THE IDEAL OF RELATIONS FOR THE RING OF INVARIANTS OF n POINTS ON THE LINE: INTEGRALITY RESULTS

BENJAMIN HOWARD, JOHN MILLSON, ANDREW SNOWDEN AND RAVI VAKIL*

ABSTRACT. Consider the projective coordinate ring of the GIT quotient $(\mathbb{P}^1)^n /\!/ \mathrm{SL}(2)$, with the usual linearization, where n is even. In 1894, Kempe proved that this ring is generated in degree one. In [HMSV2] we showed that, over $\mathbb{Q}$, the relations between degree one invariants are generated by a class of quadratic relations — the *simplest binomial relations* — with the exception of $n = 6$, where there is a single cubic relation. The purpose of this paper is to show that these results hold over $\mathbb{Z}[\frac{1}{12!}]$, and to suggest why they may be true over $\mathbb{Z}[\frac{1}{6}]$.

CONTENTS

1. INTRODUCTION

Let L be a set of even cardinality n (what we will call an *even set*). Let R_L denote the projective coordinate ring of the space of L points on $\mathbb{P}^1$, modulo the diagonal action of $\mathrm{SL}(2)$:

$$R_L = \bigoplus_{k=0}^{\infty} R_L^{(k)}, \qquad R_L^{(k)} = \Gamma(X_L, \mathscr{O}_{X_L}(k))^{\mathrm{SL}(2)}$$

where $X_L = \mathrm{Hom}(L, \mathbb{P}^1)$ and $\mathscr{O}_{X_L}(1)$ is the exterior tensor product of L copies of $\mathscr{O}_{\mathbb{P}^1}(1)$. Kempe showed that R_L is generated by its first graded piece $V_L = R_L^{(1)}$; a proof can be found in [HMSV2, §2.3]. Let I_L be the *ideal of relations* between degree one invariants, the kernel of the surjection $\mathrm{Sym}(V_L) \to R_L$.

In [HMSV2, Thm. 9.1], we showed that the quadratic *simplest binomial relations* (elements of $I_L^{(2)}$, defined in §2.2) generate I_L over $\mathbb{Z}[\frac{1}{n!}]$ for $n \geq 8$. The main result of this paper is the following:

Theorem 1.1 (Main theorem). *Let L be an even set of cardinality at least 8. Then I_L is generated over $\mathbb{Z}[\frac{1}{12!}]$ by the simplest binomial relations.*

In [HMSV2, §8] we proved that I_L is generated over $\mathbb{Z}[\frac{1}{12!}]$ by quadratics. We also proved in [HMSV2, §9] that the "simple binomial relations" (defined in §2.2) span the same space as the simplest binomial relations over $\mathbb{Z}[\frac{1}{2}]$. Thus to prove Theorem 1.1 it suffices to prove:

Theorem 1.2. *Let L be an even set. Then $I_L^{(2)}$ is spanned over $\mathbb{Z}[\frac{1}{12!}]$ by the simple binomial relations.*

We established this statement over $\mathbb{Z}[\frac{1}{n!}]$ in [HMSV2, §9] by a short argument using the representation theory of the symmetric group S_n. The use of representation theory is the reason that we required $n!$ to be invertible. In this paper, we prove the result over $\mathbb{Z}[\frac{1}{12!}]$ by a totally different — and much more involved — inductive argument. The work lies in establishing the inductive step:

Theorem 1.3. *Let L be an even set of cardinality at least 10 but not 12. Assume that for all sets L' of smaller even cardinality $I_{L'}^{(2)}$ is generated over $\mathbb{Z}[\frac{1}{2N}]$ by simple binomial relations. Then the same is true for $I_L^{(2)}$.*

As remarked above, the results of [HMSV2] show that $I_L^{(2)}$ is spanned by simple binomial relations over $\mathbb{Z}[\frac{1}{12!}]$ whenever L has cardinality at most 12. Thus Theorem 1.3 shows that this statement holds for L of any cardinality, which establishes Theorem 1.2.

Remark 1.4. The reader may wonder about the importance of the simplest binomial relations. Aside from aesthetic reasons, there is the following one: the simplest binomial relations form a single orbit under the action of the symmetric group. Thus Theorem 1.1 shows that I_L is a "principal S_n-ideal" over $\mathbb{Z}[\frac{1}{12!}]$. Perhaps most important, they naturally arise in much more general circumstances in the retrogeneration conjecture of the third author.

1.1. **Improving Theorem 1.1.** We suspect that Theorem 1.1 may in fact hold over $\mathbb{Z}[\frac{1}{6}]$. Here are some comments along these lines:

- When L has cardinality 8, Theorem 1.1 is true over $\mathbb{Z}[\frac{1}{6}]$. In fact, $I_L^{(2)}$ is then spanned over $\mathbb{Z}[\frac{1}{2}]$ by the simplest binomial relations (this follows from [HMSV1, Proposition 2.10] which gives a computer aided proof that the simple binomial relations span $I_L^{(2)}$ over $\mathbb{Z}$) while I_L is generated over $\mathbb{Z}[\frac{1}{3}]$ by quadratics (see [HMSV4]).
- Let L have cardinality 10. Then Theorem 1.2 shows that the simplest binomial relations span $I_L^{(2)}$ over $\mathbb{Z}[\frac{1}{2}]$, since this is the case for all smaller cardinalities. However, we cannot conclude from this that $I_L^{(2)}$ is generated by simplest binomial relations over $\mathbb{Z}[\frac{1}{6}]$ because we only know that I_L is generated by quadratics over $\mathbb{Z}[\frac{1}{10!}]$.
- By the results of [HMSV2], to prove Theorem 1.1 over $\mathbb{Z}[\frac{1}{6}]$ it would suffice to verify the following two statements: (1) $I_L^{(3)}$ is generated by quadratics (i.e., the map $V_L \otimes I_L^{(2)} \to I_L^{(3)}$ is surjective) over $\mathbb{Z}[\frac{1}{6}]$ when L has cardinality 10 or 12; (2) $I_L^{(2)}$ is spanned over $\mathbb{Z}[\frac{1}{6}]$ by simple binomial relations when L has cardinality 12.

The above comments show that to confirm our suspicion that Theorem 1.1 holds over $\mathbb{Z}[\frac{1}{6}]$ one needs only perform a few calculations. However, these computations are quite large: for instance, when L has cardinality 12, the space $I_L^{(3)}$ has dimension 339,240. Thus to carry this out on a computer would require considerable thought.

