On the barotropic compressible Navier-Stokes equations

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Abstract

We consider barotropic compressible Navier-Stokes equations with density dependent viscosity coefficients that vanish on vacuum. We prove the stability of weak solutions in periodic domain $\Omega = T^N$ and in the whole space $\Omega = \mathbb{R}^N$, when $N = 2$ and $N = 3$. The pressure is given by $p(\rho) = \rho^\gamma$ and our result holds for any $\gamma > 1$. Note that our notion of weak solutions is not the usual one. In particular we require some regularity on the initial density (which may still vanish). On the other hand, the initial velocity must satisfy only minimal assumptions (a little more than finite energy). Existence results for such solutions can be obtained from this stability analysis.

1 Introduction

This paper is devoted to the Cauchy problem for compressible Navier-Stokes equations with viscosity coefficients vanishing on vacuum. Let $\rho(t,x)$ and $u(t,x)$ denote the density and the velocity of a barotropic compressible viscous fluid (as usual, ρ is a non-negative function and u is a vector valued function, both defined on a subset Ω of $\mathbb{R}^N$). Then, the Navier-Stokes equations for barotropic compressible viscous fluids (often refered to as isentropic compressible Navier-Stokes equations) read (see [LL59]):

$$\begin{aligned} &\partial_t\rho + \mathrm{div}\,(\rho u) = 0 \\ &\partial_t(\rho u) + \mathrm{div}\,(\rho u \otimes u) + \nabla_x p - \mathrm{div}\,(2\,h\,D(u)) - \nabla(g\,\mathrm{div}\,u) = 0 \end{aligned} \tag{1}$$

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where $p(\rho) = \rho^\gamma$, $\gamma > 1$, denotes the pressure, $D(u) = \frac{1}{2}[\nabla u + {}^t\nabla u]$ is the strain tensor and h and g are the two Lamé viscosity coefficients (depending on the density ρ) satisfying

$$h > 0 \qquad 2h + Ng \geq 0 \tag{2}$$

(h is sometime called the shear viscosity of the fluid, while g is usually referred to as the second viscosity coefficient). One of the major difficulty of compressible fluid mechanics is to deal with vacuum. For that reason, the first results were obtained with initial data bounded away from zero. The existence of solutions defined globally in time for Navier-Stokes equations was first addressed in dimension one for smooth data by Kazhikov and Shelukhin [KS77] and for discontinuous data by Serre [Ser86] and Hoff [Hof87] (still with densities bounded away from zero). Those results have been generalized to higher dimensions by Matsumura and Nishida [MN79] for smooth data close to equilibrium and by Hoff [Hof95b], [Hof95a] in the case of discontinuous data.

Concerning large initial data that may vanish, Lions showed in [Lio98] the existence of weak solutions defined globally in time for $\gamma \geq 3/2$ when $N = 2$ and $\gamma \geq 9/5$ when $N = 3$. This result has been extended later by Feireisl, Novotny, and Petzeltova to the range $\gamma > 3/2$ in [FNP01], and recently by Feireisl for variational solutions of the full system of the Navier-Stokes equations with viscosity coefficients depending on the temperature [Fei04]. Other results provide the full range $\gamma > 1$ under symmetry assumptions on the initial datum (see for instance Jiang and Zhang [JZ03]), or, as in Vaigant and Kazhikhov [VK95] under the assumption that the second viscosity coefficient (g here) grows at least like ρ^β with $\beta > 3$ for large ρ. All those results do not require to be far from the vacuum. However they rely strongly on the assumption that the first viscosity coefficient is bounded from below by a positive constant. This assumption allows to get some estimates on the gradient of the velocity field but is not always physically realistic.

