Stability of Generalized Transition Fronts

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Abstract

We study the qualitative properties of the generalized transition fronts for the reaction-diffusion equations with the spatially inhomogeneous nonlinearity of the ignition type. We show that transition fronts are unique up to translation in time and are globally exponentially stable for the solutions of the Cauchy problem. The results hold for reaction rates that have arbitrary spatial variations provided that the rate is uniformly positive and bounded from above.

1 Introduction and main results

Generalized transition fronts

Scalar reaction-diffusion equations of the form

$$u_t = u_{xx} + f(u) \tag{1.1}$$

often admit special solutions of the form $u(t,x) = \Phi(x-ct)$ called traveling waves. These waves play an important role in the behavior of other solutions to (1.1) – they are attractors for solutions of the Cauchy problem for (1.1) with a large class of initial data. This was observed first by Kolmogorov, Petrvoskii and Piskunov in [15] and by Fisher in [8] in the 1930's in the case when $f(u) = u(1-u)$. Later in [11, 12, 13, 14], existence and stability of traveling waves was established for the ignition nonlinearities $f(u)$ which are non-negative for $u \in [0,1]$ and vanish on an interval $0 \le u \le \theta_0$ with some ignition temperature $\theta_0 \in (0,1)$.

More recently, there have been many studies of reaction-diffusion equations when the reaction rate (or the diffusivity matrix) varies periodically in space (see [1, 25] for detailed references), with similar existence and stability results established both for the Fisher-KPP and ignition type nonlinearities. When the reaction rate is spatially periodic, the role of planar fronts is played by pulsating fronts, which are global in time solutions of (1.1) that vary periodically in time when observed in a reference frame moving with a constant speed.

A natural extension of the aforementioned studies is the question of existence and stability of special solutions of reaction-diffusion equations of the form

$$u_t = u_{xx} + f(x,u) \tag{1.2}$$

without any structural assumptions on the spatial variations of the reaction $f(x,u)$ such as periodicity, almost periodicity, etc. The first definition of the notion of a generalized front without the

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periodicity assumption was given by Matano [16] and was later formalized by Shen in [24]. The idea of that definition is that the shape of a generalized front is a continuous function of the current environment. Shen has established in [24] some general criteria for the existence of such special solutions.

In [2], Berestycki and Hamel give an alternative definition of generalized traveling fronts which is somewhat easier to use in practice. In the context of the present paper it can be stated as follows. Consider a reaction-diffusion equation

$$u_t = u_{xx} + f(x,u), \quad x \in \mathbb{R}, \quad t \in \mathbb{R} \tag{1.3}$$

with a function $f(x,u) = g(x)f_0(u)$. Here $g(x)$, $x \in \mathbb{R}$, is a uniformly bounded, Lipschitz continuous, uniformly positive function:

$$0 < g^{min} \le g(x) \le g^{max} < +\infty \tag{1.4}$$

and $f_0(u)$ is an ignition type nonlinearity. This means that $f_0(s)$ is a Lipschitz function which vanishes outside an interval $(\theta_0, 1)$, $\theta_0 \in (0,1)$ and is positive for $s \in (\theta_0, 1)$:

$$f_0(s) = 0 \text{ for } s \in [0, \theta_0]; \quad f_0(s) > 0 \text{ for } s \in (\theta_0, 1);\ f_0'(1) < -\beta < 0. \tag{1.5}$$

According to Berestycki and Hamel [2, 24], a global in time solution $u(t,x)$ of (1.3) is a transition front if $0 < u(t,x) < 1$, and there exists a continuous function (the interface) $X(t)$, such that for any $\varepsilon > 0$ there exists a distance N_ε so that for all $t \in \mathbb{R}$ we have

$$u(t,x) > 1 - \varepsilon \text{ for } x < X(t) - N_\varepsilon \text{ and } u(t,x) < \varepsilon \text{ for } x > X(t) + N_\varepsilon. \tag{1.6}$$

A similar notion of a wave-like solution was used by Shen in [24]. Roughly speaking, this means that the width of the interface connecting the limit values $u^- = 1$ and $u^+ = 0$ is uniformly bounded in time. In the present situation $X(t)$ is the position of the right-moving interface, that is, the largest real number satisfying $u(t, X(t)) = \theta_0$:

$$X(t) = \sup\{x \in \mathbb{R} : u(t,x) = \theta_0\}. \tag{1.7}$$

The definition of a transition front in [2] is actually much more general than what we described above and applies to other problems. In particular, it applies to domains with non-periodic boundaries, and it has been shown in [3] that bistable reaction-diffusion equation in a domain with a star-shaped obstacle admits transition fronts.

Existence of the transition fronts for (1.3) has been proved in [19, 20] – to the best of our knowledge, this is the only example of an equation with spatially variable non-periodic coefficients where such a result was obtained so far. The transition front solution of (1.3) constructed in [19, 20], in addition, has the following properties:

P1. The transition front is monotonic in time: $u_t(t,x) > 0$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}$.

P2. The interface speed is bounded from above and from below: there exist two constants C_{min} and C_{max} so that

$$0 < C_{min} \le \dot{X}(t) \le C_{max} < \infty \text{ for all } t \in \mathbb{R}. \tag{1.8}$$

P3. For $\Sigma_R(t) = \{x : \ |x - X(t)| \le R\}$, the constant

$$\delta_R = \inf_{t \in \mathbb{R}} \inf_{x \in \Sigma_R(t)} u_t(t,x) > 0 \tag{1.9}$$

is positive for all $R > 0$.

P4. The transition front $u(t,x)$ is exponentially decaying ahead of the interface: it satisfies

$$u(t, x+X(t)) \le v(x) \le \theta_0 e^{-cx} \qquad \forall\, x>0, \quad t\in\mathbb{R}, \tag{1.10}$$

and

$$u(t, x+X(t)) \ge v(x) \qquad \forall\, x<0, \quad t\in\mathbb{R}. \tag{1.11}$$

Here $c>0$ is a constant, and $v(x)$ is a monotonically decreasing function such that

$$0<v(x)<1 \text{ and } v'(x)<0 \text{ for all } x\in\mathbb{R},\ v(+\infty)=0,\ v(-\infty)=1,$$

while at the origin we have $v'(0)<-p<0$ for some constant $p>0$.

Stability of generalized transition fronts

The main results of the present paper are the global stability and uniqueness of the transition fronts constructed in [19, 20]. Let $z(t,x)$ be the solution of the Cauchy problem for (1.3):

$$\begin{aligned} &z_t = z_{xx} + f(x,z), \quad x\in\mathbb{R}, \quad t\ge 0, \\ &z(0,x) = z_0(x). \end{aligned} \tag{1.12}$$

We will consider the cases when the initial data $z_0(x)$ is either front-like or compactly supported.

Front-like initial data

The first theorem concerns solutions of (1.12) which are a perturbation of the transition front. We define $\mathcal{P}_\alpha$ to be the class of admissible perturbations:

$$\mathcal{P}_\alpha = \left\{\rho\in C(\mathbb{R}) \mid \lim_{x\to\pm\infty}\rho(x)=0, \quad \rho(x)\le Ce^{-\alpha x} \text{ for some } C>0\right\} \tag{1.13}$$

with some $\alpha>0$.

Theorem 1.1 *Let $u(t,x)$ be a transition front solution of (1.3) satisfying (P.1-P.4) above, and let $z(t,x)$ satisfy (1.12) with the initial data $z_0(x)$ of the form $z_0(x)=u(t_0,x)+\rho(x)$ with $\rho(x)\in\mathcal{P}_{\alpha_0}$, for some $\alpha_0>0$, and such that $0\le z_0(x)\le 1$. There exist $\omega>0$ and $C>0$ such that: there exists a phase shift $\bar\tau\in\mathbb{R}$ so that*

$$\sup_{x\in\mathbb{R}} |z(t,x)-u(t+\bar\tau,x)| \le Ce^{-\omega t} \textit{ for } t>0. \tag{1.14}$$

Here the constants C and $\bar\tau$ depend on the initial data z_0, while the rate ω depends only on the parameter α_0.

We prove this result in two steps: (i) trapping the solution between two large translates in time of the transition front, modulo small corrections, (ii) showing that the difference between the required translates converges to zero exponentially in time, first with a rate dependent on the initial data, and, finally, after getting the translates close to each other, proving a local exponential convergence result with a uniform rate.

Here, step (i) is essentially inspired from Fife-McLeod [7], modified in [21] to take the degeneracy of f as $x\to+\infty$ into account. Also, because the wave is now monotonic in t rather than x the shifts become time-shifts rather than space shifts. For all time, the solution is therefore sandwiched between two large translates of the wave, up to an exponentially decreasing error. Some extra care has to be given to the fact that the nonlinear term is zero to the right of the front: this is why we work in the class $\mathcal{P}_\alpha$.

Step (ii) is a quantitative version of [21]: the inspiration is the same as the proof of "the Harnack inequality implies Hölder regularity" in the theory of elliptic equations. This is a classical elliptic argument which can be found, for instance in [9]. See also a (much more elaborate) version of this argument to prove $C^{1,\alpha}$ regularity of free boundaries in [5]. It was used to prove some exponential behavior in elliptic equations in [4], and a version of this argument was used to prove exponential stability of waves in nonlocal equations in [6]. The argument is easy to understand: once the trapping step (i) is available, the Harnack inequality enables us to trap, at each time T, $2T$, ..., nT,... (T large and chosen in terms of the Harnack constant of the equation) the solution between two translates of the wave, the shifts being at each time step diminished by a fixed factor $q \in (0,1)$. This yields the exponential convergence. An alternative proof of exponential convergence could go through first showing convergence along a sub-sequence of times, as in [21] or [1], at the expense of replacing space shifts by time shifts. The next step would be proving some local exponential stability result in the spirit of [23] and [18], the time-dervative of the wave playing the role of the slow bundle.

Compactly supported initial data

The second stability result concerns convergence to travelling fronts of solutions with compactly supported initial data. As in the homogeneous case, as soon as the part of the initial data z_0 above the ignition temperature θ_0 is large enough, the solution develops a pair of travelling fronts. Let us denote, in this result only, by $u^r(t,x)$ the transition front (traveling from left to right) of (1.3), and by $u^l(t,x)$ the transition front traveling from right to left, that is, u^l tends to 0 as $x \to -\infty$, and to 1 as $x \to +\infty$. The result is then

Theorem 1.2 *Let $u^l(t,x)$ and $u^r(t,x)$ be the left- and right-going transition front solutions of (1.3) satisfying (P.1-P.4) (redefined appropriately for the left-going front), and let $z(t,x)$ satisfy (1.12) with the initial data $z_0(x)$ that is compactly supported. There exists $L > 0$ so that if $z_0(x) > (1+\theta_0)/2$ on an interval $(a, a+L)$ with some $a \in \mathbb{R}$, then there exist $\bar\tau^l \in \mathbb{R}$ and $\bar\tau^r \in \mathbb{R}$ so that*

$$\sup_{x>0} |z(t,x) - u^l(t+\bar\tau^l, x) - u^r(t+\bar\tau^r, x) + 1| \leq Ce^{-\omega t} \text{ for } t > 0. \tag{1.15}$$

Once Theorem 1.1 is available, Theorem 1.2 is an adaptation of [21], Theorem 2.3 (construction of sub/super-solutions) combined with Theorem 1.1. In order to keep the length of the paper reasonable we omit its proof.

Uniqueness of the transition fronts

Uniqueness of transition fronts in general is presented as an open question in [2]. Let us explain why it is nontrivial. Theorem 1.1 seems, at first sight, sufficient to imply uniqueness immediately. It would, indeed, be so if the coefficients of the equation had some kind of recurrence, such as periodicity, or ergodicity. We could then take two possibe transition fronts, show that both satisfy properties P.1-P.4 (a non-trivial exercise in itself), and look at them as solutions of the Cauchy problem emanating from each of them. As both are stable, according to Theorem 1.1, this would tell us that, at $t = +\infty$, one wave is a time-translate of the other. The trouble is that if we take a sequence of times $t_n \to +\infty$, pass to the reference frame centered to the front position at $t = t_n$, and let $n \to +\infty$, the coefficients of the equation in this reference frame obtained in the limit may become very different from those of the original one. In the end we would conclude that one wave of some asymptotic equation is a time-translate of the other, which is information that we do not really know how to use. Another natural attempt, shifting the starting times t_n for the Cauchy problem for the two transition fronts backward in time, $t_n \to -\infty$, and hoping to have them approach each other arbitrarily closely by a fixed time t_0 will not help to immediately resolve the issue. That would require a control over the constants in the exponential convergence rate which may potentially "worsen" as we shift the starting time (and hence the interface location) back. We were able to carry out this approach in an ergodic random medium, but not in general.

