion follows. ■

For an analytic function $h(z) \in L^\infty$, define the analytic Toeplitz operator T_h on H^2 by $T_h g = Phg$ for all $g \in H^2$ where P is the orthogonal projection of L^2 onto H^2. Then it is easy to show that $T_f^* K_w = \overline{f(w)} K_w$.

Lemma 4.1.7 *If T_h is an analytic Toeplitz operator such that $C_t T_h = T_h C_t$ for every $t > 0$, then T_h is a constant multiple of the identity.*

Proof We prove that T_h^* is a constant multiple of the identity from which the result follows. According to [2, Theorem 9.2], the operator C_t^* can be factored as $C_t^* = C_{\sigma_t}^* T_g^*$ for a linear fractional self-map of the disk $\sigma_t(z) = \frac{e^{-t}z}{1-(1-e^{-t})z}$ and bounded linear fractional map $g(z) = \frac{1}{1-(1-e^{-t})z}$. Let $\alpha \in \mathbb{D}$, then

$$\begin{aligned}
T_h^* C_\varphi K_\alpha &= T_h^* C_{\sigma_t}^* T_g^* K_\alpha \\
&= T_h^* C_{\sigma_t}^* \overline{g(\alpha)} K_\alpha \\
&= \overline{g(\alpha)} T_h^* K_{\sigma_t(\alpha)} \\
&= \overline{g(\alpha)}\, \overline{h(\sigma_t(\alpha))} K_{\sigma_t(\alpha)} \\
C_\varphi T_h^* K_\alpha &= C_{\sigma_t}^* T_g^* T_h^* K_\alpha \\
&= \overline{g(\alpha)}\, \overline{h(\alpha)} K_{\sigma_t(\alpha)}
\end{aligned}$$

Since $g(z)$ is non-zero in $\mathbb{C}$ and the kernel functions are never zero, this implies that

$$h(\sigma_t(\alpha)) = h(\alpha)$$

for all $t > 0$ and all $\alpha \in \mathbb{D}$. Since the image $\sigma_t(\mathbb{D})$ is a disk internally tangent to the unit disk at 1, this implies that $h(z)$ is constant in all of $\mathbb{D}$ and the result follows. ■

It is worth observing that the commutant of $\{C_t : t > 0\}$ is not the same as the commutant of a single operator.

Lemma 4.1.8 *There is a bounded operator A such that $AC_1 = C_1 A$ but $AC_{\sqrt{2}} \neq C_{\sqrt{2}} A$.*

Proof The proof will follow that found in [5, Proposition 1]. Using Theorem 2.1.5, there are functions σ_i such that $\sigma_1 \circ \varphi_1 = e^{-1}\sigma_1$ and $\sigma_2 \circ \varphi_{\sqrt{2}} = e^{-\sqrt{2}}\sigma_2$. Define $F_1(w) = \exp(-2\pi i \log w)$ and $F_2(w) = \exp(-\sqrt{2} \cdot 2\pi i \log w)$ and $f_i = F_i \circ \sigma_i$. This gives analytic Toeplitz operators T_{f_i} such that $T_{f_1} C_{\varphi_1} = C_{\varphi_1} T_{f_1}$ and $T_{f_2} C_{\varphi_{\sqrt{2}}} = C_{\varphi_{\sqrt{2}}} T_{f_2}$. Now claim that $T_{f_1} C_{\varphi_{\sqrt{2}}} \neq C_{\varphi_{\sqrt{2}}} T_{f_1}$. Assume the contrary, then it follows that $f_1 \circ \varphi_{\sqrt{2}} = \varphi_{\sqrt{2}}$ and evaluate both sides at $z = 0$. The left hand side yields $e^{\sqrt{2} \cdot 2\pi i}$ and the right becomes $1 - e^{-\sqrt{2}}$, thus providing a contradiction and the conclusion follows. ■