2. Background

2.1. **The ring and the ideal.** We now recall the graphical description of the ring R_L. For more details see [HMSV2, §2]. Let $\mathscr{G}_L^{\mathrm{reg}}$ denote the set of all regular directed graphs on the vertex set L. "Regular" means that each vertex has the same valence. We allow our graphs to have loops (edges from a vertex to itself) and multiple edges between the same vertices. We give $\mathscr{G}_L^{\mathrm{reg}}$ the structure of a semi-group by defining $\Gamma \cdot \Gamma'$ to be the graph on L whose edge set is the disjoint union of those of Γ and Γ'. For a graph Γ we let X_Γ denote the corresponding element of the semi-group algebra $\mathbb{Z}[\mathscr{G}_L^{\mathrm{reg}}]$.

The ring R_L is defined to be the quotient of $\mathbb{Z}[\mathscr{G}_L^{\mathrm{reg}}]$ by the following three types of relations:

- *Loop relation:* $X_\Gamma = 0$ if Γ has a loop.
- *Sign relation:* $X_\Gamma = -X_{\Gamma'}$ if Γ' is obtained from Γ be reversing the direction of a single edge.
- *Plücker relation:* Let Γ be an element of $\mathscr{G}_L^{\mathrm{reg}}$ and let $\overrightarrow{ab}$ and $\overrightarrow{cd}$ be two edges of Γ. Let Γ' (resp. Γ'') be the graph obtained by replacing $(\overrightarrow{ab}, \overrightarrow{cd})$ in Γ with $(\overrightarrow{ad}, \overrightarrow{cd})$ (resp. $(\overrightarrow{ac}, \overrightarrow{bd})$). Then $X_\Gamma = X_{\Gamma'} + X_{\Gamma''}$.

We denote the image of X_Γ in R_L still by X_Γ. We give R_L a grading by letting $R_L^{(k)}$ be the span of the X_Γ with Γ of regular degree k.

Assume now that L is an even set. The ring R_L is then generated by its first graded piece $V_L = R_L^{(1)}$ (Kempe's theorem). The space V_L is spanned by those X_Γ where Γ is a *matching*, that is, a regular graph of degree one. We let I_L be the *ideal of relations*, that is, the kernel of the surjection $\mathrm{Sym}(V_L) \to R_L$. We let W_L be the second graded piece of R_L, namely $R_L^{(2)}$, and we let B_L be the kernel of the map $V_L^{\otimes 2} \to W_L$. The sequence

$$0 \to B_L \to V_L^{\otimes 2} \to W_L \to 0$$

is exact. The space of quadratic relations $I_L^{(2)}$ is the image of B_L under the map $V_L^{\otimes 2} \to \mathrm{Sym}^2(V_L)$.

2.2. Simple binomial relations.

Let U be a subset of L of cardinality four and put $L' = L \setminus U$. Let Δ and Δ' be matchings on U and let Γ and Γ' be matchings on L'. We then have the following obvious element of $I_L^{(2)}$:

$$X_{\Gamma\Delta} X_{\Gamma'\Delta'} - X_{\Gamma\Delta'} X_{\Gamma'\Delta}.$$

(Here $\Gamma\Delta$, etc., denotes the union of Γ and Δ, a matching with vertex set L.) We regard this as an element of B_L as well by inserting the $\otimes$ symbol in between the X's in the above. We call these elements the *simple binomial relations*. As remarked in the introduction, these were shown in [HMSV2] to span $I_L^{(2)}$ when certain primes are inverted; the goal of this paper is to show that one needs to invert fewer primes than was done there. (The "simplest binomial relations" referred to in the introduction are the simple binomial relations where Γ is a union of a single 4-cycle with a bunch of 2-cycles; they will not play an important role in the paper.)

Pure tensors in $V_L^{\otimes 2}$ can be thought of as graphs on L in which each edge has been assigned one of two colors in such a manner that each vertex belongs to exactly one edge of each color. Elements of $\mathrm{Sym}^2(V_L)$ can be thought of in a similar manner. See [HMSV2, §5.1] for a more complete discussion. With this mode of thinking, one can depict the simple binomial relations in a very visually pleasing way. For example, here is one on 10 points:

The 4-cycles are formed out of Δ and Δ' while the 6-cycles are formed out of Γ and Γ'.

There is a natural map $\boxtimes : V_L \otimes V_{L'} \to V_{L \amalg L'}$ taking $X_\Gamma \otimes X_\Delta$ to $X_{\Gamma\Delta}$. We called this *outer multiplication* in [HMSV2, §7.1]. The simple binomial relations may be defined in terms of outer multiplication. Let U be a set of size four. Then B_U is spanned by the commutativity relations $X_\Delta \otimes X_{\Delta'} - X_{\Delta'} \otimes X_\Delta$. The simple binomial relations in B_L are then exactly the outer products of arbitrary elements of $V_{L'}^{\otimes 2}$ with elements of B_U, where $L = L' \amalg U$.

2.3. The straightening algorithm.

Fix an embedding of L into the unit circle in the plane. We call a graph Γ on L *planar* if its edge do not cross when drawn as chords in the circle. The following theorem, also due to Kempe, will be key. It is well-known, so we omit a proof. For details see [HMSV1, Props. 2.5, 2.6].

Proposition 2.1 (Kempe). *The X_Γ, with Γ planar, span R_L over $\mathbb{Z}$. The only relations between these elements are the sign relations. Thus if for each undirected planar graph one chooses a direction on the edges then the resulting X_Γ's form a basis for R_L over $\mathbb{Z}$.*

To write an arbitrary X_Γ in terms of the planar basis one uses the *straightening algorithm*: simply take any two edges of Γ which cross and apply the Plücker relation to them; repeat. This algorithm terminates because the total lengths of the edges in the resulting graphs is less than the total lengths of the edges in Γ, since the lengths of the diagonals in a quadrilateral exceeds that of any pair of opposite edges. This shows that the planar graphs span over $\mathbb{Z}$.

3. Reduction of Theorem 1.3 to Proposition 3.4

The goal of this section is to reduce the proof of Theorem 1.3 to that of a different statement, Proposition 3.4. The statement of this proposition is less transparent than that of Theorem 1.3; however, in the next section we will reduce the proof of Proposition 3.4 to that of a very concrete statement.