The above explains why we are going to spend some time on uniqueness of the transition fronts, which, despite aforementioned difficulties, holds in general, without the ergodicity assumption or randomness:

Theorem 1.3 *Let $\phi(t,x)$ and $\psi(t,x)$ be two transition front solutions of (1.3). Then there exists $h \in \mathbb{R}$ so that we have $\phi(t,x) = \psi(t-h,x)$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}$.*

The proof of Theorem 1.3 proceeds in three steps. First, we show that any transition front solution of (1.3) is monotonic in time – this is known, of course, for the fronts that were constructed in [19, 20] but not in general. Monotonicity of transition fronts in time was also announced in [2], we present an alternative proof for the convenience of the reader. The next step is to show that time monotonicity leads to a uniform exponential decay to zero ahead of the front. Finally, we show that monotonicity in time, exponential decay estimates and stability of the transition front imply uniqueness.

We remark that we denote by the same letter C various constants appearing throughout the paper.

The main results of this paper may be generalized to nonlinearities $f(x,u)$ of ignition type, not of the form $g(x)f_0(u)$, provided that the ignition temperature (that may be allowed to vary in space) stays away from $u = 0$ and $u = 1$, and bounds similar to (1.4) hold. We do not pursue such generalization here to keep the presentation simple.

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2 Global exponential stability

In this section we prove Theorem 1.1. Section 2.1 contains the proof of Proposition 2.1 which sandwiches solution of the Cauchy problem between two time translates of the transition front modulo small exponential corrections. It also provides an important control of the "total needed shifts" in terms of the "initially needed shifts" in estimate (2.11). Section 2.2 contains the core of the proof of Theorem 1.1. This part itself consists of two steps. First, we use an induction argument to establish a version of local stability: there exists a time T which depends on the initial data so that at the time $t = T$ the difference between the necessary "sandwiching" shifts is smaller than a fixed number M_0 (we set $M_0 = 1$ for simplicity). After that we use the same induction to show that, with an appropriately chosen time-step T', which no longer depends on the initial data, the difference between the forward and backward in time shifts needed to sandwich the solution at the time $t_n = T + nT'$ decays exponentially in n. Exponential convergence to a time-shift of a transition front follows from this easily.

2.1 Sub and super-solutions

We begin with a proposition similar to an argument found in the work of Fife and McLeod [7]. As we wish to use this result iteratively in the next section, we take some extra care in formulating the precise statement. Let $u(t,x)$ be a generalized front. Consider $L_0 > 1$ sufficiently large so that:

$$\begin{array}{ll} \forall x \le X(t) - L_0/2, & u(t,x) \ge (1+\theta_1)/2 \\ \forall x \ge X(t) + L_0/2, & u(t,x) \le \theta_0/2 \end{array} \tag{2.1}$$

where θ_1 is chosen so that $f_0'(u) \le f_0'(1)/2$ if $u \ge \theta_1$. Recall that $X(t)$ is the position of the interface of $u(t,x)$ at time t. Given such L_0, which will remain fixed from now on, and will not depend on the initial data, we let $\Gamma_{L_0,\alpha}(x)$ be a monotonically decreasing smooth function such that $0 \le \Gamma_{L_0,\alpha}(x) \le 1$, and

$$\Gamma_{L_0,\alpha}(x) = \begin{cases} 1, & \text{for } x \in (-\infty, L_0-1], \\ e^{-\alpha(x-L_0)}, & \text{for } x \in [L_0+1, +\infty], \end{cases} \tag{2.2}$$

with the constant $\alpha = \min(\alpha_0/2, C_{min}/4)$. Here α_0 is the exponential decay rate of the initial data z_0 in Theorem 1.1 and the constant C_{min} is as in property P.2 of the transition front. We will drop below the subscripts in the notation for $\Gamma(x)$ to make notation less cumbersome.

Consider now $z_0(x)$, an initial datum for the Cauchy problem (1.12), such that the difference $z_0 - u(0,.) \in \mathcal{P}$. As u_t is uniformly positive around the interface and $z_0 - u(0,\cdot)$ vanishes exponentially fast away from the interface, we may find two shifts $\xi_0^+ > \xi_0^-$ and a correction

$$\varepsilon \le \varepsilon_0 = \min\left(\theta_0/4, (1-\theta_1)/4, \gamma_0\right) \tag{2.3}$$

such that

$$u(\xi_0^-, x) - \varepsilon\Gamma(x - X(\xi_0^-)) \le z(0,x) \le u(\xi_0^+, x) + \varepsilon\Gamma(x - X(\xi_0^+)). \tag{2.4}$$

(Later it will be apparent that the difference $\xi_0^+ - \xi_0^-$ controls the size of the constant C in Theorem 1.1.) Let us now explain what the constant γ_0 in (2.3) is. Given L_0, find δ_{2L_0} as in (1.9), and set

$$B = (2K_f + C_\Gamma)/\delta_{2L_0}, \text{with } C_\Gamma = \|\Gamma\|_{C^2(\mathbb{R})}. \tag{2.5}$$

Here K_f is the Lipschitz constant for f:

$$K_f = \sup_{x\in\mathbb{R}} \sup_{\substack{a,b\in[0,1] \\ a\neq b}} \frac{|f(x,a) - f(x,b)|}{|a-b|}. \tag{2.6}$$

Then, we define $\gamma_0 = 1/(4B)$, which ensures that

$$\varepsilon_0 B \le 1/4. \tag{2.7}$$

Without loss of generality we may assume that $\xi_0^- \le 0 \le \xi_0^+$. Moreover we impose

$$0 \le \varepsilon \le \xi_0^+ - \xi_0^-, \tag{2.8}$$

something that we may easily get by enlarging $|\xi_0^+|$ and $|\xi_0^-|$.

We will make use of the following quantities below: a constant $\beta > 0$ chosen so that

$$\beta \le \inf_{x\in\mathbb{R}} \inf_{s\in(\theta_1,1)} |\partial_s f(x,s)| \le g^{min}|f_0'(1)|/2, \tag{2.9}$$

and $\nu > 0$ and $\omega > 0$ defined as

$$\nu := \frac{\alpha C_{min}}{2} - \alpha^2 > 0, \quad \omega = \min(\beta, \nu, K_f). \tag{2.10}$$

The following proposition establishes uniform bounds trapping $z(t,x)$ between two translates of the front modulo small corrections.

Proposition 2.1 *Assume that (2.3)-(2.8) hold. There exists a constant $K_0 > 0$ which depends on the initial data z_0 only via α_0 such that if $\varepsilon \in (0, \varepsilon_0)$, then we can find two real numbers $\xi_1^- < \xi_1^+$ with the following properties:*

$$\xi_1^+ \leq \xi_0^+ + \varepsilon K_0, \quad \xi_1^- \geq \xi_0^- - \varepsilon K_0, \tag{2.11}$$

and for all $t \geq 0$, $x \in \mathbb{R}$,

$$u(t + \xi_1^-, x) - q(t)\Gamma(x - X(t + \xi_1^-)) \leq z(t, x) \leq u(t + \xi_1^+, x) + q(t)\Gamma(x - X(t + \xi_1^+)), \tag{2.12}$$

with $q(t) = \varepsilon e^{-\omega t}$.

Proof. As usual, each side of (2.12) results from the construction of a sub/super solution.

Supersolution

We will construct a super-solution for $z(t, x)$ of the form

$$\bar{u}(t, x) = u(t + \zeta(t), x) + q(t)\Gamma(x - X(t + \zeta(t))). \tag{2.13}$$

Here the function $q(t)$ is

$$q(t) = \varepsilon e^{-\omega t} \tag{2.14}$$

with the constant ω as in (2.10), and the function $\zeta(t)$ is set to be

$$\zeta(t) = \xi_0^+ + \frac{B\varepsilon(1 - e^{-\omega t})}{\omega} \leq \xi_0^+ + K_0\varepsilon, \tag{2.15}$$

with the constant B given by (2.5), and $K_0 = B/\omega$.

Now, we verify that with the above choice of parameters $\bar{u}(t, x)$ is a super-solution for $z(t, x)$. Initially, at $t = 0$, we have

$$\bar{u}(0, x) = u(\xi_0^+, x) + \varepsilon\Gamma(x - X(\xi_0^+)) \geq z_0(x),$$

according to (2.4).

Next, we check that our choice of the parameters turns $\bar{u}(t, x)$ into a super-solution for (1.12). To this end we compute

$$\mathcal{N}(\bar{u}) := \bar{u}_t - \bar{u}_{xx} - f(x, \bar{u}),$$

and show that this is non-negative for all $t \geq 0$ and $x \in \mathbb{R}$ wherever $\bar{u} \leq 1$.

Step 1. Behind the front. Behind the front, for $x < X(t + \zeta(t)) - L_0$ we have

$$\Gamma(x - X(t + \zeta(t))) \equiv 1,$$

so in this region

$$\begin{aligned} \mathcal{N}(\bar{u}) &:= \bar{u}_t - \bar{u}_{xx} - f(x, \bar{u}) = u_t - u_{xx} + \dot{\zeta}u_t + \dot{q} - f(x, u + q) \\ &= \dot{\zeta}u_t + \dot{q} + f(x, u) - f(x, u + q) \geq \dot{\zeta}u_t + \dot{q} + \beta q \geq \dot{\zeta}u_t + \dot{q} + \omega q \geq 0. \end{aligned} \tag{2.16}$$

The next to last inequality above holds as long as $\bar{u} \leq 1$, due to our definition of L_0 since $u(t + \zeta(t), x) \in ((1 + \theta_1)/2, 1)$ for $x < X(t + \zeta(t)) - L_0$ and $q(t) \geq 0$. The last inequality in (2.16) holds because $u_t > 0$, and $\dot\zeta(t) > 0$ as can be seen immediately from (2.15) while (2.14) implies that $\dot q + \omega q = 0$.

Step 2. Ahead of the front. Ahead of the front, in the region where $x > X(t+\zeta(t)) + L_0 + 1$ we have

$$\Gamma(x - X(t + \zeta(t))) = e^{-\alpha(x-X(t+\zeta(t))-L_0)}.$$

We also note that by definitions of L_0 and ε_0 ((2.1) and (2.3)), we have both $u(t+\zeta(t), x) \leq \theta_0/2 \leq \theta_0$ and $\bar u(t, x) \leq u(t + \zeta(t), x) + \varepsilon \leq \theta_0$ in the region where $x > X(t + \zeta(t)) + L_0 + 1$ and thus

$$f(x, \bar u(t, x)) = f(x, u(t + \zeta(t), x)) = 0 \tag{2.17}$$

is satisfied in this region.

With (2.17) in hand we compute

$$\begin{aligned}\mathcal{N}(\bar u) = \bar u_t - \bar u_{xx} - f(x, \bar u) &= u_t - u_{xx} + \dot\zeta u_t + \dot q(t)e^{-\alpha(x-X(t+\zeta(t))-L_0)}\\ &+ \alpha(1 + \dot\zeta)\dot X(t + \zeta(t))q(t)e^{-\alpha(x-X(t+\zeta(t))-L_0)} - q(t)\alpha^2 e^{-\alpha(x-X(t+\zeta(t))-L_0)}\\ &= \dot\zeta u_t + e^{-\alpha(x-X(t+\zeta(t))-L_0)}\left[\dot q(t) + \alpha(1 + \dot\zeta)\dot X(t)q - q(t)\alpha^2\right] \geq 0.\end{aligned}$$

The last inequality above holds because $\dot\zeta(t) > 0$, $u_t > 0$ and we also have

$$\dot q(t) + \alpha(1 + \dot\zeta)\dot X(t)q - q\alpha^2 \geq \left[-\omega + \alpha\dot X(t) - \alpha^2\right] q \geq \left[\alpha C_{min} - \nu - \alpha^2\right] q \geq 0, \tag{2.18}$$

due to our choice of the constant ν in (2.10), the definition of ω and the fact that $\dot\zeta > 0$.