Theorem 4.1.9 *A bounded operator, B, is in the commutant of $\{C_t : t > 0\}$ if and only if for every k with real part greater than -1/2, there exists a complex number μ_k so that $B(1-z)^k = \mu_k (1-z)^k$.*

Proof Let B be a bounded operator on H^2 and f_k be the vector in H^2 such that $f_k(z) = (1-z)^k$. For the sufficient condition, assume that for every k with real part at least -1/2 there exists a complex number μ_k such that $Bf_k = \mu_k f_k$. Then for $z \in \mathbb{D}$,

$$\begin{aligned}(C_t B f_k)(z) &= C_t (B f_k)(z) \\ &= C_t \mu_k f_k(z) \\ &= \mu_k C_t f_k(z) \\ &= \mu_k e^{-tk} f_k(z)\end{aligned}$$

where the last equality follows since f_k is an eigenvector for C_t for all $t > 0$ with eigenvalue e^{-tk}. Similarly, $(BC_t f_k)(z) = e^{-tk} \mu_k f_k(z)$ hence $BC_t = C_t B$ on $\operatorname*{span}_{\operatorname{Re}(k) > -1/2} \{f_k\}$. But by Proposition 4.1.5, this implies that $B \in \{C_t : t > 0\}'$.

For the necessary condition, assume that $B \in \{C_t : t > 0\}'$ and k is such that $\operatorname{Re}(k) > -1/2$. Then

$$C_t B f_k = B C_t f_k = B e^{-tk} f_k = e^{-tk} B f_k$$

so that Bf_k is an eigenvector for C_t with eigenvalue e^{-tk} for all $t > 0$, hence by Lemma 4.1.6, $(Bf_k)(z) = \mu_k f_k(z)$ for some complex number μ_k. ■

Since the value of μ_k varies analytically with k as shown in the proof of Theorem 4.1.10, we will associate the commuting operator with the function $m(k) = \mu_k$.

■

Theorem 4.1.10 *Let A be a bounded operator on H^2. If A commutes with the semigroup of operators $\{C_t : t > 0\}$, then there exists a bounded analytic function, $m_A(k)$, on the half-plane $\{k : \mathrm{Re}\,(k) > -1/2\}$ such that $\|m_A\| \leq \|A\|$ and $A(1-z)^k = m_A(k)(1-z)^k$ for each $z \in \mathbb{D}$ and k in the half-plane. If there exists a bounded analytic function, $m_A(k)$, on the half-plane $\{k : \mathrm{Re}\,(k) > -1/2\}$ such that $A(1-z)^k = m_A(k)(1-z)^k$ for every k in the half-plane, then $A \in \{C_t : t > 0\}'$.*

Proof This is the content of Theorem 4.1.9 with the additional claim that m_A is analytic and $\|m_A\| \leq \|A\|$. To prove that $m_A(k)$ is analytic, we prove that it is weakly analytic which will imply that it is norm analytic. Therefore, let $u \in H^2$ and compute $\lim\limits_{\epsilon\to 0}\frac{1}{\epsilon}(\langle m_A(k+\epsilon)(1-z)^{k+\epsilon}, u\rangle - \langle m_A(k)(1-z)^k, u\rangle) = \lim\limits_{\epsilon\to 0}\frac{1}{\epsilon}(\langle A(1-z)^{k+\epsilon}, u\rangle - \langle A(1-z)^k, u\rangle)$ as follows.

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\frac{1}{\epsilon}(\langle A(1-z)^{k+\epsilon}, u\rangle - \langle A(1-z)^k, u\rangle) \\
&\qquad = \left\langle \lim_{\epsilon\to 0}\frac{1}{\epsilon}((1-z)^{k+\epsilon} - (1-z)^k), A^*u \right\rangle \\
&\qquad = \left\langle \frac{d}{dk}((1-z)^k), A^*u \right\rangle \\
&\qquad = \langle \ln(1-z)(1-z)^k, A^*u \rangle \\
&\qquad = \ln(1-z)\,\langle (1-z)^k, A^*u \rangle
\end{aligned}
$$

By choosing an appropriate branch for $\ln(1-z)$, this will insure that the log function is analytic on its domain. Therefore the limit is an analytic function of z, hence this shows that $m_A(k)$ is weakly analytic, thus norm analytic [9].