3.1. **Partitions.** Let L be an even set. By a *partition* of L we mean a collection $\mathscr{U}$ of subsets of L such that $\mathscr{U}$ has cardinality at least 2, each element of $\mathscr{U}$ has even cardinality and L is the disjoint union of the elements of $\mathscr{U}$. For a partition $\mathscr{U}$ we define

$$V_{\mathscr{U}} = \bigotimes_{U \in \mathscr{U}} V_U, \qquad W_{\mathscr{U}} = \bigotimes_{U \in \mathscr{U}} W_U, \qquad B_{\mathscr{U}} = \sum_{U \in \mathscr{U}} B_U \otimes V_{\mathscr{U} \setminus \{U\}}^{\otimes 2},$$

the sum taking place inside of $V_{\mathscr{U}}^{\otimes 2}$ in the last definition. Note that the sequence

$$0 \to B_{\mathscr{U}} \to V_{\mathscr{U}}^{\otimes 2} \to W_{\mathscr{U}} \to 0$$

is exact. We now put

$$\widetilde{V}_L^{(2)} = \bigoplus_{\mathscr{U}} V_{\mathscr{U}}^{\otimes 2}, \qquad \widetilde{W}_L = \bigoplus_{\mathscr{U}} W_{\mathscr{U}}, \qquad \widetilde{B}_L = \bigoplus_{\mathscr{U}} B_{\mathscr{U}},$$

the sum taken over all partitions $\mathscr{U}$. Note that the sequence

$$0 \to \widetilde{B}_L \to \widetilde{V}_L^{(2)} \to \widetilde{W}_L \to 0$$

is exact.

A subset $U \subset L$ is *closed* with respect to a graph Γ if every edge of Γ which contains a vertex in U is completely contained in U. A partition $\mathscr{U}$ is *closed* with respect to Γ if each of its pieces is. One may interpret $\widetilde{W}_L$ as the free module on tuples $(\Gamma, \mathscr{U})$ where Γ is a degree two graph on L and $\mathscr{U}$ is a closed partition of L. One may perform Plücker relations on two edges only if they both lie within the same piece of the partition. Similarly, one may interpret $\widetilde{V}_L^{(2)}$ as the free module on tuples $(\Gamma, \mathscr{U})$ where Γ is a colored graph of degree two such and $\mathscr{U}$ is closed with respect to Γ. One is allowed to perform Plücker relations on two edges of the same color which lie in the same piece of the partition. The map $\widetilde{V}_L^{(2)} \to \widetilde{W}_L$ is then given by forgetting the coloring.

3.2. **The main diagram.** We have maps $\widetilde{V}_L^{(2)} \to \widetilde{V}_L^{\otimes 2}$ and $\widetilde{W}_L \to W_L$, given by forgetting the partition. A key point is the following:

Lemma 3.1. *Assume the cardinality of L is at least six. Then the maps $\widetilde{V}_L^{(2)} \to V_L^{\otimes 2}$ and $\widetilde{W}_L \to W_L$ are surjective over $\mathbb{Z}[\frac{1}{2}]$*

Proof. It is shown in [HMSV2, §6.3] that $V_L^{\otimes 2}$ and W_L are spanned over $\mathbb{Z}[\frac{1}{2}]$ by graphs which are unions of 2- and 4-cycles. This can also be seen (at least for W_L) from the identities (I2) and (I3) in the appendix. Thus if L has cardinality at least 6 then one can write any element of $V_L^{\otimes 2}$ as a sum of X_{Γ_i} where each Γ_i is a 2-colored graph which is disconnected, and thus belongs to some $V_{\mathscr{U}}$. The same reasoning applies to W_L. $\square$

We denote the kernel of $\widetilde{V}_L^{(2)} \to V_L^{\otimes 2}$ by P_L and the kernel of $\widetilde{W}_L \to W_L$ by Q_L. We now have the commutative diagram:

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
P_L & \xrightarrow{\alpha_L} & Q_L \\
\downarrow & & \downarrow \\
0 \to \widetilde{B}_L \to & \widetilde{V}_L^{(2)} \to & \widetilde{W}_L \to 0 \\
\downarrow{\scriptstyle\beta_L} & \downarrow & \downarrow \\
0 \to B_L \to & V_L^{\otimes 2} \to & W_L \to 0 \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

in which the rows are exact. The columns are exact over $\mathbb{Z}[\frac{1}{2}]$ if L has cardinality at least six, by the previous lemma.

Lemma 3.2. *Let L be an even set of cardinality at least six. Assume (1) β_L is surjective; and (2) $B_{L'}$ is spanned over $\mathbb{Z}[\frac{1}{2N}]$ by simple binomial relations for all proper even subsets L' of L. Then B_L is spanned over $\mathbb{Z}[\frac{1}{2N}]$ by simple binomial relations.*

Proof. The image of β_L is exactly the space spanned by outer products of elements of $B_{L'}$ and elements of $V_{L\setminus L'}^{\otimes 2}$, where L' is a proper even subset of L. Hypothesis (2) ensures that $B_{L'}$ is spanned by outer products of the commutativity relation on four points. Thus the image of β_L is spanned by the outer products of commutativity relations on four points. Since β_L is surjective by (1), we see that B_L is spanned by outer products of commutativity relations on four points, i.e., by simple binomial relations. $\square$

Lemma 3.3. *Let L be an even set of cardinality at least six. Then β_L is surjective over $\mathbb{Z}[\frac{1}{N}]$ if and only if α_L is surjective over $\mathbb{Z}[\frac{1}{N}]$.*

Proof. This is a simple diagram chase. $\square$

We thus see that to prove Theorem 1.3 we only need to show that α_L is surjective when L has cardinality at least 10 but not 12. We will prove this by giving explicit (and simple) generators for Q_L and then verifying that these generators are in the image of α_L.

3.3. Generators for Q_L. There is an obvious class of relations in Q_L, which we call the *merging relations*: the relations $(\Gamma, \mathscr{U}) = (\Gamma, \mathscr{U}')$ where Γ is a graph, $\mathscr{U}$ is a partition with at least three pieces and closed with respect to Γ and $\mathscr{U}'$ is obtained from $\mathscr{U}$ by merging (taking the union of) two of its pieces. We then have the following proposition, the proof of which accounts for most of our effort.

Proposition 3.4. *If the cardinality of L is at least 10 but not 12 then the merging relations span Q_L over $\mathbb{Z}[\frac{1}{2}]$.*

The following lemma shows that Proposition 3.4 implies Theorem 1.3.

Lemma 3.5. *The merging relations are in the image of the map α_L (over $\mathbb{Z}$).*

Proof. Let $(\Gamma, \mathscr{U}) - (\Gamma, \mathscr{U}')$ be a merging relation. By using the same Plücker relations on each copy of Γ, this relation can be written as a sum of merging relations in which Γ only has even cycles. We can then 2-color Γ to get a graph Γ' in $V_L^{\otimes 2}$. Since we did not change the underlying graph, $\mathscr{U}$ and $\mathscr{U}'$ are still closed with respect to Γ'. Thus $(\Gamma', \mathscr{U}) - (\Gamma', \mathscr{U}')$ is an element of P_L. We have thus lifted the original merging relation through α_L. $\square$

Remark 3.6. We will see below that when the cardinality of L is equal to 12 there is a single relation (up to symmetry) that one can include, called the odd cycle exchange relation (§4.1), which together with the merging relations spans all of Q_L. However, it is not clear if this relation is in the image of α_L.