Step 3. The middle region. Finally, we look at the region around the interface where $|x - X(t + \zeta(t))| < 2L_0$. There we have

$$\begin{aligned}&\mathcal{N}(\bar u) = \bar u_t - \bar u_{xx} - f(x, \bar u) = u_t - u_{xx} - f(x, u) + \dot\zeta u_t + f(x, u) - f(x, \bar u) \\ &+\dot q(t)\Gamma(x - X(t + \zeta(t))) - q(t)(1 + \dot\zeta)\dot X(t + \zeta(t))\Gamma_x(x - X(t + \zeta(t))) - q(t)\Gamma_{xx}(x - X(t + \zeta(t))).\end{aligned} \tag{2.19}$$

The right side above can be bounded from below as follows: the first three terms on the right vanish, the term $\dot\zeta u_t$ is bounded from below using (1.9) by $\dot\zeta\delta_{2L_0}$, while $|f(x, u) - f(x, \bar u)| \leq K_f q(t)$, where K_f is the Lipschitz constant for f defined in (2.6). The second line of (2.19) is treated as follows: we can estimate the first term from below by $(-\omega q(t))$, the second term is non-negative because $\dot\zeta > 0$ while $\Gamma_x \leq 0$, and, finally, the last term is bounded from below by $(-C_\Gamma q(t))$, where C_Γ is as in (2.5). Putting these considerations together we arrive at

$$\mathcal{N}(\bar u) \geq \dot\zeta\delta_{2L_0} - K_f q(t) - \omega q(t) - C_\Gamma q(t) \tag{2.20}$$

The right side of (2.20) is non-negative under the condition

$$\dot\zeta \geq \frac{K_f + \omega + C_\Gamma}{\delta_{2L_0}} q(t). \tag{2.21}$$

This condition is ensured by our choice of the constant B in (2.5) and the requirement that $\omega \leq K_f$ in (2.10), as $B \geq (K_f + \omega + C_\Gamma)/\delta_{2L_0}$. It follows now that $z(t, x) \leq \bar u(t, x)$ for all $t \geq 0$ and thus, in particular, (2.12) holds with $\xi_1^+ = \lim_{t\to+\infty} \zeta(t)$. Moreover, (2.15) implies that ξ_1^+ satisfies

$$\xi_1^+ \leq \xi_0^+ + K_0\varepsilon. \tag{2.22}$$

Sub-solution

Now, we construct a sub-solution for $z(t,x)$ of the form

$$\tilde{u}(t,x) = u(t-\zeta(t),x) - q(t)\Gamma(x - X(t-\zeta(t))).$$

Here the functions $q(t)$ and $\zeta(t)$ are as in the super-solution construction, except that $\zeta(t)$ is now defined as

$$\zeta(t) = -\xi_0^- + \frac{B\varepsilon(1-e^{-\omega t})}{\omega}.$$

We will compute

$$\mathcal{N}(\tilde{u}) := \tilde{u}_t - \tilde{u}_{xx} - f(x,\tilde{u}),$$

and show that this is non-positive for our choice of the parameters defining $\tilde{u}$. The computation is very similar to that for the super-solution, we provide the details for the convenience of the reader.

First, at $t = 0$, as before we have $z_0(t,x) \geq \tilde{u}(t,x)$. Therefore, we only need to verify that $\mathcal{N}(\tilde{u}) \leq 0$ for all $t \geq 0$ and $x \in \mathbb{R}$, wherever $\tilde{u} \geq 0$.

Step 1. Behind the front. Behind the front, for $x < X(t-\zeta(t)) - L_0$ the function Γ is constant:

$$\Gamma(x - X(t-\zeta(t))) \equiv 1.$$

We also observe that, due to the definition of L_0, we have

$$u(t-\zeta(t),x) \geq \tilde{u}(t,x) \geq u(t-\zeta(t),x) - \varepsilon \geq \frac{1+\theta_1}{2} - \varepsilon \geq \theta_1 \tag{2.23}$$

is satisfied for $x < X(t-\zeta(t)) - L_0$, since $\varepsilon < \varepsilon_0 \leq (1-\theta_1)/4$ by (2.3). Therefore, we have

$$f(x,u) - f(x,\tilde{u}) \leq -\beta q \tag{2.24}$$

and thus

$$\begin{aligned}\mathcal{N}(\tilde{u}) &:= \tilde{u}_t - \tilde{u}_{xx} - f(x,\tilde{u}) = u_t - u_{xx} - f(x,u) - \dot{\zeta}u_t - \dot{q}(t) + f(x,u) - f(x,\tilde{u}) \\ &= -\dot{\zeta}u_t - \dot{q}(t) + f(x,u) - f(x,\tilde{u}) \leq -\dot{\zeta}u_t - \dot{q}(t) - \beta q \leq -\dot{\zeta}u_t - \dot{q}(t) - \omega q = -\dot{\zeta}u_t \leq 0,\end{aligned}$$

due to our choice of $\zeta(t)$ and $q(t)$.

Step 2. Ahead of the front. Ahead of the front, for $x > X(t-\zeta(t)) + L_0 + 1$ we have

$$\Gamma(x - X(t-\zeta(t))) = e^{-\alpha(x - X(t-\zeta(t)) - L_0)}.$$

The definition of L_0 implies that $\tilde{u}(t,x) \leq u(t-\zeta(t),x) \leq \theta_0/2$ and thus

$$f(\tilde{u}(t,x)) = f(x,u(t-\zeta(t),x)) = 0 \tag{2.25}$$

is satisfied for $x > X(t) + L_0 + 1$. With this condition we compute

$$\begin{aligned}&\mathcal{N}(\tilde{u}) = \tilde{u}_t - \tilde{u}_{xx} - f(x,\tilde{u}) = u_t - u_{xx} - \dot{\zeta}u_t(t-\zeta(t),x) \\ &-\dot{q}(t)e^{-\alpha(x-X(t-\zeta(t))-L_0)} - \alpha(1-\dot{\zeta})\dot{X}(t)q(t)e^{-\alpha(x-X(t-\zeta(t))-L_0)} + q(t)\alpha^2 e^{-\alpha(x-X(t-\zeta(t))-L_0)} \\ &= -\dot{\zeta}u_t - e^{-\alpha(x-X(t)-L_0)}\left[\dot{q}(t) + \alpha(1-\dot{\zeta})\dot{X}(t)q(t) - q(t)\alpha^2\right] \\ &\leq -e^{-\alpha(x-X(t)-L_0)}\left[-\omega + \alpha(1-\dot{\zeta})\dot{X}(t) - \alpha^2\right]q(t) \\ &\leq -e^{-\alpha(x-X(t)-L_0)}\left[-\omega + \alpha(1-B\varepsilon)C_{min} - \alpha^2\right]q(t) \\ &\leq -e^{-\alpha(x-X(t)-L_0)}\left[-\omega + \frac{3\alpha C_{min}}{4} - \alpha^2\right]q(t) \leq 0,\end{aligned}$$

due to the requirement that $\omega \le \nu$ in (2.10) and condition (2.7) on B and ε.

Step 3. The middle region. In the region near the interface $-2L_0 < x - X(t-\zeta(t)) < 2L_0$, we compute

$$\mathcal{N}(\tilde u) = \tilde u_t - \tilde u_{xx} - f(x,\tilde u) = u_t - u_{xx} - f(x,u) - \dot\zeta u_t + f(x,u) - f(x,\tilde u) \tag{2.26}$$
$$-\dot q(t)\Gamma(x - X(t-\zeta(t))) + q(t)(1-\dot\zeta)\dot X(t)\Gamma_x(x - X(t-\zeta(t))) + q(t)\Gamma_{xx}(x - X(t-\zeta(t))).$$

As in (2.20) we may bound this sum from above as follows: in the first line we note that $u_t \ge \delta_{2L_0}$ in the region of interest, while $|f(x,u) - f(x,\tilde u)| \le K_f q(t)$. In the second line of (2.26) we use the definition of $q(t)$, the non-positivity of Γ_x and the fact that $\dot\zeta \le 1$ to deduce that

$$\mathcal{N}(\tilde u) \le -\dot\zeta\delta_{2L_0} + K_f q(t) + \omega q(t) + C_\Gamma q(t), \tag{2.27}$$

with δ_{2L_0} as in (1.9) and C_Γ defined in (2.5). The sum (2.27) will be non-positive under the condition

$$\dot\zeta \ge \frac{K_f + \omega + C_\Gamma}{\delta_{2L_0}} q(t), \tag{2.28}$$

that is, provided that

$$B \ge \frac{K_f + \omega + C_\Gamma}{\delta_{2L_0}},$$

which is ensured by our choice of B in (2.5). Therefore, $\tilde u(t,x)$ is a sub-solution for $z(t,x)$. Moreover, $\xi_1^- = -\sup_{t\to+\infty}|\zeta(t)|$ satisfies

$$\xi_1^- \ge \xi_0^- - K_0\varepsilon.$$

This finishes the proof of Proposition 2.1. □

2.2 Convergence to a transition front: proof of Theorem 1.1

As we have mentioned, proving Theorem 1.1 now amounts to improving the distance between the waves controlling the solution $z(t,x)$ from below and above in (2.12). This is done in two steps described in the following proposition. In the first step one brings the two bounding fronts sufficiently close, and in the second one shows that after that they are within a fixed distance from each other, the shifts converge exponentially.

Proposition 2.2 *Assume that the initial data $z_0(x)$ is as in Theorem 1.1 and that $\xi_0^+ > \xi_0^-$ are defined by (2.4). Let $\varepsilon_0 > 0$ be as in Proposition 2.1. Then the following hold:*
(i) There exists a time $T > 0$ and two shifts $\xi_T^+ > \xi_T^-$ so that $|\xi_T^+ - \xi_T^-| \le 1$, and

$$u(T+\xi_T^-, x) - q_0\Gamma(x - X(T+\xi_T^-)) \le z(T,x) \le u(T+\xi_T^+, x) + q_0\Gamma(x - X(T+\xi_T^+)), \tag{2.29}$$

with $0 \le q_0 \le \min(\varepsilon_0, \xi_0^+ - \xi_0^-)$ and the function $\Gamma(x)$ defined in (2.2) . The time T depends on the initial data $z_0(x)$ only through the decay rate α_0 and the difference $\xi_0^+ - \xi_0^-$.
(ii) There exists a time step $T' > 0$ and constants $K_1 > 0$ and $\gamma \in (0,1)$ which do not depend on z_0, so that for $t \ge t_n = T + nT'$ we have

$$u(t+\xi_n^-, x) - q_n e^{-\omega(t-t_n)} \le z(t,x) \le u(t+\xi_n^+, x) + q_n e^{-\omega(t-t_n)}, \tag{2.30}$$

with the sequences $\xi_n^+ \ge \xi_n^-$ such that $0 \le q_n \le \min(\epsilon_0, \xi_n^+ - \xi_n^-)$,

$$0 \le \xi_n^+ - \xi_n^- \le K_1\gamma^n, \tag{2.31}$$

and

$$\xi_{n+1}^+ \le \xi_n^+ + K_1(\xi_n^+ - \xi_n^-), \quad \xi_{n+1}^- \ge \xi_n^- - K_1(\xi_n^+ - \xi_n^-). \tag{2.32}$$

The main point here is (2.31): the difference between the shifts goes down by a fixed factor after each time step T', which, of course, is the core of the exponential convergence. The reason why we need to split this proposition in two steps is that unless we know that the "starting" difference $\xi_0^+ - \xi_0^-$ at the time $t = T$ is sufficiently small, we can not get the exponential rate of convergence independent of the initial data. This, of course, is common in such situations: first, solutions have to get in a prescribed ball and then they start approaching each other at what essentially is a "linearized" rate. Because the time T depends only on the difference $\xi_0^+ - \xi_0^-$ and the decay rate α_0, the constant C in Theorem 1.1 depends on the initial data only through the difference $\xi_0^+ - \xi_0^-$ and the decay rate α_0.

It is easy to see from (2.31) and (2.32) that the sequence ξ_n^+ is bounded from above and the sequence ξ_n^- is bounded from below. As $\xi_n^- \le \xi_n^+$, both sequences are bounded from above and below. It is also easy to see that (2.31) and (2.32) preclude each of the sequences ξ_n^- and ξ_n^+ from having two different limit points. Hence, each of them converges, and (2.31) implies that the limits $\bar{\xi}^+$ and $\bar{\xi}^-$ of ξ_n^+ and ξ_n^-, respectively, coincide: $\bar{\xi}^+ = \bar{\xi}^-$. We set $\bar{\tau} = \bar{\xi}^+$ and observe from (2.31), (2.32) that

$$\xi_n^- - C\gamma^n \le \xi_{n+1}^- \le \xi_{n+1}^+ \le \xi_n^+ + C\gamma^n, \tag{2.33}$$

with a constant C which depends only on K_1. This together with (2.31), in turn implies that $|\xi_{n+1}^- - \xi_n^-| \le C\gamma^n$. It follows that $|\bar{\tau} - \xi_n^-| \le C\gamma^n$ and, similarly $|\bar{\tau} - \xi_n^+| \le C\gamma^n$. Finally, we note that $0 \le q_n \le \xi_n^+ - \xi_n^- \le C\gamma^n$.

Therefore, we have the following situation for $t \ge t_n = T + nT'$:

$$u(t + \xi_n^-, x) - C\gamma^n e^{-\omega(t-t_n)} \le z(t,x) \le u(t + \xi_n^+, x) + C\gamma^n e^{-\omega(t-t_n)}.$$

As $|\bar{\tau} - \xi_n^\pm| \le C\gamma^n$, we deduce that for $t \ge T + nT'$ the following bound holds:

$$u(t + \bar{\tau}, x) - C\gamma^n \left[1 + e^{-\omega(t-t_n)}\right] \le z(t,x) \le u(t + \bar{\tau}, x) + C\gamma^n \left[1 + e^{-\omega(t-t_n)}\right],$$

and thus $|z(t,x) - u(t + \bar{\tau}, x)| \le C\gamma^n$ for $t \ge T + nT'$. This proves the exponential convergence stated in Theorem 1.1. □

Proof of Proposition 2.2(i)

It remains to prove Proposition 2.2 to finish the proof of Theorem 1.1. The proofs of the two parts in this proposition are very similar, the difference being that in the proof of the first part constants depend on the initial data z_0 while in the second part they do not.