Now note that for k with $\mathrm{Re}\,(k) > -1/2$,

$$\|A\| \geq \frac{\|A(1-z)^k\|}{\|(1-z)^k\|} = \frac{\|m_A(k)(1-z)^k\|}{\|(1-z)^k\|} = \frac{|m_A(k)|\|(1-z)^k\|}{\|(1-z)^k\|} = |m_A(k)|$$

■

Theorem 4.1.11 *If $A \in \{C_t : t > 0\}'$ and A is polynomially compact, then A is a multiple of the identity operator.*

Proof Suppose that A is polynomially compact and $A \in \{C_t : t > 0\}'$. Then there is a non-zero polynomial p such that $p(A)$ is compact and $\sigma_p(p(A))$ is a sequence converging to zero or a finite set including 0. By the spectral mapping theorem then, $\sigma_p(A)$ consists of finitely many sequences with finitely many limit points. Since $m_A(k)$ is analytic on an open half-plane, it maps connected sets to connected sets, and the range of h_A is a subset of the point spectrum of A, it follows that $m_A(k)$ is constant. Thus by Theorem 4.1.9 A is a multiple of the identity. ■

Corollary 4.1.12 *There are no bounded projections in $\{C_t : t > 0\}'$. Furthermore, there is no common reducing subspace for the semigroup of operators $\{C_t : t > 0\}$.*

Proof Projections are polynomially compact using the polynomial $p(x) = x - x^2$. ■

Proposition 4.1.13 *If $C_\psi C_t = C_t C_\psi$, then $\psi(z) = e^{-s}z + (1 - e^{-s})$ for some $s > 0$ and conversely.*

Proof The map $\varphi_t(z)$ is a univalent map in the half-plane dilation model for iteration, therefore the intertwining σ is also univalent by Theorem 2.1.5. In addition, the unit disk is a fundamental set for $\varphi_t(z)$. According to [10], a function $g(z)$ is in the pseudo-iteration semi-group of $\varphi_t(z)$ if it can be represented as $g(z) = \sigma^{-1}(\Psi(\sigma(z)))$ where $\Psi(z)$ is a linear fractional map commuting (in the sense $C_f g = C_g f$) with multiplication by e^{-t}. Therefore $\Psi(z) = \alpha z$ for some $\alpha \in \mathbb{C}$. Note that $0 < |\alpha| < 1$ or else $g(z)$ is constant ($\alpha = 0$) or undefined as a self map of the disk ($|\alpha| > 1$). Since the map $\sigma(z) = 1 - z$, it follows that $g(z) = \alpha z + (1 - \alpha)$. Since $g : \mathbb{D} \to \mathbb{D}, \alpha \in \mathbb{R}$ and the proof is complete. ■

5. THE NEWTON SPACE

In [8], the authors show that certain composition operators on the Newton space are unitarily equivalent to multiplication operators on the Hardy space. In order to give a basic description of the Newton space, we first define the Newton polynomials. The following notation, which is the notation of [8], will be needed. For $n \in \mathbb{N}$ and $\alpha \in \mathbb{C}$, let $(\alpha)_n = (\alpha)(\alpha+1)\cdots(\alpha+n-1)$, also called the rising factorial. Note that this gives the standard factorial as $n! = (1)_n$. Then we define the n-th Newton polynomial as

$$N_n(w) = \frac{(-w)_n}{(1)_n}$$

Let $\mathbb{P}$ denote the open right half-plane $\{w : \operatorname{Re} w > -1/2\}$. Then the Newton space can be represented as the set

$$N^2(\mathbb{P}) = \{F(w) = \sum_{n=0}^{\infty} a_n N_n(w) : \sum_{n=0}^{\infty} |a_n|^2 < \infty\}$$