4. Reduction of Proposition 3.4 to Proposition 4.10

The goal of this section is to reduce the proof of Proposition 3.4 to that of two much more concrete statements, given in Proposition 4.10.

4.1. The odd cycle exchange relation. We first discuss a class of relations called the *odd cycle exchange* relations. Let Γ be a regular degree 2 graph on L and let $\{U_i\}_{i=1}^4$ be four sets of odd cardinality which are closed with respect to Γ. Put $\mathscr{U} = \{U_1 \cup U_2, U_3 \cup U_4\}$ and $\mathscr{U}' = \{U_1 \cup U_3, U_2 \cup U_4\}$. The odd cycle exchange relation is then

$$(\Gamma, \mathscr{U}) = (\Gamma, \mathscr{U}').$$

Note that if the cardinality of L is less than 12 then there are no odd cycle exchange relations because L cannot be split up into four sets off odd cardinality at least 3. On the other hand, when L has cardinality at least 14 the odd cycle exchange relations are not new:

Proposition 4.1. *If the cardinality of L is at least 14 then the odd cycle exchange relations are linear combinations of merging relations over $\mathbb{Z}[\frac{1}{2}]$.*

Proof. Let Γ and $\{U_i\}_{i=1}^4$ as above be given. First consider the case where Γ contains more than one cycle in one of the U_i, say in U_1. We can thus write $U_1 = V \cup U_1'$ where V has even cardinality, U_1' has odd cardinality and U and V are closed with respect to Γ. We now have the sequence of merging relations

$$
\begin{aligned}
(\Gamma, \{U_1 \cup U_2, \ \ U_3 \cup U_4\}) &= (\Gamma, \{V, \ \ U_1' \cup U_2, \ \ U_3 \cup U_4\}) \\
&= (\Gamma, \{V, \ \ U_1' \cup U_2 \cup U_3 \cup U_4\}) \\
&= (\Gamma, \{V, \ \ U_1' \cup U_3, \ \ U_2 \cup U_4\}) \\
&= (\Gamma, \{U_1 \cup U_3, \ \ U_2 \cup U_4\})
\end{aligned}
$$

which realizes the odd cycle exchange relation.

We now handle the general case. Since the cardinality of L is at least 14, one of the U_i, say U_1, has at least five vertices. Now, by using only Plücker relations in U_1 one may write $\Gamma = \sum a_i \Gamma_i$ where each Γ_i has more than one cycle in U_1 and a_i belong to $\mathbb{Z}[\frac{1}{2}]$. This is essentially proved in [HMSV2, §6.3], but also follows from identity (I2) of the appendix. Since we only used Plücker relations within U_1, all the Plücker relations are allowable with respect to $\mathscr{U} = \{U_1 \cup U_2, U_3 \cup U_4\}$ and $\mathscr{U}' = \{U_1 \cup U_3, U_2 \cup U_4\}$. We thus have the following equality in $\widetilde{W}_L$:

$$
(\Gamma, \mathscr{U}) - (\Gamma, \mathscr{U}') = \sum \left[(\Gamma_i, \mathscr{U}) - (\Gamma_i, \mathscr{U}') \right]
$$

This expresses the odd cycle exchange relation in which we are interested (the one on the left) in terms of odd cycle exchange relations in which the graph on U_1 has more than one cycle. By the previous paragraph, the right side lies in the span of the merging relations. $\square$

When the cardinality of L is equal to 12 there is only one odd cycle exchange relation, up to symmetry:
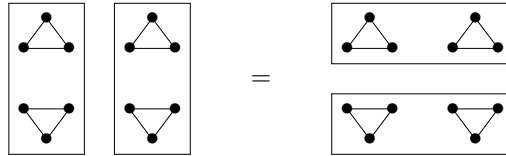
Clearly the proof of Proposition 4.1 does not apply to this relation. We do not know if this relation lies in the span of the merging relations.

4.2. **The space** W_L'. Let W_L'' (resp. W_L') be the quotient of $\widetilde{W}_L$ by merging relations (resp. merging and odd cycle exchange relations). Consider the maps

$$
\widetilde{W}_L \to W_L'' \to W_L' \to W_L.
$$

The first two maps are surjective; the third is surjective after 2 is inverted. Proposition 3.4 is equivalent to the map $W_L'' \to W_L$ being an isomorphism over $\mathbb{Z}[\frac{1}{2}]$ when L has cardinality at least 10 but not 12. We have already shown that $W_L'' \to W_L'$ is an isomorphism over $\mathbb{Z}[\frac{1}{2}]$ when the cardinality of L is at least 10 but not 12. Thus it suffices to show that $W_L' \to W_L$ is an isomorphism over $\mathbb{Z}[\frac{1}{2}]$ when the cardinality of L is at least 10 and not 12. In fact, we will prove this even when the cardinality of L is equal to 12.

To prove this result, we need a way to think about elements of W_L'. The basic idea is that in quotienting $\widetilde{W}_L$ by the merging and odd cycle exchange relations we have completely forgotten the partition data. We now elaborate on this idea.

Definition 4.2. (a) A degree two graph is *forbidden* if it is connected or the union of two odd cycles. Graphs which are not forbidden are *allowable*.
(b) Let Γ be an allowable graph. A pair of edges (e, e') of Γ is *allowable* if it meets the following conditions:

- If Γ is the union of two cycles then e and e' lie in the same cycle.
- If Γ is the union of three cycles, two of which are odd, then e and e' lie in cycles of the same parity.
- Otherwise, there is no condition.

(c) A Plücker relation on an allowable pair of edges is called an *allowable Plücker relation*.

Note that applying an allowable Plücker relation to an allowable graph results in allowable graphs. The description of W_L' which we seek is the following. The proof is straightforward, and left to the reader.

Proposition 4.3. *(a) Let Γ be an allowable graph. Choose a partition $\mathscr{U}$ of L which is closed with respect to Γ. Then the image of $(\Gamma, \mathscr{U})$ under the map $\widetilde{W}_L \to W'_L$ is independent of choice of $\mathscr{U}$. We denote the image by X'_Γ*
(b) The X'_Γ with Γ allowable span W'_L and satisfy the allowable Plücker relations.
(c) The map $W'_L \to W_L$ takes X'_Γ to X_Γ.