Given the initial data z_0 we use Proposition 2.1 to find two shifts $\xi_0^\pm$ so that

$$u(t + \xi_0^-, x) - q_0 e^{-\omega t}\Gamma(x - X(t + \xi_0^-)) \le z(t,x) \le u(t + \xi_0^+, x) + q_0\Gamma(x - X(t + \xi_0^+)) \tag{2.34}$$

for all $t \ge 0$. We are going to find a time-step s_0 which depends on the function $z_0(x)$ so that for $t \ge s_n = ns_0$ we have

$$u(t+\xi_n^-, x) - q_n e^{-\omega(t-s_n)}\Gamma(x - X(t+\xi_n^-)) \le z(t,x) \le u(t+\xi_n^+, x) + q_n e^{-\omega(t-s_n)}\Gamma(x - X(t+\xi_n^+)), \tag{2.35}$$

with the constants $\xi_n^+ \ge \xi_n^-$ satisfying

$$\xi_0^- - 1 \le \xi_n^- \le \xi_n^+ \le \xi_0^+ + 1, \quad 0 \le \xi_n^+ - \xi_n^- \le D\gamma_0^n, \quad 0 \le q_n \le \xi_n^+ - \xi_n^-, \tag{2.36}$$

and with a constant $\gamma_0 \in (0,1)$ which depends on z_0. Then we will take $T_0 = Ns_0$ and $\xi_T^\pm = \xi_N^\pm$ with a sufficiently large N to satisfy the conditions in part (i) of Proposition 2.2.

The induction is initialized with (2.34) and Proposition 2.1. Let us assume now that (2.35) holds and try to do the induction step from n to $n+1$. We first consider the set

$$\Omega_R = \{(t,x) \in \mathbb{R}^2 \ : \ |x - X(t + (\xi_0^+ + \xi_0^-)/2)| \leq R\} \tag{2.37}$$

with $R > 2L_0 > 0$ chosen sufficiently large so that Ω_R contains all points (t,x) where both $u(t + \xi_0^- - 1, x) \leq (1+\theta_1)/2$ and $u(t + \xi_0^+ + 1, x) \geq \theta_0/2$ hold. Due to property P4, R depends only on the difference $\xi_0^+ - \xi_0^-$. By the induction assumption, $\xi_n^\pm$ satisfies $\xi_0^- - 1 \leq \xi_n^- \leq \xi_n^+ \leq \xi_0^+ + 1$ so that either

$$u(t + \xi_n^-, x) \geq \frac{(1+\theta_1)}{2} \quad \text{or} \quad u(t+\xi_n^+, x) \leq \frac{\theta_0}{2} \tag{2.38}$$

holds for all $(t,x) \in \mathbb{R}^2 \setminus \Omega_R$, for all n.

We will use $u_n^+(t,x)$ and $u_n^-(t,x)$ to denote the functions $u(t+\xi_n^+, x)$ and $u(t+\xi_n^-, x)$, respectively. From Property P3, we know that for all $(t,x) \in \Omega_{3R}$ we have

$$u_n^+(t,x) - u_n^-(t,x) \geq K_2(\xi_n^+ - \xi_n^-) \tag{2.39}$$

with the constant $K_2 = \delta_{3R}$ being independent of n. Define $\bar{s}_n = s_n + \bar{s}$ with $\bar{s} = (\log(p/K_2))/\omega$ so that the exponentially in time decaying terms in (2.35) are small compared to $K_2(\xi_n^+ - \xi_n^-)$ in Ω_{3R} for $t \geq \bar{s}_n$. Here $p > 1$ is a large constant that we will set later. Then, if $\epsilon_n = (\xi_n^+ - \xi_n^-)/(p\delta_{3R})$,

$$u_n^+(t+\epsilon_n, x) \geq u(t + \xi_n^+, x) + \epsilon_n \delta_{3R} \geq z(t,x)$$

and

$$u_n^-(t - \epsilon_n, x) \leq u(t + \xi_n^-, x) - \epsilon_n \delta_{3R} \leq z(t,x)$$

if $t \geq \bar{s}_n$ and $(t,x) \in \Omega_{3R}$. So, the functions $u_n^+(t+\epsilon_n, x) - z(t,x)$ and $u_n^-(t-\epsilon_n, x) - z(t,x)$ are strictly positive and strictly negative, respectively, within the region Ω_{3R} wherever $t \geq \bar{s}_n$. This enables us to apply the Harnack inequality, since $u_n^+(t+\epsilon_n, x)$, $u_n^-(t-\epsilon_n, x)$, and z satisfy the same equation.

Equation (2.39) implies that for any $t \geq \bar{s}_n$, either

$$\sup_{\Omega_{2R} \cap \{t\} \times \mathbb{R}} (u_n^+(t+\epsilon_n, x) - z(t,x)) \geq \frac{K_2}{2}(\xi_n^+ - \xi_n^-) \tag{2.40}$$

or

$$\sup_{\Omega_{2R} \cap \{t\} \times \mathbb{R}} (z(t,x)) - u_n^-(t-\epsilon_n, x)) \geq \frac{K_2}{2}(\xi_n^+ - \xi_n^-). \tag{2.41}$$

must hold. For some constants $\sigma \geq 1$ and $\tau \geq 1$ to be chosen later, we apply the Harnack inequality to obtain a constant $q_0 \in (0,1)$ – maybe extremely small, and depending on the initial data z_0 via the shifts $\xi_0^\pm$, and on the parameter τ – such that

$$\inf_{\Omega_R \cap [\bar{s}_n + \sigma, \bar{s}_n + \sigma + \tau] \times \mathbb{R}} (u_n^+(t+\epsilon_n, x) - z(t,x)) \geq \frac{q_0 K_2}{2}(\xi_n^+ - \xi_n^-) \tag{2.42}$$

if (2.40) holds at $s = \bar{s}_n + \sigma/2$, and

$$\inf_{\Omega_R \cap [\bar{s}_n + \sigma, \bar{s}_n + \sigma + \tau] \times \mathbb{R}} (z(t,x)) - u_n^-(t-\epsilon_n, x)) \geq \frac{q_0 K_2}{2}(\xi_n^+ - \xi_n^-) \tag{2.43}$$

if (2.41) holds at $s = \bar{s}_n + \sigma/2$. It is important to observe here that the Harnack constant q_0 depends on σ and τ, but it does not depend on the factor p or on s_n. Assume from now on that (2.40) and

(2.42) hold, the other case can be treated in a similar fashion. Set $r_0 = q_0 K_2/2$. Due to property P3, we know that for $t \in [\bar{s}_n + \sigma, \bar{s}_n + \sigma + \tau]$, $d \in (0,1)$, and $(t,x) \in \Omega_R$:

$$\begin{aligned} u_n^+(t - dr_0(\xi_n^+ - \xi_n^-), x) - z(t,x) &\geq u_n^+(t + \epsilon_n, x) - z(t,x) - C(\epsilon_n + dr_0(\xi_n^+ - \xi_n^-)) \\ &= u_n^+(t + \epsilon_n, x) - z(t,x) - C(\xi_n^+ - \xi_n^-)(\frac{1}{p\delta_{3R}} + dr_0) \\ &\geq (\xi_n^+ - \xi_n^-)\left(r_0 - C(\frac{1}{p\delta_{3R}} + dr_0)\right), \end{aligned}$$

with $C = \sup_{t,x} u_t$. Since r_0 is independent of the factor p, we may choose p large and d small, independently of n, so that the right hand side is positive for $(t,x) \in \Omega_R$ and $t \in [\bar{s}_n + \sigma, \bar{s}_n + \sigma + \tau]$:

$$u_n^+(t - dr_0(\xi_n^+ - \xi_n^-), x) - z(t,x) \geq 0 \tag{2.44}$$

Now, let us consider what happens when $(t,x) \notin \Omega_R$.

1. Behind the front. We first look at the part of $\mathbb{R} \setminus \Omega_R$ which is behind the front:

$$\tilde{\Omega}_R^- := \left\{(t,x) : x \leq X(t + (\xi_0^+ + \xi_0^-)/2) - R, \quad t \in [\bar{s}_n + \sigma, \bar{s}_n + \sigma + \tau]\right\}.$$

By our choice of R, $u(t + \xi_0^- - 1, x) \geq (1+\theta_1)/2$ in this region. Because $dr_0 < 1/2$, we have

$$\xi_0^- - 1 \leq \xi_n^- \leq \xi_n^+ - dr_0(\xi_n^+ - \xi_n^-),$$

and thus

$$u_n^+(t - dr_0(\xi_n^+ - \xi_n^-), x) \geq u(t + \xi_n^-, x) \geq u(t + \xi_0^- - 1, x) \geq (1+\theta_1)/2$$

in $\tilde{\Omega}_R^-$. Moreover, for $(t,x) \in \tilde{\Omega}_R^-$ we have

$$z(t,x) \geq u(t + \xi_n^-, x) - q_n e^{-\omega(t - s_n)} \geq (1+\theta_1)/2 - (\xi_0^+ - \xi_0^-)e^{-\omega\sigma} \geq \theta_1,$$

provided that we choose σ sufficiently large, independently of n. Hence, the function

$$v(t,x) := u_n^+(t - dr_0(\xi_n^+ - \xi_n^-), x) - z(t,x)$$

satisfies

$$v_t - v_{xx} = f(u(t + \xi_n^+ - dr_0(\xi_n^+ - \xi_n^-), x) - f(z(t,x)) = a(t,x)v, \tag{2.45}$$

with $a(t,x) \leq -\beta$ in $\tilde{\Omega}_R^-$.

Moreover, because $u_t > 0$, the boundary of $\tilde{\Omega}_R^-$ is a smooth curve. The function $v(t,x)$ is non-negative on this curve because of (2.44). Initially, at time $t = \bar{s}_n + \sigma$ we have, with $C = \sup_{t,x} u_t$:

$$\begin{aligned} v(\bar{s}_n + \sigma, x) &= u(\bar{s}_n + \sigma + \xi_n^+ - dr_0(\xi_n^+ - \xi_n^-), x) - z(\bar{s}_n + \sigma, x) \\ &\geq -q_n e^{-\omega\sigma} - Cdr_0(\xi_n^+ - \xi_n^-) \geq -(\xi_n^+ - \xi_n^-)e^{-\omega\sigma} - Cdr_0(\xi_n^+ - \xi_n^-) \end{aligned}$$

for all $x \in \mathbb{R}$. We used here the induction assumption $0 \leq q_n \leq \xi_n^+ - \xi_n^-$. Consequently, applying the comparison principle to (2.45) in the temporal-spatial domain $\tilde{\Omega}_R^-$ we conclude that

$$v(t,x) \geq -C\left[e^{-\omega\sigma} + dr_0\right](\xi_n^+ - \xi_n^-)e^{-\beta(t - \bar{s}_n - \sigma)} \text{ for } (t,x) \in \tilde{\Omega}_R^-. \tag{2.46}$$

2. Ahead of the front. Consider next the region in $\mathbb{R} \setminus \Omega_R$ ahead of front:

$$\tilde{\Omega}_R^+ := \left\{(t,x) : x \geq X(t + (\xi_0^+ + \xi_0^-)/2) + R, \quad t \in [\bar{s}_n + \sigma, \bar{s}_n + \sigma + \tau]\right\}.$$

By our choice of R, $u(t+\xi_0^+ + 1, x) \le \theta_0/2$ in this region. Again, we define the difference

$$v(t,x) = u_n^+(t - dr_0(\xi_n^+ - \xi_n^-), x) - z(t,x)$$

and bound it from below as follows. Note that both $u_n^+(t - dr_0(\xi_n^+ - \xi_n^-), x)$ and $z(t,x)$ are below the ignition threshold in $\tilde{\Omega}_R^+$, for a sufficiently large time-step σ:

$$u_n^+(t - dr_0(\xi_n^+ - \xi_n^-), x) \le u(t+\xi_n^+, x) \le u(t+\xi_0^+ + 1, x) \le \theta_0/2$$

in $\tilde{\Omega}_R^+$, and

$$z(t,x) \le u(t+\xi_n^+, x) + (\xi_n^+ - \xi_n^-)e^{-\omega(t-s_n)} \le u(t+\xi_0^+ + 1, x) + (\xi_0^+ - \xi_0^-)e^{-\omega\sigma} \le \theta_0$$

if $\sigma > 0$ is large enough (depending on the difference $\xi_0^+ - \xi_0^-$) and $(t,x) \in \tilde{\Omega}_R^+$. Therefore, we have

$$v_t - v_{xx} = 0 \text{ in } \tilde{\Omega}_R^+. \tag{2.47}$$

Moreover, at the time $t = \bar{s}_n + \sigma$ we have

$$\begin{aligned} v(\bar{s}_n + \sigma, x) &= u(\bar{s}_n + \xi_n^+ + \sigma - dr_0(\xi_n^+ - \xi_n^-), x) - z(\bar{s}_n + \sigma, x) \\ &\ge -q_n e^{-\omega\sigma - \alpha(x - X(\bar{s}_n + \sigma + \xi_n^+) - L_0)} - C dr_0(\xi_n^+ - \xi_n^-)e^{-\alpha(x - X(\bar{s}_n + \sigma + \xi_n^+))} \\ &\ge -C\left[e^{-\omega\sigma} + dr_0\right](\xi_n^+ - \xi_n^-)e^{-\alpha(x - X(\bar{s}_n + \sigma + \xi_n^+))} \end{aligned}$$

for all $x \in \mathbb{R}$. We used above the exponential decay estimate for the time derivative of $u(t,x)$:

$$|u(t,x) - u(t-s,x)| = s u_t(t - s', x) \le C s e^{-\alpha(x - X(t-s'))} \le C s e^{-\alpha(x - X(t))},$$

for $s \ge 0$, with some $s' \in (0,s)$. The boundary of $\tilde{\Omega}_R^+$ is a smooth curve in $\mathbb{R}^2$ on which the function $v(t,x)$ is non-negative because of (2.44). Furthermore, just as in the proof of Proposition 2.1, we notice that the function $q(t,x) = -e^{-\alpha(x - X(t+\xi_n^+)) - \nu t}$ is a sub-solution to the equation (2.47) for v, where ν is defined in (2.10). It follows that

$$v(t,x) \ge -C(\xi_n^+ - \xi_n^-)(e^{-\omega\sigma} + dr_0)e^{-\alpha(x - X(t+\xi_n^+)) - \nu(t - \bar{s}_n - \sigma)} \text{ for } (t,x) \in \tilde{\Omega}_R^+. \tag{2.48}$$