The main difficulty when dealing with vanishing viscosity coefficients on vacuum is that the velocity cannot even be defined when the density vanishes. The first result handling this difficulty is due to Bresch, Desjardins and Lin [BDL03]. They showed the stability of weak solutions for the following Korteweg's system of equations:

$$\begin{aligned} &\partial_t\rho + \operatorname{div}(\rho u) = 0 \\ &\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla_x p - \nu \operatorname{div}(\rho\, D(u)) = \kappa\rho\nabla\Delta\rho. \end{aligned} \tag{3}$$

The result was later improved by Bresch and Desjardins in [BD03] to include the case of vanishing capillarity ($\kappa = 0$), but with an additional quadratic friction term $r\rho|u|u$ (see also [BD02]). The key point in those papers is to show that the structure of the diffusion term provides some regularity for the density thanks to a new mathematical entropy inequality. However, those estimates are not enough to treat the case without capillarity and friction effects $\kappa = 0$ and $r = 0$ (which corresponds to equation (1) with $h(\rho) = \rho$ and $g(\rho) = 0$).

The main difficulty, to prove the stability of the solutions of (1), is to pass to the limit in the term $\rho u \otimes u$ (which requires the strong convergence of $\sqrt{\rho}u$). Note that this is easy when the viscosity coefficients are bounded below by a positive constant. On the other hand, the new bounds on the gradient of the density make the control of the pressure term far simpler than in the case of constant viscosity coefficients.

Our result is in the same spirit as the one of Bresch, Desjardins and Lin and makes use of the same mathematical entropy, first discovered by Bresch and Desjardins in [BD02] for the particular case where $h(\rho) = \rho$ and $g(\rho) = 0$, and later generalized by Bresch and Desjardins [BD04] to include the case of any viscosity coefficients $h(\rho), g(\rho)$ satisfying the relation:

$$g(\rho) = 2\rho h'(\rho) - 2h(\rho). \tag{4}$$

The precise formulation of the inequality that we use can be found in a recent paper by Bresch and Desjardins [BD05] where the full system of Navier-Stokes equations for heat conducting fluid is being investigated. We recall the proof in appendix for the confort of the reader.

Our main contribution is to show the stability of some weak solutions of (1) under some conditions on the viscosity coefficients (including (4)) but without any additional regularizing terms. The interest of our result lie primarily in the fact that our conditions allow for viscosity coefficients that vanish on the vacuum set. It includes the case $h(\rho) = \rho$, $g(\rho) = 0$ (when $N = 2$ and $\gamma = 2$, we recover the Saint Venant model for Shallow water), but our conditions on h and g will exclude the case of constant viscosity $h(\rho) = \mu$, $g(\rho) = \xi$. Indeed, it is readily seen that (4) implies that $g(\rho) = \xi = -2\mu$, and thus $2\mu + \xi = 0$. In this border line case we thus lose all informations on the derivatives of u. It is worth pointing out that while we can gain regularity on the density with this new estimate, we have to lose regularity on the velocity (on the vacuum set).

Note that the main difficulty will be to establish the compactness of $\sqrt{\rho}u$ in L^2 strong, and the key ingredient to achieve this is an additional estimate which bounds ρu^2 in a space better than $L^\infty(0,T;L^1(\Omega))$ (namely $L^\infty(0,T;L\log L(\Omega))$).

For the sake of simplicity we will consider the case $\Omega = \mathbb{R}^N$ and the case of bounded domain with periodic boundary conditions, namely $\Omega = T^N$. For the same reason we consider only power pressure laws although the result could be extend to non monotonic pressure law of the form of [Fei02]. Note that the result holds for any power $\gamma > 1$ under appropriate assumptions on h and g.

Naturally the main motivation to the study of the stability of weak solutions is to obtain existence results. Classically this can be done by constructing a sequence of solutions of approximated systems of equations satisfying the appropriate a priori estimates. In our framework, this can be achieved by proceeding as in [BDL03].

In the next section, we state the assumptions on the viscosity coefficients, we define precisely the notion of "weak solutions" and we state our main results. In Section 3, we recall the well known physical energy inequality and state the key estimates. The proof of Theorem 2.1 is detailed in Section 4. For the sake of completeness, we recall in Appendix A the proof of the mathematical entropy inequality of Bresch and Desjardins.