This appears as [8, Theorem 2.1] as well as the fact that the Newton polynomials form an orthonormal basis for N^2, which is a non-trivial result that follows from [11, Theorem 1.2]. Thus N^2 is a Hilbert space of analytic functions. In order to avoid possible confusion, the variables z, α will be used to represent points of the unit disk for functions in the Hardy space H^2 while the variables w, λ will denote points in $\mathbb{P}$ for functions in N^2. It turns out that the Newton space is also a reproducing kernel Hilbert space with kernel functions given by

$$k_\lambda(w) = \frac{\Gamma(w + \bar{\lambda} + 1)}{\Gamma(w+1)\Gamma(\bar{\lambda}+1)}$$

For notational purposes, we will denote the functions $(1-z)^k$ for $z \in \mathbb{D}$ and $k \in \mathbb{P}$ by $v_{\bar{k}}(z)$. The following result shows that a familiar operator has a simple representation in this space.

Proposition 5.0.14 *The operator* $\Delta : N^2 \to N^2$ *defined by* $(\Delta f)(w) = f(w) - f(w+1)$ *is the backwards shift on the orthonormal basis* $\{N_n\}_{n\geq 0}$ *of* N^2.

Their main result is [8, Theorem 4.1]:

Theorem 5.0.15 *Let* S_ζ *be the operator defined by* $(S_\zeta f)(w) = f(w+\zeta)$ *for* $\operatorname{Re}\zeta \geq 0$. *Then the map* $U : N^2 \to H^2$ *defined by* $U(N_n) = \chi_n$ *where* $\chi_n(z) = z^n$ *has the following properties:*

1. *For* $\alpha \in \mathbb{D}, f_\alpha(w) = (1-\alpha)^w \in N^2$ *and* $(Uf_\alpha)(z) = (1-\overline{\alpha}z)^{-1}$ *for all* $z \in \mathbb{D}$
2. *For* $w \in \mathbb{P}, v_w(z) = (1-z)^w \in H^2$ *and* $UK_w = v_{\overline{w}}$
3. *If* M_z *is the operator of multiplication by* z *on* H^2, *then* $M_zU = U\Delta^*$
4. *For* $\zeta \in \mathbb{C}$ *with* $\operatorname{Re}(\zeta) \geq 0, M_{(1-z)^{\bar{\zeta}}}U = UC^*_{S_\zeta}$

Since every separable Hilbert space is isomorphic to every other separable Hilbert space, the existence of the map U is not surprising, however this theorem provides additional structure to the isomorphism. In particular, the first two claims show how kernel functions are mapped between the spaces and the latter two claims show that multiplication operators are unitarily equivalent to the adjoints of shift operators.

An alternative description of the Newton space is given by considering the closure of the polynomials in $L^2(\mathbb{C}, \mu)$ where μ is a probability measure on $\mathbb{C}$. In the case where μ is a probability measure with finite moments,

$$\int_{\mathbb{C}} |z|^n \, d\mu(z) < \infty$$

This construction yields many spaces depending on the measure. In particular, both H^2 and N^2 can be constructed in this manner. By choosing $d\mu(re^{i\theta}) = \delta_1(r)d\theta$ to be the normalized arc-length on the circle, the space constructed is $H^2(\mathbb{D})$. On the other hand, if $d\mu(x+iy) = \frac{1}{2\pi}\frac{|\Gamma(x+iy)|^2}{\Gamma(2x+2)}\, dy d\gamma(x)$ where $\gamma(x)$ is the discrete measure on $\mathbb{R}$ with unit masses at $\{-1/2 + n/2 : n = 1, 2, \dots\}$, then the Newton space $N^2(\mathbb{P})$ is constructed. This construction and the proof of Lemma 5.0.16 rely on the properties of μ.