The statement that $W'_L \to W_L$ is an isomorphism over $\mathbb{Z}[\frac{1}{2}]$ may now be rephrased as the following statement: given a collection of allowable graphs $\{\Gamma_i\}$ such that $\sum a_i X_{\Gamma_i} = 0$ when all Plücker relations are allowed, the identity still holds if we restrict ourselves to the allowable Plücker relations (and are allowed to invert 2). We will prove this by establishing a form of the straightening algorithm for the space W'_L.

Remark 4.4. In statement (b) of the proposition, it is in fact true that the allowable Plücker relations generate all the linear relations between the X'_Γ. We will not make use of this fact, and so do not provide a proof. In fact, it will follow from our proof that $W'_L \to W_L$ is an isomorphism: our proof shows that the map from the space of allowable graphs modulo allowable Plücker relations to W_L is an isomorphism.

4.3. **Quasi-planar graphs.** Fix once and for all an embedding of L into the unit circle in the plane. We thus have a notation of planar graphs and Kempe's basis theorem (Proposition 2.1). It is clear that allowable planar graphs cannot form a basis for W'_L, since $W'_L \to W_L$ is surjective and there are planar graphs which are forbidden. Thus to prove something analogous to Kempe's theorem for W'_L we need to allow some non-planar graphs. We choose to allow graphs which are non-planar in the most mild manner possible:

Definition 4.5. We say that a graph Γ on L is *quasi-planar* if either (1) it is planar; or (2) Γ has a doubled edge e such that it becomes planar when e is removed, and e crosses exactly two edges.

We call any edge e as in (2) a *distinguished doubled edge* of Γ. In any quasi-planar graph there are at most two distinguished doubled edges and usually only one. A distinguished doubled edge meets exactly one other cycle of Γ.

Let Γ be a quasi-planar graph. We can obtain a planar graph from Γ by replacing the distinguished doubled edge and the unique cycle it crosses by the unique planar cycle on the same set of vertices. We call this graph the *associated planar graph* to Γ. See Figure 1 for an example. We say that two quasi-planar graphs are *equivalent* if they have the same associated planar graph.

Remark 4.6. There is a sign ambiguity in this definition since we have not said how to orient the edges in the associated planar graph. This will not matter for our purposes.
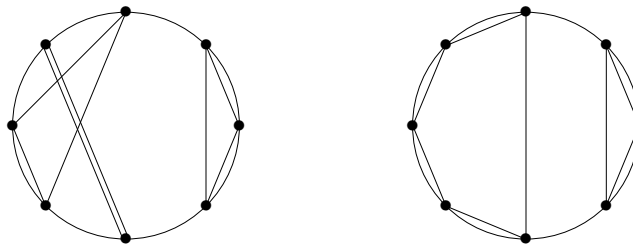

FIGURE 1. A quasi-planar graph and its associated planar graph.

Lemma 4.7. *Assume the cardinality of L is at least 8. Then every planar graph is associated to some allowable quasi-planar graph.*

The proof of this lemma is straightforward. See Figure 2 for a counterexample when L has cardinality 6.
Let Γ be a quasi-planar graph and let n be the number of cycles in Γ. We define the *level* of Γ to be n if Γ is planar and $n-1$ otherwise. Thus in the quasi-planar case we just omit the doubled edge in our count of cycles. Note that Γ and its associated planar graph have the same level.
We define a decreasing filtration on W_L (resp. W'_L) by letting $F^i W_L$ (resp. $F^i W'_L$) be the span of all planar graphs (resp. allowable quasi-planar graphs) with level at least i. It is clear that both filtrations are decreasing and separated. The filtration $F^i W_L$ is also exhaustive: $F^0 W_L = WL$ since the planar graphs span W_L. It is not clear that $F^0 W'_L = W'_L$, but we will prove this below.
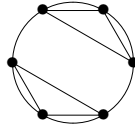
FIGURE 2. The unique planar graph which is not associated to any quasi-planar graph.

For a planar graph Γ of level i we let $\overline{X}_\Gamma$ denote the image of X_Γ in $F^i W_L / F^{i+1} W_L$. For an allowable quasi-planar graph Γ of level i we let $\overline{X}'_\Gamma$ denote the image of X'_Γ in $F^i W'_L / F^{i+1} W'_L$.

Proposition 4.8. *Let Γ be an allowable quasi-planar graph of level i and let Γ' be its associated planar graph, which also has level i. Then $X_\Gamma = \pm 2 X_{\Gamma'}$ modulo $F^{i-1} W_L$. Thus the map $W'_L \to W_L$ is filtration preserving and the image of $\overline{X}'_\Gamma$ under $\operatorname{gr} W'_L \to \operatorname{gr} W_L$ spans the same space as $\overline{X}_\Gamma$ if 2 is inverted.*

Proof. The statement is obvious if Γ is planar itself. Thus assume that Γ is non-planar and let e be a distinguished doubled edge. We apply identity (I1) of the appendix to Γ, where vertices 1 and 4 belong to e and the other two edges belong to the cycle which e meets. On the right side of the identity the first graph is simply Γ'. The remaining graphs have more cycles, and thus belong to $F^{i+1} W_L$. $\qquad\square$

As a consequence, we have:

Corollary 4.9. *Let L be an even set of cardinality at least 8. Then over $\mathbb{Z}[\frac{1}{2}]$:*
  (a) *Allowable quasi-planar graphs span W_L.*
  (b) *Equivalent allowable quasi-planar graphs span the same subspace of $\operatorname{gr} W_L$.*

In the following two sections, we establish the following proposition:

Proposition 4.10. *Let L be an even set of cardinality at least 10. Then over $\mathbb{Z}[\frac{1}{2}]$:*
  (a) *Allowable quasi-planar graphs span W'_L.*
  (b) *Equivalent allowable quasi-planar graphs span the same subspace of $\operatorname{gr} W'_L$.*

Proposition 4.10, Corollary 4.9 and Proposition 2.1 imply that $\operatorname{gr} W'_L \to \operatorname{gr} W_L$ is an isomorphism. Consequently, $W'_L \to W_L$ is an isomorphism. Thus Proposition 3.4 is implied by Proposition 4.10. (To see this in another way, Proposition 4.10 shows that the dimension of W'_L at most the number of equivalence classes of allowable quasi-planar graphs, which is equal to the number of planar graphs, which is equal to the dimension of W_L by Proposition 2.1. Thus $\dim W'_L \le \dim W_L$. Since $W'_L \to W_L$ is surjective it is therefore an isomorphism.)