Shrinking $\xi_n^+ - \xi_n^-$

Let us now summarize the above computation and put together the estimates (2.44), (2.46) and (2.48). We have shown that if (2.40) holds (rather than (2.41)) then at times $t \in (\bar{s}_n + \sigma, \bar{s}_n + \sigma + \tau)$ we have

$$u(t + \xi_n^+ - dr_0(\xi_n^+ - \xi_n^-), x) \ge z(t,x)$$

for $(t,x) \in \Omega_R$,

$$u(t + \xi_n^+ - dr_0(\xi_n^+ - \xi_n^-), x) + C\left(e^{-\omega\sigma} + dr_0\right)(\xi_n^+ - \xi_n^-)e^{-\beta(t - \bar{s}_n - \sigma)} \ge z(t,x) \tag{2.49}$$

for $(t,x) \in \tilde{\Omega}_R^-$, and

$$u(t + \xi_n^+ - dr_0(\xi_n^+ - \xi_n^-), x) + C(e^{-\omega\sigma} + dr_0)(\xi_n^+ - \xi_n^-)e^{-\alpha(x - X(t+\xi_n^+)) - \nu(t - \bar{s}_n - \sigma)} \ge z(t,x) \tag{2.50}$$

for $(t,x) \in \tilde{\Omega}_R^+$. Of course, we also still have the lower bound intact:

$$z(t,x) \ge u(t + \xi_n^-, x) - q_n e^{-\omega(t - s_n)}\Gamma(x - X(t + \xi_n^-)) \tag{2.51}$$

for all $t \geq s_n$ and $x \in \mathbb{R}$.

We may now set $s_{n+1} = \bar{s}_n + \sigma + \tau$, and the new shifts as

$$\xi^+_{n+1/2} = \xi^+_n - dr_0(\xi^+_n - \xi^-_n), \quad \xi^-_{n+1/2} = \xi^-_n,$$

and the new correction

$$q_{n+1/2} = [(\xi^+_n - \xi^-_n)e^{-\omega\sigma} + C(\xi^+_n - \xi^-_n)(e^{-\omega\sigma} + dr_0)]e^{-\omega\tau}.$$

The first term in the parentheses above comes from the lower bound in (2.51), using also the induction assumption $0 \leq q_n \leq \xi^+_n - \xi^-_n$, while the second accounts for the terms appearing in the left side of (2.49) and (2.50). Then at $t = s_{n+1}$ we have

$$\begin{aligned} &u(s_{n+1} + \xi^-_{n+1/2}, x) - q_{n+1/2}\Gamma(x - X(s_{n+1} + \xi^-_{n+1/2})) \leq z(s_{n+1}, x) \\ &\quad \leq u(s_{n+1} + \xi^+_{n+1/2}, x) + q_{n+1/2}\Gamma(x - X(s_{n+1} + \xi^+_n)). \end{aligned} \tag{2.52}$$

In order to correct the argument of the function Γ in the right side above, observe that $X(s_{n+1} + \xi^+_n) - X(s_{n+1} + \xi^+_{n+1/2}) \leq Cdr_0(\xi^+_n - \xi^-_n)$. This and the fact that Γ is non-increasing implies that for $x < X(s_{n+1} + \xi^+_n) + L_0 + 1$,

$$\begin{aligned} &\Gamma(x - X(s_{n+1} + \xi^+_n)) \\ &\leq \left[\Gamma(X(s_{n+1} + \xi^+_n) + L_0 + 1 - X(s_{n+1} + \xi^+_{n+1/2}))\right]^{-1} \Gamma(x - X(s_{n+1} + \xi^+_{n+1/2})) \\ &= e^{\alpha(X(s_{n+1}+\xi^+_n) - X(s_{n+1}+\xi^+_{n+1/2})+1)}\Gamma(x - X(s_{n+1} + \xi^+_{n+1/2})) \\ &\leq e^{\alpha}e^{\alpha Cdr_0(\xi^+_n - \xi^-_n)}\Gamma(x - X(s_{n+1} + \xi^+_{n+1/2})) \leq C\Gamma(x - X(s_{n+1} + \xi^+_{n+1/2})), \end{aligned}$$

as $dr_0(\xi^+_n - \xi^-_n) < 1/2$. For $x \geq X(s_{n+1} + \xi^+_n) + L_0 + 1$ the definition of Γ implies

$$\begin{aligned} \Gamma(x - X(s_{n+1} + \xi^+_n)) &= e^{-\alpha(x - X(s_{n+1}+\xi^+_n) - L_0)} \\ &\leq e^{\alpha Cdr_0(\xi^+_n - \xi^-_n)}e^{-\alpha(x - X(s_{n+1}+\xi^+_{n+1/2}) - L_0)} \\ &= e^{\alpha Cdr_0(\xi^+_n - \xi^-_n)}\Gamma(x - X(s_{n+1} + \xi^+_{n+1/2})) \leq C\Gamma(x - X(s_{n+1} + \xi^+_{n+1/2})). \end{aligned}$$

With these bounds (2.52) takes the form

$$\begin{aligned} &u(s_{n+1} + \xi^-_{n+1/2}, x) - q'_{n+1/2}\Gamma(x - X(s_{n+1} + \xi^-_{n+1/2})) \leq z(s_{n+1}, x) \\ &\quad \leq u(s_{n+1} + \xi^+_{n+1/2}, x) + q'_{n+1/2}\Gamma(x - X(s_{n+1} + \xi^+_{n+1/2})). \end{aligned} \tag{2.53}$$

Here we have defined $q'_{n+1/2} = Cq_{n+1/2}$.

We also have $\xi^+_{n+1/2} - \xi^-_{n+1/2} = (1 - dr_0)(\xi^+_n - \xi^-_n)$ with $dr_0 \in (0, 1)$, and, finally,

$$\begin{aligned} q'_{n+1/2} &\leq C(\xi^+_n - \xi^-_n)\left[e^{-\omega\sigma} + dr_0\right]e^{-\omega\tau} \leq C(\xi^+_n - \xi^-_n)e^{-\omega\tau} = (\xi^+_{n+1/2} - \xi^-_{n+1/2})\frac{e^{-\omega\tau}}{1 - dr_0} \\ &\leq \xi^+_{n+1/2} - \xi^-_{n+1/2}, \end{aligned}$$

provided that τ is sufficiently large. Therefore, we are in a position to apply Proposition 2.1 which implies that for $t \geq s_{n+1}$ we have

$$\begin{aligned} &u(t + \xi^-_{n+1}, x) - q_{n+1}e^{-\omega(t - s_{n+1})}\Gamma(x - X(t + \xi^-_{n+1})) \leq z(t, x) \\ &\leq u(t + \xi^+_{n+1}, x) + q_{n+1}e^{-\omega(t - s_{n+1})}\Gamma(x - X(t + \xi^+_{n+1})). \end{aligned}$$

with

$$q_{n+1} = q'_{n+1/2},\ \xi^+_{n+1} = \xi^+_{n+1/2} + K_0 q'_{n+1/2} \text{ and } \xi^-_{n+1} = \xi^-_{n+1/2} - K_0 q'_{n+1/2}.$$

The new shift ξ^+_{n+1} can be estimated as follows:

$$\begin{aligned} \xi^+_{n+1} &= \xi^+_n - dr_0(\xi^+_n - \xi^-_n) + C(\xi^+_n - \xi^-_n)(e^{-\omega\sigma} + dr_0)e^{-\omega\tau} \\ &\le \xi^+_n - (\xi^+_n - \xi^-_n)\left[dr_0 - C(e^{-\omega\sigma} + dr_0)e^{-\omega\tau}\right], \end{aligned} \tag{2.54}$$

while the other shift, ξ^-_{n+1}, is bounded as

$$\xi^-_{n+1} = \xi^-_{n+1/2} - Cq'_{n+1/2} = \xi^-_n - Cq_{n+1/2} \ge \xi^-_n - C(\xi^+_n - \xi^-_n)(e^{-\omega\sigma} + dr_0)e^{-\omega\tau}. \tag{2.55}$$

We conclude that

$$\xi^+_{n+1} - \xi^-_{n+1} \le (\xi^+_n - \xi^-_n)(1 - dr_0 + C(e^{-\omega\sigma} + dr_0)e^{-\omega\tau}).$$

Recall that r_0 depends on the Harnack constant over the time interval of length τ – therefore, we have no explicit control on how it depends on τ. Nevertheless, by taking first τ sufficiently large and then choosing σ large enough (probably, "extremely" large so that the term $e^{-\omega\sigma}$ would be smaller than Cr_0, which is already very small for τ large), we can ensure that

$$\xi^+_{n+1} - \xi^-_{n+1} \le (1-\varepsilon)(\xi^+_n - \xi^-_n) \tag{2.56}$$

with $\varepsilon = dr_0/10 > 0$. While (2.54) and (2.55) were obtained under the assumption that (2.40) rather than (2.41) holds, the geometric bound (2.56) holds in either case. Moreover, in either case we have the following bounds modifying (2.54) and (2.55), implied by (2.58):

$$\xi^-_n - \varepsilon(\xi^-_n - \xi^+_n) \le \xi^-_{n+1} \le \xi^+_{n+1} \le \xi^+_n + \varepsilon(\xi^+_n - \xi^-_n), \tag{2.57}$$

provided that τ and σ are sufficiently large. As $\varepsilon < (\xi^+_0 - \xi^-_0)/100$, it follows that the induction assumption $\xi^-_{n+1} \ge \xi^-_0 - 1$ and $\xi^+_{n+1} \le \xi^+_0 + 1$ holds at each step, and, eventually, we obtain a sequence of shifts ξ^+_n and ξ^-_n so that

$$0 \le \xi^+_n - \xi^-_n \le D(1-\varepsilon)^n. \tag{2.58}$$

This proves part (i) of Proposition 2.2.

Proof of Proposition 2.2(ii)

Part (ii) of Proposition 2.2 is proved using exactly the same induction procedure as in the proof of the first part of this proposition. Namely, we stop the original iteration process at the time T and start it completely anew, with a new time-step T'. The main difference is that this time we start the iteration after the time T so that "initially" we have $|\xi^+_T - \xi^-_T| \le 1$. Therefore, all constants appearing in the estimates of the preceding step are now independent of ξ^+_T and ξ^-_T, as they depended only on α_0 and the initial separation $\xi^+_0 - \xi^-_0$. In particular, the parameters τ' and σ', as well as r'_0 can be now chosen independent of z_0. Hence, the time step $T' = \tau' + \sigma'$ required to do the iteration after $t = T$ is independent of the initial data z_0. Therefore, the new sequence of shifts $\tilde{\xi}^+_n$ and $\tilde{\xi}^-_n$ obtained at times $s'_n = T + nT'$ satisfies

$$\xi^-_n - C\gamma^n \le \xi^-_{n+1} \le \xi^+_{n+1} \le \xi^+_n + C\gamma^n,\ \ 0 \le \xi^+_n - \xi^-_n \le C\gamma^n \tag{2.59}$$

with the constants $C > 0$ and $\gamma \in (0,1)$ independent of the initial data z_0. This finishes the proof of Proposition 2.2. □

3 Uniqueness of the transition front

As we have mentioned in the introduction, the proof of uniqueness proceeds in three steps. First, we show that any transition front (and not only those constructed in [19] and [20]) are monotonic in time. Second, we show that such a front solution is exponentially decaying in x ahead of the interface. Finally, we use these two properties together with global in time stability of the fronts to establish uniqueness.