2 Notations and main result

We assume that Ω is either the whole space $\mathbb{R}^N$ or a bounded domain with periodic boundary conditions ($\Omega = T^N$), and we consider the following system of equations:

$$\partial_t \rho + \mathrm{div}\,(\rho u) = 0 \qquad (5)$$

$$\partial_t(\rho u) + \mathrm{div}\,(\rho u \otimes u) + \nabla_x \rho^\gamma - \mathrm{div}\,(2h(\rho)D(u)) - \nabla(g(\rho)\mathrm{div}\, u) = 0, \qquad (6)$$

with initial conditions

$$\rho|_{t=0} = \rho_0 \geq 0\,, \qquad \rho u|_{t=0} = m_0. \qquad (7)$$

We now detail our assumptions on the viscosity coefficients h and g.

Conditions on $h(\rho)$ and $g(\rho)$:
First we assume that $h(\rho)$ and $g(\rho)$ are two $\mathcal{C}^2(0,\infty)$ functions satisfying:

$$g(\rho) = 2\rho h'(\rho) - 2h(\rho). \qquad (8)$$

As stated in the introduction, this relation is fundamental to get more regularity on the density. Moreover, we assume that there exists a positive constant $\nu \in (0,1)$ such that

$$h'(\rho) \geq \nu \,, \qquad h(0) \geq 0 \tag{9}$$

$$|g'(\rho)| \leq \frac{1}{\nu} h'(\rho) \tag{10}$$

$$\nu h(\rho) \leq 2h(\rho) + N g(\rho) \leq \frac{1}{\nu} h(\rho) \,. \tag{11}$$

When $\gamma \geq 3$ and $N = 3$, we also require that

$$\liminf_{\rho \to \infty} \frac{h(\rho)}{\rho^{\gamma/3+\varepsilon}} > 0, \tag{12}$$

for some small $\varepsilon > 0$.

Remark 2.1 *The functions*

$$h(\rho) = \rho, \qquad g(\rho) = 0$$

satisfy (8-11). For example in dimension 3*, any linear combination of the form* $\sum \alpha_k \rho^{n_k}$ *with* $\alpha_k > 0$*,* $n_k > 2/3$ *for all* k *(*$n_k > 1/2$ *in dimension* 2*) and* $\sup n_k \geq 1$ *is an admissible function for* $h(\rho)$*.*

Remark 2.2 *The lower estimate in (11) is trivial when* $g \geq 0$*, while the upper estimate is trivial when* $g \leq 0$*. Together they yield:*

$$|g(\rho)| \leq C_\nu h(\rho) \quad \forall \rho > 0.$$

This inequality and (10) will be necessary to pass to the limit in the term $\nabla(g(\rho_n)\, div\, u_n)$*.*

Remark 2.3 *Condition (9) makes the proof simpler, but is not optimal. However, condition (11) is necessary to control the viscosity term and together with (8), it yields*

$$\frac{N-1+\nu}{N\rho} \leq \frac{h'(\rho)}{h(\rho)} \leq \frac{N-1+1/\nu}{N\rho}, \qquad \textit{for all } \rho > 0,$$

and so

$$\begin{cases} C\rho^{(N-1)/N+\nu/N} \leq h(\rho) \leq C\rho^{(N-1)/N+1/(N\nu)}, & \rho \geq 1 \\ C\rho^{(N-1)/N+1/(N\nu)} \leq h(\rho) \leq C\rho^{(N-1)/N+\nu/N}, & \rho \leq 1 \end{cases} \tag{13}$$

In particular, we must have $h(0) = 0$. Moreover, this shows that if we do not assume (9), the "best" $h(\rho)$ we can take is $h(\rho) = \rho^{(N-1)/N+\nu/N}$. This is actually enough to prove the stability of weak solutions for all γ when $N = 2$ and for $\gamma < 3/2$ when $N = 3$. However, if we assume $h(\rho) \sim C\rho^{2/3+\nu}$ for small ρ and $h(\rho) \sim C\rho$ for large ρ, then we can consider any $\gamma \in (1, 3)$ when $N = 3$.