5. Proof of Proposition 4.10(a)

We assume throughout this section that the cardinality of L is even and at least 10. The word "graph" will always mean "regular graph of degree two." We also work over $\mathbb{Z}[\frac{1}{2}]$ throughout this section.

We now prove Proposition 4.10(a). We will often use the identities of the appendix. It is worth noting that (I2) and (I3) hold unconditionally, that is, all Plücker relations involved are always allowable. The reason for this is that these identities involve only a single cycle. We begin with a definition.

Definition 5.1. A vertex in a graph is *special* if after deleting all edges containing it the graph is planar. A cycle in a graph is called *special* if it contains a special vertex. A graph is *semi-planar* if it contains a special vertex.

A quasi-planar graph is semi-planar. A general semi-planar graph looks like the following:
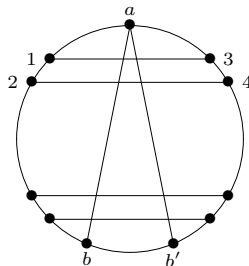
Here a is a special vertex. The vertices 1, 2, 3 and 4 are part of a planar cycle, as are the four vertices at the bottom of the circle. There may be other planar cycles which either do or do not intersect one of the two edges containing a.

Lemma 5.2. *The X'_Γ with Γ an allowable semi-planar graph span W'_L over $\mathbb{Z}[\frac{1}{2}]$.*

Proof. Elements of the form X'_Γ, with Γ an allowable graph containing a doubled edge, span W'_L over $\mathbb{Z}[\frac{1}{2}]$. This can be seen using the results of [HMSV2, §6.3] or the identity (I2). Now, if Γ has a doubled edge e then one can hold e fixed and apply the straightening algorithm to the rest of Γ. All the resulting graphs have the property that they become planar when e is removed. They are thus semi-planar. Hence X'_Γ has been expressed as a sum of semi-planar graphs. $\qquad\square$

Let Γ be a semi-planar graph and let a be a special vertex. The vertex a meets two edges. If one starts at a and follows one of these edges one will intersect some of the cycles of Γ. In fact, the set of cycles met, and the order in which they are met, is independent of the edge chosen. We call these cycles the *skewered cycles*. We call the last skewered cycle encountered (starting from a) *extreme*. We now make a few simple observations.

Lemma 5.3. *Let Γ be an allowable graph. Assume one of the following conditions is satisfied:*
- (a) *Γ has a doubled edge e which crosses at most two other edges.*
- (b) *Γ has four distinct vertices a, b, c, and d such that $\overline{ab}$, $\overline{bc}$ and $\overline{cd}$ are edges in Γ which do not cross any edge.*
- (c) *Γ contains an n-cycle ($n \geq 4$) with the property that no edge in the n-cycle crosses any edge.*
- (d) *Γ contains two 3-cycles with the property that no edge in either cycle crosses any edge.*
- (e) *Γ is semi-planar and has a special n-cycle with $n \geq 4$.*
- (f) *Γ is semi-planar and has a skewered n-cycle with $n \neq 3, 4$.*

Then X'_Γ is a sum of allowable quasi-planar graphs over $\mathbb{Z}[\frac{1}{2}]$.

Proof. (a) Fix the edge e and apply the straightening algorithm to the remainder of Γ. The key point is that in the straightening algorithm one only has to Plücker edges which cross and therefore no new crossings with e are introduced. Thus in all the graphs resulting from the straightening algorithm e will still only cross two edges. These graphs are therefore quasi-planar.

(b) First consider the case where $\overline{ad}$ is an edge in Γ. It is clear that this edge is not crossed by any other edge. Holding the 4-cycle $abcd$ fixed and applying the straightening algorithm to the rest of Γ write X'_Γ as a sum of allowable planar graphs. Now consider the case where $\overline{ad}$ is not an edge in Γ. Then a is connected to a unique vertex x other than b and d is connected to a unique vertex y other than c. It may be that $x = y$. We now apply identity (I3) with 1, 2, 3, 4, 5, 6 taken to be x, a, b, c, d, y. The first graph on the right side can be handled by the first case. The remaining graphs can be handled by (a) of this lemma.

(c) If n is even simply hold the n-cycle fixed and apply the straightening algorithm to the rest of the graph. If n is odd then it is ≥ 5. Apply identity (I2) to five consecutive vertices in the cycle. The first graph has an even cycle which does not cross any edge and has already been handled. The remaining graphs are handled by (a) of this lemma.

(d) Simply hold the two 3-cycles fixed and apply the straightening algorithm to the remainder of the graph. The key point is that since we have fixed *two* 3-cycles all the Plücker relations in the straightening algorithm are allowable.

(e) If there is a special cycle of length at least five then this cycle has four consecutive vertices none of which is the special vertex. It can be written as a sum of allowable quasi-planar graphs by part (b) of this lemma. Now consider the case where there is a special cycle of length 4. Apply the identity:

(1)
$$2 \; {}_{d}\!\!\overset{a}{\underset{c}{\bigcirc}}\!\!{}_{b} \;=\; {}_{d}\!\!\overset{a}{\underset{c}{\bigcirc}}\!\!{}_{b} \;+\; {}_{d}\!\!\overset{a}{\underset{c}{\bigcirc}}\!\!{}_{b} \;+\; {}_{d}\!\!\overset{a}{\underset{c}{\bigcirc}}\!\!{}_{b}$$

This identity can be gotten by applying the straightening algorithm to the middle term on the right. Here we take a to be the special vertex. Thus $\overline{bc}$ and $\overline{cd}$ are not crossed by any edge. Each of the graphs on the right can thus be handled by (a) of this lemma; precisely we take the edge e in (a) to be $\overline{cd}$, $\overline{bd}$ and $\overline{bc}$.

(f) Consider a skewered n-cycle. If $n = 2$ we are done by (a). Thus assume $n \geq 5$. The special cycle splits the n vertices of this cycle into two sets. If one of these sets has cardinality at least four then the result follows from (b) of this lemma. This is always the case when $n \geq 7$. Say now $n = 6$. We must handle the case where the special cycle splits the 6 vertices into two sets of size 3. Apply identity (I2) where 2, 3, 4 all lie to one side of the special cycle. The first graph on the right has two 3-cycles of the sort that can be handled by (d) of this lemma. The remaining three graphs can be handled by (a) of this lemma. Now say $n = 5$. We must handle the case where the special cycle splits the 5 vertices into a set of size 3 and a set of size 2. Apply (I2) where again 2, 3 and 4 lie to one side of the special cycle. The first graph now has a doubled edge and can be handled by (a); the remaining three graphs can be handled with (a) as before. $\square$

The following lemma completes the proof of Proposition 4.10(a).