Monotonicity of transition fronts

We recall that monotonicity in time of the transition fronts has been announced in [2]. We present the proof for the convenience of the reader.

Proposition 3.1 *Let $\psi(t,x)$ be a transition front solution of (1.3). Then $\psi_t(t,x) > 0$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}$.*

Proof. We begin with the following lemma.

Lemma 3.2 *Let $\psi(t,x)$ be a transition front solution of (1.3). There exists $h_0 > 0$ so that for any $h \geq h_0$ we have $\psi(t+h,x) \geq \psi(t,x)$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}$.*

Proof of Lemma 3.2. We recall that $\theta_1 \in (0,1)$ is such that $f_0'(u) \leq f_0'(1)/2$ for all $u \in [\theta_1, 1]$ and we set

$$\Omega = \left\{(t,x) \in \mathbb{R}^2 : \ \theta_0/2 \leq \psi(t,x) \leq (1+\theta_1)/2\right\}. \tag{3.1}$$

Let

$$\Omega_t = \{x \in \mathbb{R} : \ (t,x) \in \Omega\} \tag{3.2}$$

be the cross-section of Ω with t fixed. The definition of the transition front implies that the width of Ω_t is uniformly bounded in time: there exists $l \geq 0$ so that for all $t \in \mathbb{R}$ we have $\sup \Omega_t - \inf \Omega_t \leq l$. We will consider separately the points in Ω_t, those with $x > \sup \Omega_t$ or $x < \inf \Omega_t$, and, finally, "the bad points in the middle", that is, $x \in (\inf \Omega_t, \sup \Omega_t) \setminus \Omega_t$.

The middle region – good points. Let us first show that the conclusion of lemma holds in the set Ω: there exists $h_1 > 0$ so that we have

$$\psi(t,x) \leq \psi(t+h,x) \text{ for all } (t,x) \in \Omega \text{ and any } h \geq h_1. \tag{3.3}$$

Assume that this is false. Then there exist a sequence of real numbers h_n such that $h_n \to +\infty$ as $n \to +\infty$ and a sequence of points $(t_n, x_n) \in \Omega$ so that

$$\psi(t_n + h_n, x_n) \leq \psi(t_n, x_n) \leq (1+\theta_1)/2. \tag{3.4}$$

As $\psi(t_n, x_n) \geq \theta_0/2$, it follows from the definition of the transition front that there exists $M > 0$ so that for all $x \leq x_n - M$ we have $\psi(t_n, x) \geq \theta_1$. Consider the function $q(t,x)$ which solves the following Cauchy problem for $t \geq 0$:

$$q_t - q_{xx} = g^{min} f_0(q), \quad q(0,x) = \begin{cases} \theta_1, & x \leq 0 \\ 0, & x \geq 0. \end{cases}$$

Thus $\psi(t_n, x) \geq q(0, x - x_n + M)$ for all x and $\psi(t_n + h, x) \geq q(h, x - x_n + M)$ for all $h \geq 0$ and $x \in \mathbb{R}$. There exists a time $\eta_0 > 0$ so that for all $t \geq \eta_0$ we have

$$\inf_{x \leq M} q(t,x) \geq \frac{2+\theta_1}{3}.$$

In particular, for $h_n \geq \eta_0$ we have

$$\psi(t_n + h_n, x_n) \geq q(h_n, M) \geq \frac{2+\theta_1}{3} > \frac{1+\theta_1}{2},$$

which contradicts (3.4). Therefore, we can find $h_1 > 0$ so that (3.3) holds.

Next, we analyze what happens outside the set Ω. An argument similar to the one above shows that there exists $h_2 \geq h_1$ so that we have

$$\psi(t+h, x) \geq (1+\theta_1)/2 \quad \text{if} \quad h \geq h_2 \ \text{ and } \ x \leq \sup \Omega_t. \tag{3.5}$$

This is seen as follows: from the definition of the transition front there exists l_1 so that $\psi(t,x) \geq \theta_1$ for all $x \leq \sup \Omega_t - l_1$ and all $t \in \mathbb{R}$. Then we have $\psi(t+h,x) \geq q(h, x + l_1 - \sup \Omega_t)$ for all $h \geq 0$. Hence, in order to ensure that (3.5) holds it suffices to choose a time h_2 so that $q(h_2, y) \geq (1+\theta_1)/2$ for all $y \leq l_1$.

We will show below that the function

$$r(t,x) = \psi(t+h_2, x) - \psi(t,x) \tag{3.6}$$

is positive for all $x \in \mathbb{R}$ and $t \in \mathbb{R}$. It would then follow immediately from (3.6) that

$$\psi(t+h,x) - \psi(t,x) \geq 0 \text{ for all } h \geq h_2.$$

Note that the preceding argument proves that $r(t,x) \geq 0$ for $x \in \Omega_t$, hence we have to deal only with $x \notin \Omega_t$.

The region on the left. Consider now what happens to the left of Ω, that is, on the set of points (t,x) such that $x \leq \inf \Omega_t$. The function $r(t,x)$ satisfies

$$\begin{aligned} & r_t - r_{xx} + a(t,x) r = 0, \quad t \in \mathbb{R}, \quad x \leq \inf \Omega_t, \\ & r(t,x) \geq 0 \text{ on the set } \{t \in \mathbb{R}, \ x = \inf \Omega_t\}, \end{aligned} \tag{3.7}$$

with the function

$$a(t,x) = -g(x) \frac{f_0(\psi(t+h_2,x)) - f_0(\psi(t,x))}{\psi(t+h_2,x) - \psi(t,x)}. \tag{3.8}$$

Notice that (3.5) implies $\psi(t+h_2,x) \geq \theta_1$ for $x \leq \inf \Omega_t$. Also, $\psi(t,x) \geq (1+\theta_1)/2$ in this region. Thus, $a(t,x) \geq \beta$ for all $t \in \mathbb{R}$ and $x \leq \inf \Omega_t$, with $\beta > 0$ defined by (2.9). Assume that there exist $t \in \mathbb{R}$ and $x \in \Omega_t$ so that $r(t,x) < 0$ and thus

$$\eta_1 := \inf_{\substack{t \in \mathbb{R} \\ x \leq \inf \Omega_t}} r(t,x) < 0.$$

In this case, there exists a sequence of points (t_n, x_n), with $t_n \in \mathbb{R}$ and $x_n \leq \inf \Omega_{t_n}$, such that

$$r(t_n, x_n) < \eta_1 + \frac{|\eta_1|}{2n}.$$

The definition of the transition front implies that $r(t,x) \to 0$ as $(x - \inf \Omega_t) \to -\infty$, uniformly in t. Hence, $|x_n - \inf \Omega_{t_n}| \leq l_2$ for some $l_2 > 0$ fixed. On the other hand, since $r(t,x) \geq 0$ for $x = \inf \Omega_t$, standard regularity estimates imply that x_n are separated away from the boundary $\{x = \inf \Omega_t\}$: there exists $l_2' \in (0, l_2)$ so that $|x_n - \inf \Omega_t| \geq l_2'$ for all $n \in \mathbb{N}$. Moreover, $r(t,x)$ is uniformly small in a ball of fixed radius around (t_n, x_n):

$$r(t,x) < \eta_1/2 < 0$$

for all (t,x) in the ball $B_n = \{(t,x):\ |x-x_n|^2 + |t-t_n|^2 < l_2'\}$. Hence, the function $r(t,x)$ satisfies

$$r_t - r_{xx} \ge \beta|\eta_1|/2 > 0 \ \text{ for } \ (t,x) \in B_n, \ \ n \in \mathbb{N}.$$

However, we may find a subsequence $n_k \to +\infty$ so that the family of shifted functions

$$r_k(t,x) = r(t + t_{n_k}, x + x_{n_k})$$

converges to a limit $\bar{r}(t,x)$ uniformly in the ball $B_0 := \{(t,x):\ |x|^2 + |t|^2 < l_2'\}$. The function $\bar{r}(t,x)$ satisfies

$$\bar{r}_t - \bar{r}_{xx} \ge \beta|\eta_1|/2 > 0 \text{ for } (t,x) \in B_0$$

and attains its minimum $\bar{r}(0,0) = \eta_1$, inside B_0. Due to the maximum principle, this is a contradiction. Hence, we have $r(t,x) \ge 0$ for all $x \le \inf \Omega_t$.

The region on the right. Now, let us consider what happens to the right of the set Ω_t, where $x \ge \sup \Omega_t$ (and thus $\psi(t,x) \le \theta_0/2$). There the function $r(t,x)$ satisfies

$$r_t - r_{xx} = g_0(x)[f_0(\psi(t+h_2,x)) - f_0(\psi(t,x))] = g_0(x) f_0(\psi(t+h_2,x)) \ge 0. \tag{3.9}$$

Moreover $r(t,x) \ge 0$ on the boundary $\{t \in \mathbb{R},\ x = \sup \Omega_t\}$, due to (3.3). Now, we argue as before: assume that

$$\eta_2 := \inf_{\substack{t \in \mathbb{R} \\ x \ge \sup \Omega_t}} r(t,x) < 0.$$

Then we can find a sequence of points (t_n, x_n) such that $t_n \in \mathbb{R}$, $x_n \ge \sup \Omega_{t_n}$, and

$$r(t_n, x_n) < \eta_2 + \frac{|\eta_2|}{2n}.$$

Let $d_n = \operatorname{dist}((t_n,x_n), \Omega)$. For each n, we have $r(t_n,x_n) < \eta_2/2$ and thus $\psi(t_n,x_n) \ge |\eta_2|/2$. Therefore, we see from the definition of the transition front that the distance $|x_n - \sup \Omega_{t_n}|$ must stay bounded as $n \to \infty$. That is, there exists $l_3 > 0$ such that $d_n \le l_3$ for all n. Also, regularity estimates for ψ (and thus for r) imply that there exists $l_3' \in (0, l_3)$ so that $d_n \ge l_3'$. Now, consider the balls $B_n = \{(t,x):\ |x-x_n|^2 + |t-t_n|^2 < d_n^2\}$. Inside each such ball, $r(t,x)$ satisfies

$$r_t - r_{xx} \ge 0 \text{ for } (t,x) \in B_n.$$

Again, we choose a subsequence $n_k \to +\infty$ so that $d_{n_k} \to d_0 \in [l_3', l_3]$ and the shifted functions $r_k(t,x) = r(t + t_{n_k}, x + x_{n_k})$ converge to a limit $\bar{r}(t,x)$ uniformly in $B_0 := \{(t,x):\ |x|^2 + |t|^2 < d_0^2\}$. The function $\bar{r}(t,x)$ satisfies

$$\bar{r}_t - \bar{r}_{xx} \ge 0 \text{ for } (t,x) \in B_0$$

and attains its minimum $\bar{r}(0,0) = \eta_2$, inside B_0. The maximum principle implies $\bar{r}(x,t) \equiv \eta_2$ throughout $\bar{B}_0$. However, this cannot happen. Indeed, by the definition of d_n there is a sequence of points $(\hat{t}_n, \hat{x}_n) \in \partial B_n$ such that $(\hat{t}_n, \hat{x}_n) \in \Omega$ and thus $r(\hat{t}_n, \hat{x}_n) \ge 0$. Hence, $\bar{r}(t,x) \ge 0$ at some point on the boundary of B_0, so $\bar{r}(t,x)$ cannot be equal to $\eta_2 < 0$ throughout $\bar{B}_0$. Thus, $r(t,x) \ge 0$ for $x \le \sup \Omega_t$.