Although the previous result is significant, the result of principal interest for this work is found in the proof of [8, Theorem 3.1] and is stated here as

Lemma 5.0.16 *Every bounded analytic function on $\mathbb{P}$ induces a bounded multiplication operator on $N^2(\mathbb{P})$.*

Proof A function ψ that is bounded and analytic on $\mathbb{P}$, has non-tangential limits almost everywhere with respect to Lebesgue measure on the boundary line of $\mathbb{P}$. Since the restriction of μ to this line is absolutely continuous with respect to Lebesgue measure, we may extend ψ to the closure of $\mathbb{P}$, yielding a function in $L^\infty(\mu)$, which in turn induces a bounded multiplication operator M_ψ on $L^2(\mu)$. Since N^2 can be viewed as a subspace of $L^2(\mu)$ which is invariant for M_ψ, M_ψ will be bounded there as well. ■

Therefore the set of bounded analytic functions on $\mathbb{P}$ is in one-to-one correspondence with operators in the commutant of $\{C_t : t > 0\}$.

Proposition 5.0.17 *If A commutes with C_t for every $t > 0$, then $\|M_{m_A}\|_{\mathcal{B}(N^2)} = \|m_A\|_{H^\infty(N^2)} = \|A\|_{\mathcal{B}(H^2)}$.*

Proof Firstly, $\|M_{m_A}\|_{\mathcal{B}(N^2)} = \|A\|_{\mathcal{B}(H^2)}$ since A is unitarily equivalent to M_{m_A}. Now $\|M_{m_A} f\|^2_{N^2} = \langle M_{m_A} f, M_{m_A} f\rangle_{N^2} = \int_0^\infty m_A(t) f(t) \overline{m_A(t) f(t)} e^{-t} dt \leq \|m_A\|^2_\infty \|f\|^2_2$ [11] so $\|M_{m_A}\| \leq \|m_A\|_\infty$. On the other hand, $\|M^*_{m_A}\| \geq |M^*_{m_A} \dfrac{K_\alpha}{\|K_\alpha\|}| = |\overline{m_A(\alpha)}|$ implies that $\|M_{m_A}\| \geq \|m_A\|_\infty$. Therefore $\|M_{m_A}\|_{B(N^2)} = \|m_A\|_{H^\infty(N^2)}$. ■

In addition to the preservation of norms between the above spaces, the following theorem answers a question implicit in the paper [12] of Cowen and Gallardo-Guitérrez that proves the hyperbolic composition operators are unitarily equivalent to a class of adjoints of analytic multiplication operators on the Hardy space H^2.

Theorem 5.0.18 *The composition operator C_t is unitarily equivalent to the adjoint of multiplication by $m_t \in N^2$ given by $m_t(w) = e^{-tw}$ for $w \in \mathbb{P}$. Furthermore, the function $m_t(w)$ is analytic in w.*

Proof The computation $C_t(1-z)^w = (1-\varphi_t(z))^w = (-(e^{-t}z-e^{-t}))^w = e^{-tw}(1-z)^w$ shows that $m_t(w) = e^{-tw}$ for C_t. Since $C_t v_{\overline{w}} = e^{-t\overline{w}} v_{\overline{w}} = \langle U m_t, v_{\overline{w}} \rangle_{H^2} v_{\overline{w}}$, and the inner product on H^2 is conjugate analytic in the second component, it follows that $m_t(w)$ is analytic in w. ■

Theorem 5.0.19 *$H^\infty(\mathbb{P}) \cong \{C_t : t > 0\}'$ as Banach algebras.*

Proof Let $A \in \{C_t : t > 0\}'$ and recall that $A(1-z)^k = m_A(k)(1-z)^k$ for all $k \in \mathbb{P}$ and the algebraic structure the commutant inherits as a subalgebra of $\mathcal{B}(H^2)$. The correspondence $\Phi(A) = m_A$ is one-to-one and onto, therefore through routine calculation, this correspondence is linear. It is isometric by Proposition 5.0.17, and an isomorphism. ■

Corollary 5.0.20 *The algebra $\{C_t : t > 0\}'$ is a commutative Banach algebra.*

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