Weak solutions

We say that (ρ, u) is a weak solution of (5-6) on $\Omega \times [0, T]$, with initial conditions (7) if

$$
\begin{aligned}
&\rho \in L^\infty(0, T, L^1(\Omega) \cap L^\gamma(\Omega)),\\
&\sqrt{\rho} \in L^\infty(0, T; H^1(\Omega)),\\
&\sqrt{\rho}\, u \in L^\infty(0, T; (L^2(\Omega))^N),\\
&h(\rho)D(u) \in L^2(0, T; (W^{-1,1}_{\mathrm{loc}}(\Omega))^{N\times N}), \quad g(\rho)\mathrm{div}\, u \in L^2(0, T; W^{-1,1}_{\mathrm{loc}}(\Omega)),
\end{aligned}
$$

with $\rho \geq 0$ and $(\rho, \sqrt{\rho}u)$ satisfying

$$
\begin{cases} \partial_t \rho + \mathrm{div}\,(\sqrt{\rho}\sqrt{\rho}u) = 0 \\ \rho(0, x) = \rho_0(x) \end{cases} \qquad \text{in } \mathcal{D}',
$$

and if the following equality holds for all $\varphi(t, x)$ smooth test function with compact support such that $\varphi(T, \cdot) = 0$:

$$
\begin{aligned}
&\int_\Omega m_0 \cdot \varphi(0, \cdot)\, dx + \int_0^T\!\!\int_\Omega \sqrt{\rho}(\sqrt{\rho}u)\partial_t\varphi + \sqrt{\rho}u \otimes \sqrt{\rho}u : \nabla\varphi\, dx\\
&\quad + \int_0^T\!\!\int_\Omega \rho^\gamma \mathrm{div}\,\varphi\, dx - \big\langle 2h(\rho)D(u)\,,\, \nabla\varphi\big\rangle - \big\langle g(\rho)(\mathrm{div}\, u)\,,\, (\mathrm{div}\,\varphi)\big\rangle = 0,
\end{aligned} \tag{14}
$$

where the diffusion terms make sense when written as

$$
\begin{aligned}
&\big\langle 2h(\rho)D(u)\,,\, \nabla\varphi\big\rangle =\\
&\quad = -\int \frac{h(\rho)}{\sqrt{\rho}}(\sqrt{\rho}u_j)\partial_{ii}\varphi_j\, dx\, dt - \int (\sqrt{\rho}u_j)2h'(\rho)\partial_i\sqrt{\rho}\;\partial_i\varphi_j\, dx\, dt\\
&\qquad - \int \frac{h(\rho)}{\sqrt{\rho}}(\sqrt{\rho}u_i)\partial_{ji}\varphi_j\, dx\, dt - \int (\sqrt{\rho}u_i)2h'(\rho)\partial_j\sqrt{\rho}\;\partial_i\varphi_j\, dx\, dt,
\end{aligned}
$$

and

$$\langle g(\rho)(\operatorname{div} u)\,,\,(\operatorname{div}\varphi)\rangle =$$
$$= -\int \frac{g(\rho)}{\sqrt{\rho}}(\sqrt{\rho}u_i)\partial_{ij}\varphi_j\,dx\,dt - \int(\sqrt{\rho}u_i)2g'(\rho)\partial_i\sqrt{\rho}\partial_j\varphi_j\,dx\,dt.$$

In particular, the fact that the diffusion term $2h(\rho)D(u)$ (and $g(\rho)\operatorname{div} u$) lies in $L^2(0,T;(W^{-1,1}_{\rm loc}(\Omega))^{n\times n})$ will follow from the fact that

$$h'(\rho)\nabla\sqrt{\rho} \in L^\infty(0,T;L^2_{\rm loc}(\Omega))\,, \quad \text{and} \quad h(\rho)/\sqrt{\rho} \in L^\infty(0,T;L^2_{\rm loc}(\Omega)),$$

and similar conditions on $g(\rho)$. This will be provided by assumptions (10), (9) and (13).