The middle region – the bad points. Finally, consider the set D of points (t,x) such that $\inf \Omega_t < x < \sup \Omega_t$ but $(t,x) \notin \Omega$. Recall from (3.5) that $\psi(t+h_2,x) \ge (1+\theta_1)/2$ for all (t,x) in this region. So, if $r(t,x) < 0$ at some point $(t,x) \in D$, then we still have $\psi(t,x) > (1+\theta_1)/2$ at this point, as well. Assume that

$$\eta_3 := \inf_{(t,x) \in D} r(t,x) < 0$$

and choose a sequence $(t_n, x_n) \in D$ so that

$$r(t_n, x_n) < \eta_3 + \frac{|\eta_3|}{2n}.$$

We may now proceed in the same way we treated the case $x < \inf \Omega_t$. Indeed, as $r(t,x) \geq 0$ for $x \in \Omega_t$, we may find balls B_n of a fixed radius l_4 around the points (t_n, x_n) which are contained in the set D, and in which $r(t,x) < \eta_3/2$, but $\psi(t,x) \geq \theta_1$ (and hence $\psi(t+h_2, x) \geq \theta_1$ as well). Inside each ball $B_n = \{(t,x) : \ |x - x_n|^2 + |t - t_n|^2 < l_4\}$, the function $r(t,x)$ satisfies

$$r_t - r_{xx} \geq \beta|\eta_3|/2 > 0 \text{ for } (t,x) \in B_n.$$

By taking an appropriate subsequence we derive a contradiction to the maximum principle, as before. This shows that $r(t,x) \geq 0$ for all t and x, and concludes the proof of Lemma 3.2. □

The end of the proof of Proposition 3.1. We may now define $\bar{h}$ as the smallest $h \geq 0$ such that for any $k \geq \bar{h}$ we have $\psi(t+k, x) \geq \psi(t,x)$. In order to show that $\bar{h} = 0$ (and thus conclude the proof of Proposition 3.1) assume that $\bar{h} > 0$. The maximum principle implies that $\psi(t+\bar{h}, x) > \psi(t,x)$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}$. Define the set $\tilde{\Omega}$

$$\tilde{\Omega} = \left\{(t,x) \in \mathbb{R}^2 \ : \ \inf \Omega_t \leq x \leq \sup \Omega_t\right\} \tag{3.10}$$

where Ω_t is still defined by (3.1)-(3.2). We claim that

$$\inf_{(t,x)\in\tilde{\Omega}} \psi(t+\bar{h}, x) - \psi(t,x) > 0, \tag{3.11}$$

Indeed, otherwise, there would exist a sequence $(t_n, x_n) \in \tilde{\Omega}$ so that $r(t_n, x_n) \to 0$, while we also have

$$r(t,x) = \psi(t+\bar{h}, x) - \psi(t,x) \geq 0$$

everywhere. Consider the function

$$r_n(t,x) = \psi(t + t_n + \bar{h}, x + x_n) - \psi(t + t_n, x + x_n) \geq 0.$$

Then $r_n(0,0) \to 0$ as $n \to +\infty$. The Harnack inequality and the regularity of r_n implies that $r_n(t,x) \to 0$ locally uniformly on $\mathbb{R} \times \mathbb{R}$. However, there exists a subsequence $n_k \to +\infty$ so that both $\psi_{n_k}(t,x) = \psi(t - t_{n_k}, x - x_{n_k})$ and $g_{n_k}(x) = g(x - x_{n_k})$ converge locally uniformly to the corresponding limits $\bar{\psi}(t,x)$ and $\bar{g}(x)$, which satisfy $\bar{g}(x) \geq g^{min}$, and

$$\bar{\psi}_t - \bar{\psi}_{xx} = \bar{g}(x) f_0(\bar{\psi}), \quad (t,x) \in \mathbb{R}^2. \tag{3.12}$$

Moreover, as $(x_n, t_n) \in \tilde{\Omega}$, we have $\inf_x \bar{\psi}(t,x) \leq \theta_0/2$ and $\sup_x \bar{\psi}(t,x) \geq (1+\theta_1)/2$ for all $t \in \mathbb{R}$. Finally, as $r_n(t,x) \to 0$, we have $\bar{\psi}(t+\bar{h}, x) = \bar{\psi}(t,x)$ for all t and x. Therefore, if $\bar{h} > 0$ then $\bar{\psi}(t,x)$ is actually a non-constant, space-time global and periodic in time solution of (3.12). This is impossible, however, since $\bar{\psi}(0,x) \geq \theta_1$ for $x \leq x_0$, with a suitable shift x_0 and thus $\bar{\psi}(t,x) \to 1$ as $t \to \infty$. Hence, (3.11) holds.

We deduce from (3.11) and uniform a priori bounds on ψ_t that if $\gamma > 0$ is sufficiently small, then

$$\inf_{(t,x)\in\tilde{\Omega}} \psi(t+\bar{h}-\gamma, x) - \psi(t,x) > 0. \tag{3.13}$$

However, we may now use the same argument as in the proof of Lemma 3.2 to show that if γ is small enough, then (3.13) also holds for the points (t,x) satisfying either $x < \inf \Omega_t$ or $x > \sup \Omega_t$. Notice that the "bad points in the middle" no longer need a special treatment as they are included in the set $\tilde{\Omega}$. To show that (3.13) holds for the points on the left, we need to take γ small enough so that

$$\psi(t+\bar{h}-\gamma,x)\geq\theta_1 \tag{3.14}$$

for $x<\inf\Omega_t$. Since $\psi(t+\bar{h},x)>\psi(t,x)\geq(1+\theta_1)/2$ for all (t,x) satisfying $x<\inf\Omega_t$, uniform bounds on ψ_t imply that (3.14) holds if γ is small. Then we apply the argument as before. For the points on the right, $x>\sup\Omega_t$, the argument applies without modification. Hence (3.13) holds for all points $(t,x)\in\mathbb{R}^2$ if γ is sufficiently small. This contradicts the minimality of $\bar{h}$. Hence, $\bar{h}=0$ and thus $\psi(t,x)$ is monotonic in t. □

We now obtain a global lower bound on the interface speed from Proposition 3.1:

Corollary 3.3 *There exists $\delta>0$ such that $\dot{X}(t)\geq\delta$ for all $t\in\mathbb{R}$.*

Proof. We already proved that $\psi_t>0$. We claim that for every $\varepsilon>0$ there exists $c_\varepsilon>0$ so that $\psi_t(t,x)\geq c_\varepsilon$ for all (t,x) such that $\psi(t,x)\in(\varepsilon,1-\varepsilon)$. This is shown by the argument we used to establish (3.11) at the end of the proof of Proposition 3.1. If there were no such c_ε then we would find a sequence of points (t_n,x_n) such that $\psi(t_n,x_n)\in(\varepsilon,1-\varepsilon)$ and $\psi_t(t_n,x_n)\leq 1/n$. Then we could extract a subsequence of functions $\psi_k(t,x)=\psi(t+t_{n_k},x+x_{n_k})$ converging locally uniformly to a function $\bar{\psi}(t,x)$ which satisfies the equation

$$\bar{\psi}_t-\bar{\psi}_{xx}=\bar{g}(x)f_0(\bar{\psi}),$$

with a function $\bar{g}(x)\geq g^{min}>0$ and $\bar{\psi}(t,x)$ being nonconstant in x. Since

$$\frac{\partial\psi_k(0,0)}{\partial t}\to 0,$$

and $\psi_t>0$, we may apply the Harnack inequality to ψ_t and conclude that $\bar{\psi}_t(t,x)=0$ for all t and x. Thus, the function $\bar{\psi}=\bar{\psi}(x)$ satisfies

$$-\bar{\psi}_{xx}=\bar{g}(x)f_0(\bar{\psi}).$$

However, due to the properties of f_0, there can be no non-constant solution to this equation with $\bar{\psi}\in[0,1]$. So, there exists such a c_ε, as claimed.

Since $|\psi_x|$ is bounded uniformly in t and x, and $\psi_x(t,X(t))\leq 0$, the lower bound on ψ_t at the point $(t,X(t))$ implies that

$$\dot{X}(t)=-\frac{\psi_t(t,X(t))}{\psi_x(t,X(t)}\geq\delta$$

with some $\delta>0$ independent of $t\in\mathbb{R}$. □

The exponential decay

The time-monotonicity of the transition fronts implies exponential decay ahead of the front.

Proposition 3.4 *Let $\psi(t,x)$ be a transition front solution of (1.3) and let $\xi(t)=\sup\{x:\ \psi(t,x)=\theta_0/2\}$. There exists $r>0$ so that for all $t\in\mathbb{R}$ and $x>\xi(t)$ we have*

$$\psi(t,x)\leq\frac{\theta_0}{2}e^{-r(x-\xi(t))}.$$

Proof. The same argument used to prove Corollary 3.3 shows that $\dot{\xi}(t) \geq r$ for some $r > 0$ independent of $t \in \mathbb{R}$. The function $s(t,x) = 2\psi(t, x + \xi(t))/\theta_0$ satisfies

$$s_t - \dot{\xi} s_x - s_{xx} = 0, \quad s(t,0) = 1,$$

and $s(t,x) \to 0$ as $x \to +\infty$, uniformly in $t \in \mathbb{R}$. We claim that $s(t,x) \leq e^{-rx}$ for all $t \in \mathbb{R}$ and $x \geq 0$. Indeed, consider the difference

$$p(t,x) = e^{-rx} - s(t,x),$$

which satisfies

$$p_t - \dot{\xi} p_x - p_{xx} \geq 0, \quad p(t,0) = 0, \tag{3.15}$$

and $p(t,x) \to 0$ as $x \to +\infty$, uniformly in $t \in \mathbb{R}$. Given $\varepsilon > 0$, consider an interval $[0, M]$, with $M = M(\varepsilon) > 0$ such that $p(t, M) > -\varepsilon$ for all $t \in \mathbb{R}$. There exists a time $T(M, \varepsilon) > 0$ so that any solution of (3.15) on the interval $0 \leq x \leq M$ with the Cauchy data $p(t_0, x)$ such that $|p(t_0, x)| \leq 1$ and the boundary condition $p(t, M) \geq -\varepsilon$, satisfies $p(t_0 + T(M, \varepsilon), x) \geq -2\varepsilon$. Thus, as $p(t,x)$ is a global in time solution of (3.15) and ε may be chosen arbitrarily small, we have $p(t,x) \geq 0$ for all $t \in \mathbb{R}$ and $x \geq 0$. □

The proof of uniqueness

We may now finish the proof of Theorem 1.3. Let $\phi(t,x)$ be the transition front satisfying properties P.1-P.4., and assume that there exists another transition front $\psi(t,x)$. Without loss of generality we may assume that $\phi(0,0) = \psi(0,0) = \theta_0$. We will prove that this normalization implies $\psi(t,x) = \phi(t,x)$ for all $(t,x) \in \mathbb{R}^2$. Let us denote by $X_\phi(t)$ and $X_\psi(t)$ the interface positions corresponding to ϕ and ψ, respectively: $\phi(t, X_\phi(t)) = \theta_0$ and $\psi(t, X_\psi(t)) = \theta_0$. The key steps are the following lemmas.

Lemma 3.5 *There is $M > 0$ such that*

$$|X_\phi(t) - X_\psi(t)| \leq M, \quad \text{for all } t \in \mathbb{R}. \tag{3.16}$$

Lemma 3.6 *There exists $h_0 > 0$ so that for all $(t,x) \in \mathbb{R}^2$ we have $\phi(t - h_0, x) \leq \psi(t,x)$.*

Proof of Lemma 3.5

We first prove the result for $t \leq 0$. Suppose that there is a sequence $t_n \leq 0$ such that $t_n \to -\infty$ and

$$|X_\phi(t_n) - X_\psi(t_n)| \to +\infty$$

as $n \to \infty$. Assume first that $X_\phi(t_n) - X_\psi(t_n) \to +\infty$. It follows from property P.2 bounding $\dot{X}_\phi(t)$ that $X_\phi(t_n - \tau) - X_\psi(t_n) \to +\infty$ as well, for any fixed number $\tau > 0$. Now, if $\xi(t) = \xi_\psi(t)$ is defined as in Proposition 3.4, the difference $X_\phi(t_n - \tau) - \xi_\psi(t_n)$ also diverges, since $\xi_\psi(t)$ must stay within a bounded distance from $X_\psi(t)$ because $\psi(t,x)$ is a transition front. Therefore, due to the exponential decay of $\psi(t_n, x)$ beyond the point $x = \xi_\psi(t_n)$, we see that for any $\varepsilon > 0$ we may take n large enough so that

$$\psi(t_n, x) \leq \begin{cases} \phi(t_n - \tau, x) + \varepsilon, & \text{for } x \leq X_\phi(t_n - \tau) + L_0 \\ \phi(t_n - \tau, x) + \varepsilon e^{-\alpha(x - X_\phi(t_n - \tau))} & \text{for } x \geq X_\phi(t_n - \tau) + L_0, \end{cases} \tag{3.17}$$

if $\alpha < r/2$ and L_0 is defined as in Proposition 2.1. Therefore, Proposition 2.1 implies that if n is sufficiently large we have

$$X_\psi(t) \leq X_\phi(t - \tau) + 1 \text{ for all } t > t_n.$$

Using this condition and Corollary 3.3, and normalization $X_\phi(0) = 0$, we see that

$$X_\psi(0) \le X_\phi(-\tau) + 1 \le -\tau\delta + 1 < X_\phi(0)$$

if $\tau > \delta^{-1}$. This contradicts the normalization $X_\psi(0) = X_\phi(0)$.

On the other hand, if $X_\phi(t_n) - X_\psi(t_n) \to -\infty$, then $X_\phi(t_n + \tau) - X_\psi(t_n) \to -\infty$ as well for all $\tau \in \mathbb{R}$ fixed. Then for any $\varepsilon > 0$ we may use Proposition 2.1 to bound ψ below as

$$\psi(t_n, x) \ge \begin{cases} \phi(t_n + \tau, x) - \varepsilon, & \text{for } x \le X_\phi(t_n + \tau) + L_0 \\ \phi(t_n + \tau, x) - \varepsilon e^{-\alpha(x - X_\phi(t_n + \tau))} & \text{for } x \ge X_\phi(t_n + \tau) + L_0, \end{cases} \tag{3.18}$$

for n sufficiently large. It follows that $X_\psi(0) \ge X_\phi(\tau) - 1 \ge \delta\tau - 1 > X_\phi(0)$, if $\tau > \delta^{-1}$, which contradicts the normalization $X_\psi(0) = X_\phi(0)$. This establishes (3.16) for $t \le 0$.