Lemma 5.4. *Let Γ be an allowable semi-planar graph. Then X'_Γ is a sum of allowable quasi-planar graphs over $\mathbb{Z}[\frac{1}{2}]$.*

Proof. The idea is to proceed inductively on the number of skewered cycles. Thus let Γ be given. In the generic case, simple apply the straightening algorithm to the special cycle and the extreme skewered edge. All the resulting graphs will be semi-planar and have fewer skewered cycles. This procedure does not work in some small cases because the straightening algorithm may involve forbidden Plücker relations. In fact, this procedure fails only in the following two cases: Γ has two even cycles; or Γ has two odd cycles, one even cycle and the special cycle and extreme skewered cycle have opposite parity. We now handle each of these cases, breaking the second into two subcases.

First assume that Γ has only two even cycles. By Lemma 5.3(e), if the special cycle has length at least four then we are done. Thus we may assume that the special cycle is a doubled edge. But now the other cycle has length at least 8 (since there are at least 10 vertices by assumption) and the result follows from Lemma 5.3(f).

Now assume that Γ has three cycles, that the special cycle has even length and the extreme skewered cycle has odd length. By Lemma 5.3 it suffices to treat the case where the special cycle is a doubled edge and the extreme skewered cycle is a 3-cycle. Thus the remaining cycle has length at least 5. If it is skewered we are done by Lemma 5.3(f) otherwise we are done by Lemma 5.3(c).

Finally assume that Γ has three cycles, that the special cycle has odd length and the extreme skewered cycle has even length. By Lemma 5.3 if suffices to treat the case where the special cycle is a 3-cycle and the extreme skewered cycle is a 4-cycle. If there are at least 12 vertices in total then there is an n-cycle with $n \geq 5$ and we are done as before. However, if there are 10 vertices then we have to do some work. The special cycle and extreme skewered cycle look like:
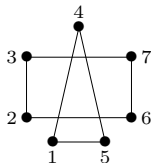
Here 4 is the special vertex. We cannot apply the straightening algorithm, as this would involve forbidden Plücker relations. Instead, we apply (1) to the square above. We now have
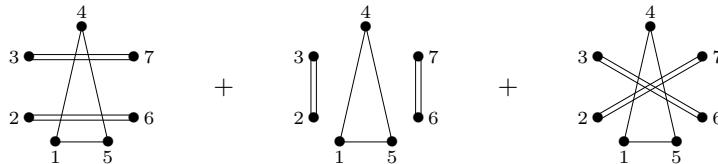
The first two graphs can be handled by Lemma 5.3(a). In the third graph, we hold $\overline{27}$ fixed and apply the straightening algorithm to the remaining edges. This, of course, involves only allowable Plücker relations. The result is:
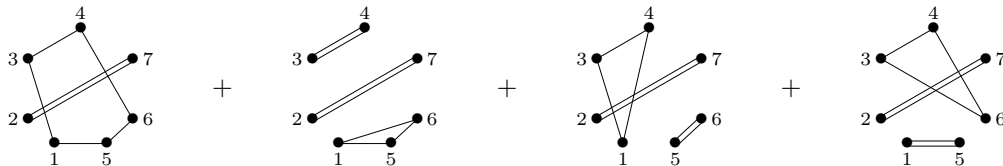
Each of these graphs has 4 as a special vertex. Thus in all of them the doubled edge $\overline{27}$ only crosses the edges which are drawn. Each of the above graphs is therefore a sum of allowable quasi-planar graphs by Lemma 5.3(a). $\square$

6. Proof of Proposition 4.10(b)

We maintain the assumptions from the beginning of the last section.

We now prove Proposition 4.10(b). We handle the three cases of level 1, level 2 and level at least 3 separately. We begin by considering graphs of level 3 or more.

Lemma 6.1. *Let Γ be an allowable quasi-planar graph of level at least 3 and let Γ' be its associated planar graph. Then Γ' is allowable and $\overline{X}'_\Gamma$ and $\overline{X}'_{\Gamma'}$ span the same subspace of $\operatorname{gr} W'_L$. In particular, any two equivalent allowable quasi-planar graphs of level at least 3 span the same subspace of $\operatorname{gr} W'_L$.*

Proof. Simply apply identity (I1) to the doubled edge in Γ. The first graph is Γ' and the rest have higher level. $\square$

We now we consider graphs of level 2. There are three possibilities for such a graph Γ:

- Γ has two odd cycles and one doubled edge which crosses one of the odd cycles.
- Γ has two even cycles and one doubled edge which crosses one of the even cycles.
- Γ has two even cycles.

In the latter two cases, the same reasoning as Lemma 6.1 applies. We are thus reduced to the first case. We call such a graph "odd."

Let Γ be an odd level 2 allowable quasi-planar graph. Let e be the 2-cycle, let C be the cycle which crosses e and let C' be the other cycle. The vertices used in C and e form an interval, as do the vertices used in C'. We call these two intervals the *distinguished intervals* of Γ. It follows from the definition that two odd graphs are equivalent if and only if they have the same distinguished intervals.

Lemma 6.2. *Let Γ be an odd level 2 allowable quasi-planar graph and let c, d and e be consecutively connected vertices which do not form a 3-cycle. Let Γ' be the unique odd graph which is equivalent to Γ and for which $\overline{ce}$ is the doubled edge. Then $\overline{X}'_\Gamma$ and $\overline{X}'_{\Gamma'}$ span the same subspace of $\operatorname{gr} W'_L$.*

Proof. We apply (I2) to Γ with 2, 3 and 4 taken to be c, d and e. The last two graphs on the right of the identity are allowable planar since they have 2-cycles on consecutive vertices. The first graph can also be written as a sum of allowable planar graphs (it is quasi-planar and has four cycles). This leaves the second graph. This graph has two doubled edges: the original one and $\overline{ce}$. We apply identity (I1) to the original doubled edge. The first graph on the right is Γ' and the remaining have higher level. This gives an expression $\overline{X}'_\Gamma = \pm 2\overline{X}'_{\Gamma'}$ and proves the lemma. $\square$

Corollary 6.3. *Two equivalent allowable quasi-planar graphs of level 2 span the same subspace of $\operatorname{gr} W'_L$.*

Proof. Let Γ be a given odd level 2 graph and let $\overline{\Gamma}$ be its associated planar graph. Let $n < m$ be the lengths of the two cycles in $\overline{\Gamma}$; these are odd numbers whose sum is at least 10. We first consider the case where n and m are each at least 5. Pick three consecutively connected vertices c_0, d_0 and e_0 in $\overline{\Gamma}$. Now, let $\overline{ce}$ be the distinguished doubled edge of Γ. If c and e lie in the same component of $\overline{\Gamma}$ as c_0 apply the lemma to move the doubled edge to the other component. Now one can again apply the lemma to move the doubled edge of Γ to $\overline{c_0 d_0}$. This shows that any allowable quasi-planar graphs with associated planar $\overline{\Gamma}$ spans the same space in $\operatorname{gr} W'_L$ as the unique allowable quasi-planar graph associated to $\overline{\Gamma}$ and with distinguished doubled edge $\overline{c_0 d_0}$.