For $t \ge 0$, the normalization $X_\psi(0) = X_\phi(0)$ and the exponential decay of ψ implies that for any $\varepsilon > 0$,

$$\psi(0, x) \le \begin{cases} \phi(\tau, x) + \varepsilon, & \text{for } x \le X_\phi(\tau) + L_0 \\ \phi(\tau, x) + \varepsilon e^{-\alpha(x - X_\phi(\tau))} & \text{for } x \ge X_\phi(\tau) + L_0, \end{cases} \tag{3.19}$$

if τ is sufficiently large. Hence, Proposition 2.1 implies that by choosing ε small and then τ large, we get

$$X_\psi(t) \le X_\phi(t + \tau) + 1 \le X_\phi(t) + 1 + \tau C^{max}$$

for all $t > 0$. Similarly, for $\varepsilon > 0$,

$$\psi(0, x) \ge \begin{cases} \phi(-\tau, x) - \varepsilon, & \text{for } x \le X_\phi(-\tau) + L_0 \\ \phi(-\tau, x) - \varepsilon e^{-\alpha(x - X_\phi(-\tau))} & \text{for } x \ge X_\phi(-\tau) + L_0, \end{cases} \tag{3.20}$$

if τ is sufficiently large. Again, Proposition 2.1 implies that by choosing ε small and τ large we obtain

$$X_\psi(t) \ge X_\phi(t - \tau) - 1 \ge X_\phi(t) - 1 - \tau C^{max}$$

for all $t > 0$. This establishes (3.16) for $t \ge 0$ and completes the proof of Lemma 3.5. □

Proof of Lemma 3.6

As in the proof of Proposition 3.1 we first establish the claim in the domain

$$\Omega = \left\{ (t, x) \in \mathbb{R}^2 : \ \frac{\theta_0}{2} \le \phi(t, x) \le \frac{1 + \theta_1}{2} \right\}.$$

We let $\Omega_t = \{x \in \mathbb{R} : \ (t, x) \in \Omega\}$ be the cross-section of Ω with t fixed, which must have uniformly bounded width: $\sup \Omega_t - \inf \Omega_t \le w$. From Lemma 3.5, we know that $|X_\phi(t) - X_\psi(t)| < M$ for all $t \in \mathbb{R}$. Since $X_\phi(t) \in \Omega_t$ for all $t \in \mathbb{R}$, this implies that that

$$\inf_{(t,x) \in \Omega} \psi(t, x) \ge \inf_t \left(\inf_{|x| \le w + M} \psi(t, x + X_\psi(t)) \right) > \varepsilon \tag{3.21}$$

for some $\varepsilon > 0$. The strict lower bound follows from the Harnack inequality applied to ψ. If this were not so, there would be a sequence of points (t_n, x_n) with $|x_n|$ uniformly bounded such that $\psi(t_n, x_n + X_\psi(t_n)) < 1/n$ as $n \to \infty$. Then the Harnack inequality and the regularity of ψ imply that the functions $\zeta_n(t, x) = \psi(t + t_n, x + x_n + X_\psi(t_n))$ converge to zero locally uniformly on $\mathbb{R} \times \mathbb{R}$. In particular, it would follow that $\psi(t_n, X_\psi(t_n)) \to 0$, a contradiction. Thus, (3.21) has to hold for some $\varepsilon > 0$.

The lower bound $\dot{X}_\phi(t) \geq \delta$ implies that $X_\phi(t-h) \leq X_\phi(t) - \delta h$ for all $h > 0$. Hence, as ϕ is a transition front, we may take h sufficiently large so that $\phi(t-h,x) < \varepsilon$ for all $(t,x) \in \Omega$. We now see from (3.21) that

$$\phi(t-h,x) < \varepsilon < \psi(t,x) \text{ for all } (t,x) \in \Omega$$

if h is large enough.

The rest of the argument proceeds as in the proof of Lemma 3.2. We claim that there exists h_1 such that $\psi(t+h_1,x) \geq (1+\theta_1)/2$ for all $x \leq \sup \Omega_t$ and all $t \in \mathbb{R}$. The lower bound $\dot{X}_\psi(t) \geq \delta$ implies that $X_\psi(t+h_1) - \delta h_1 \geq X_\psi(t)$. Therefore, if $x \leq \sup \Omega_t$, we must have

$$x \leq X_\psi(t+h_1) - \delta h_1 + w + M,$$

which implies that

$$\inf_{x \leq \sup \Omega_t} \psi(t+h_1,x) \geq \inf_{x < -\delta h_1 + w + M} \psi(t+h_1, X_\psi(t+h_1) + x) \tag{3.22}$$

If h_1 is sufficiently large, the right hand side is larger than $(1+\theta_1)/2$, since ψ is a transition front.

We choose now $\bar{h} = \max(h_0, h_1)$ and set $r(t,x) = \psi(t+\bar{h},x) - \phi(t,x)$. This function satisfies

$$r_t - r_{xx} = a(t,x)r$$

at all points (t,x) where $x < \inf \Omega_t$, with

$$a(t,x) = g(x)\frac{f(\psi(t+\bar{h},x)) - f(\phi(t,x))}{\psi(t+\bar{h},x) - \phi(t,x)} \leq -\beta.$$

in this region. Now, $r(t,x)$ can not have a negative infimum in the region $x < \inf \Omega_t$ for the same reason as in the proof of Lemma 3.2 for the "region on the left". The regions "on the right" and "bad points in the middle" are also treated in exactly the same way as in the aforementioned lemma. We conclude that $r(t,x) \geq 0$ for all t and x. □

End of the proof of Theorem 1.3

Using Lemma 3.6 we find $\bar{h}$ which is the smallest of all h so that $\phi(t,x) \leq \psi(t+h,x)$ for all t and x. We will now show that $\bar{h} = 0$. As we have $\phi(t,x) \leq \psi(t+\bar{h},x)$, if there exists a point (t,x) so that $\phi(t,x) = \psi(t+\bar{h},x)$ then $\phi(t,x) \equiv \psi(t+\bar{h},x)$ by the maximum principle. Then $X_\phi(0) = X_\psi(0)$ implies that $\bar{h} = 0$. Hence, unless $\bar{h} = 0$ we must have $\phi(t,x) < \psi(t+\bar{h},x)$ everywhere.

Now, by using an argument similar to that which proved (3.13) we deduce that if $\bar{h} > 0$ then there exists $\gamma > 0$ so that

$$\inf_{(t,x) \in \tilde{\Omega}^-} \psi(t+\bar{h}-\gamma,x) - \phi(t,x) > 0 \tag{3.23}$$

where

$$\tilde{\Omega}^- = \{t \leq 0, \quad x \in \mathbb{R}, \quad \inf \Omega_t \leq x \leq \sup \Omega_t\}.$$

Indeed, suppose that (3.23) fails for all $\gamma > 0$. Then we may find a sequence of points $(t_n, x_n) \in \tilde{\Omega}^-$ such that

$$\lim_{n\to\infty} \psi(t_n+\bar{h},x_n) - \phi(t_n,x_n) = 0,$$

and $t_n \to -\infty$. Because $(t_n,x_n) \in \tilde{\Omega}^-$, the difference $|X_\phi(t_n) - x_n|$ is uniformly bounded. The functions

$$r_n(t,x) = \psi(t+t_n+\bar{h},x+x_n) - \phi(t+t_n,x+x_n) > 0$$

satisfy an equation of the form

$$r_t - r_{xx} + a(t,x)r = 0.$$

Since $r_n(0,0) \to 0$ as $n \to \infty$, the Harnack inequality and the regularity of r_n implies that $r_n(t,x)$ converges to zero locally uniformly as $n \to \infty$. This and the fact that ψ decays exponentially ahead of the interface $X_\psi(t_n)$ implies that for any $\varepsilon > 0$,

$$\psi(t_n + \bar{h}, x) \le \begin{cases} \phi(t_n, x) + \varepsilon, & \text{for } x \le X_\phi(t_n) + L_0 \\ \phi(t_n, x) + \varepsilon e^{-\alpha(x - X_\phi(t_n))} & \text{for } x \ge X_\phi(t_n) + L_0, \end{cases} \tag{3.24}$$

holds if n is sufficiently large and $\alpha < r/2$. In this case Proposition 2.1 would imply that $X_\psi(t+\bar{h}) \le X_\phi(t) + C\varepsilon$ for all $t > t_n$ if n is sufficiently large. Since $t_n \to -\infty$, we may let ε be arbitrarily small and conclude that $X_\psi(0) < X_\psi(\bar{h}) \le X_\phi(0)$, which is a contradiction to the normalization $X_\psi(0) = X_\phi(0)$ if $\bar{h} > 0$. Thus, there exists $\gamma > 0$ such that (3.23) holds.

However, we now may argue as at the end of the proof of Proposition 3.1 to show that actually $\psi(t + \bar{h} - \gamma, x) > \phi(t,x)$ for all $(t,x) \in (-\infty, 0] \times \mathbb{R}$ if $\gamma > 0$ is small enough. That is, (3.23) also holds for the points (t,x) satisfying $t \le 0$ and either $x < \inf \Omega_t$ or $x > \sup \Omega_t$. The only difference now is that we treat $\tilde{\Omega}^-$ instead of $\tilde{\Omega}$. As before, the "bad points in the middle" are included in the set $\tilde{\Omega}^-$. To show that (3.23) holds for the points on the left, we need to take γ small enough so that

$$\psi(t + \bar{h} - \gamma, x) \ge \theta_1 \tag{3.25}$$

for $x < \inf \Omega_t$ and $t \le 0$. Since $\psi(t + \bar{h}, x) > \phi(t,x) \ge (1 + \theta_1)/2$ for all (t,x) in this region, uniform bounds on ψ_t imply that (3.25) holds if γ is small for $x < \inf \Omega_t$ and $t \le 0$. Then we apply the argument as before, considering the function $r(t,x) = \psi(t + \bar{h} - \gamma, x) - \phi(t,x)$. If

$$\eta_1 := \inf_{\substack{t \le 0 \\ x < \inf \Omega_t}} r(t,x) < 0, \tag{3.26}$$

then we find a sequence of points (t_n, x_n) such that $r(t_n, x_n) \to \eta_1$. Taking an appropriate subsequence n_k, the functions $r_k(t,x) = r(t + t_{n_k}, x + x_{n_k})$ converge locally uniformly to a function $\bar{r}(t,x)$ which satisfies $\bar{r}(0,0) = \eta_1$ and

$$\bar{r}_t - \bar{r}_{xx} > 0, \quad x \in B_0^- = \{(t,x) \in \mathbb{R}^2 \mid t \le 0, |t|^2 + |x|^2 \le l\} \tag{3.27}$$

with $l \ge 0$ a sufficiently small constant. Since $\bar{r}$ attains its minimum at $(0,0)$, we obtain a contradiction, by the maximum principle.

For the points on the right, $x > \sup \Omega_t$, the previous argument also applies with little modification. The only difference is that we define the constants $d_n = \text{dist}((t_n, x_n), \tilde{\Omega}^-)$ and the half-balls $B_n^- = \{(t,x) : \ t \le 0, \ |x - x_n|^2 + |t - t_n|^2 \le d_n^2\}$. The same argument as before produces a subsequence (t_{n_k}, x_{n_k}) such that $d_{n_k} \to d_0 > 0$ and the functions

$$r_k(t,x) := \psi(t + t_{n_k} + \bar{h} - \gamma, x + x_{n_k}) - \phi(t + t_{n_k}, x + x_{n_k})$$

converge locally uniformly to a function $\bar{r}(t,x)$ satisfying

$$\bar{r}_t - \bar{r}_{xx} \ge 0, \quad \text{for } (t,x) \in B_0^- \tag{3.28}$$

where $B_0^- = \{(t,x) : t \le h_0, |x|^2 + |t|^2 \le d_0^2\}$ and $h_0 = \min(d_0, -\limsup_k t_{n_k}) \ge 0$. Since we have $\min_{B_0^-} \bar{r}(t,x) = \bar{r}(0,0) < 0$, the function $\bar{r}$ must be constant over $\bar{B}_0^-$. However, this contradicts the fact that $\bar{r} \ge 0$ at some point on the boundary ∂B_0^-, by definition of d_n.

Hence (3.23) holds for all points $(t,x) \in (-\infty, 0] \times \mathbb{R}$ if γ is sufficiently small. The maximum principle then implies that (3.23) holds for all points $(t,x) \in \mathbb{R}^2$. This contradicts the minimality of $\bar{h}$. Therefore, $\bar{h} = 0$ and $\psi(t,x) \ge \phi(t,x)$. Since $\psi(0,0) = \phi(0,0)$, the maximum principle implies uniqueness: $\phi(t,x) = \psi(t,x)$ for all $(t,x) \in \mathbb{R}^2$. □

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