We now consider the case where $n = 3$ so that $m \geq 7$. The distinguished doubled edge must lie in the m-cycle of Γ. Pick three consecutively connected vertices c_0, d_0 and e_0 in $\overline{\Gamma}$ which lie in the m-cycle. Using the lemma we can first move the doubled edge of Γ far away from c_0 and e_0 and then move it to be exactly $\overline{c_0 d_0}$. By the same reasoning at the end of the previous paragraph this completes the proof. $\square$

We finish by considering allowable quasi-planar graphs of level 1. These are graphs which have a single doubled edge and exactly one other cycle, which is planar and which meets the doubled edge. Such a graph is determined by which two vertices belong to the doubled edge.

Lemma 6.4. *Let Γ be an allowable quasi-planar graph of level 1 and let c, d and e be consecutively connected vertices. Let Γ' be the unique allowable quasi-planar level 1 graph which has $\overline{ce}$ for its doubled edge. Then $\overline{X}'_\Gamma$ and $\overline{X}'_{\Gamma'}$ span the same subspace of $\operatorname{gr} W'_L$.*

Proof. We apply (I2) to Γ with 2, 3, 4 taken to be c, d and e. Note that c, d and e necessarily lie in an n-cycle with $n \geq 8$, so (I2) can indeed be applied. The first graph on the right side has level at least two. The third and fourth graphs have a 2-cycle on consecutive vertices and so can be written as a sum of allowable planars. Now, the second graph has $\overline{ce}$ as a doubled edge and two other cycles: the original doubled edge and remainder of the original long cycle. We apply identity (I1) to the original doubled edge and the two edges it crosses. The first graph on the right is Γ' and the rest have higher level. We thus obtain an expression $\overline{X}'_\Gamma = \pm 2\overline{X}'_{\Gamma'}$, which establishes the lemma. $\square$
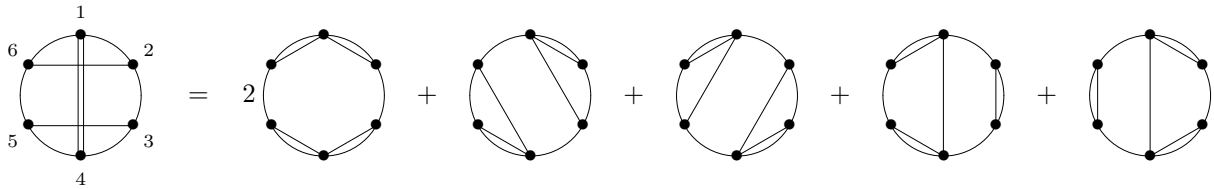
Corollary 6.5. *Two equivalent allowable quasi-planar graphs of level 1 span the same subspace of $\operatorname{gr} W'_L$.*

Proof. This goes like the second paragraph of the proof of the previous corollary. $\square$

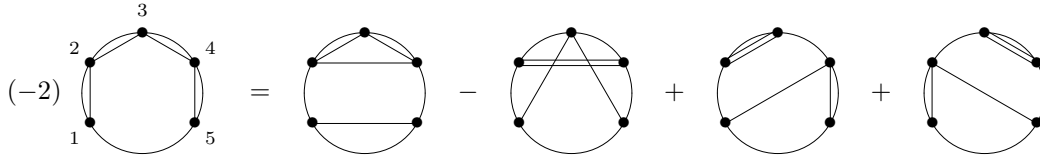## Appendix A. Long identities

In all the following identities we use the convention that edges point from smaller numbers to larger numbers.
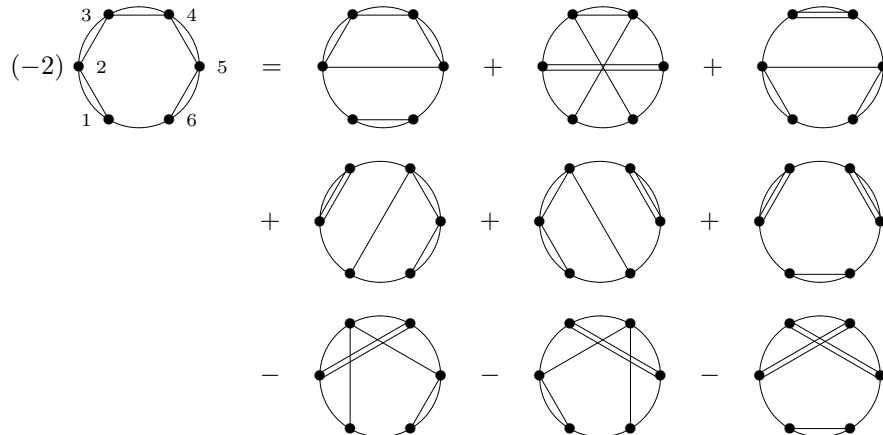
Identity (I1):

This identity can be proved by applying the straightening algorithm to the left side.

Identity (I2):

This identity can be proved by applying the straightening algorithm to the second graph on the right side

Identity (I3):

This identity is proved in [HMSV2, §6.3]. It can be proved by applying the straightening algorithm to the three graphs on the last line.

References

[HMSV1] B. Howard, J. Millson, A. Snowden and R. Vakil, *The equations for the moduli space of n points on the line*, Duke Math. J. **146** (2009), no. 2, 175–226.

[HMSV2] B. Howard, J. Millson, A. Snowden and R. Vakil, *The ideal of relations for the moduli space of n points on the line*, preprint 2009, posted on the arXiv.

[HMSV3] B. Howard, J. Millson, A. Snowden, and R. Vakil, *The relations among invariants of points on the projective line*, arXiv:0906.2437, C. R. Math. Acad. Sci. Paris, to appear.

[HMSV4] B. Howard, J. Millson, A. Snowden, and R. Vakil, *The ring of projective invariants of eight points on the line via representation theory*, arXiv:0809.1233.