Main result:

The main result of this paper is the following:

Theorem 2.1 *Assume that $\gamma > 1$ and that $h(\rho)$ and $g(\rho)$ are two $C^2(0,\infty)$ functions of ρ satisfying conditions (8)-(11) (together with (12) if $\gamma \geq 3$ and $N = 3$). Let $(\rho_n, u_n)_{n\in\mathbb{N}}$ be a sequence of weak solutions of (5-6) satisfying entropy inequalities (18), (21) and (26), with initial data*

$$\rho_n|_{t=0} = \rho_0^n(x) \quad \text{and} \quad \rho_n u_n|_{t=0} = m_0^n(x) = \rho_0^n(x)u_0^n(x),$$

where ρ_0^n and u_0^n are such that

$$\rho_0^n \geq 0, \qquad \rho_0^n \to \rho_0 \text{ in } L^1(\Omega), \quad \rho_0^n u_0^n \to \rho_0 u_0 \text{ in } L^1(\Omega), \tag{15}$$

and satisfy the following bounds (with C constant independent on n):

$$\int_\Omega \rho_0^n \frac{|u_0^n|^2}{2} + \frac{1}{\gamma-1}\rho_0^{n\gamma}\,dx < C, \qquad \int_\Omega \frac{1}{\rho_0^n}|\nabla h(\rho_0^n)|^2\,dx < C, \tag{16}$$

and

$$\int_\Omega \rho_0^n \frac{1+|u_0^n|^2}{2}\ln(1+|u_0^n|^2)\,dx < C, \tag{17}$$

Then, up to a subsequence, $(\rho_n, \sqrt{\rho_n}u_n)$ converges strongly to a weak solution of (5)-(6) satisfying entropy inequalities (18), (21) and (26) (the density ρ_n converges strongly in $\mathcal{C}^0((0,T);L^{3/2}_{loc}(\Omega))$, $\sqrt{\rho_n}u_n$ converges strongly in $L^2(0,T;L^2_{\rm loc}(\Omega))$ and the momentum $m_n = \rho_n u_n$ converges strongly in $L^1(0,T;L^1_{\rm loc}(\Omega))$, for any $T > 0$).

3 Entropy inequalities and a priori estimates

In this section, we recall the well-known energy inequality and state the main inequalities that we will use throughout the proof of Theorem 2.1.

The usual energy inequality associated with the system of equations (5-6) can be written as:

$$\frac{d}{dt}\int\left[\rho\frac{u^2}{2}+\frac{1}{\gamma-1}\rho^{\gamma}\right]dx+\int 2h(\rho)|D(u)|^2dx+\int g(\rho)(\operatorname{div}u)^2dx\leq 0. \quad (18)$$

This inequality can be established for smooth solutions of (5-6) by multiplying the momentum equation by u.

When h and g satisfies $2h(\rho)+Ng(\rho)\geq 0$ and if we have

$$\mathcal{E}_0=\int_{\Omega}\rho_0\frac{u_0^2}{2}+\frac{1}{\gamma-1}\rho_0^{\gamma}\,dx<+\infty,$$

then (18) yields:

$$\begin{aligned}&||\sqrt{\rho}\,u||_{L^\infty(0,T;L^2(\Omega))}\leq C,\\&||\rho||_{L^\infty(0,T;L^\gamma(\Omega))}\leq C.\end{aligned}\quad (19)$$

Furthermore, Hypothesis (11) gives:

$$||\sqrt{h(\rho)}D(u)||_{L^2(0,T;L^2(\Omega))}\leq C. \quad (20)$$

Finally, integrating (5) with respect to x yields the natural L^1 estimate:

$$||\rho||_{L^\infty(0,T;L^1(\Omega))}\leq C.$$

Unfortunately, it is a well-known fact that those natural estimates are not enough to prove the stability of the solutions of (5-6). In particular, the fact that ρ^γ is bounded in $L^\infty(0,T;L^1(\Omega))$ does not implies that ρ_n^γ converges to ρ^γ.

However, further estimates can be obtained by mean of the following lemma:

Lemma 3.1 *Assume that $h(\rho)$ and $g(\rho)$ are two $C^2(0,\infty)$ functions satisfying (8). Then, the following equality holds for smooth solutions of (5-6):*

$$\begin{aligned}\frac{d}{dt}\int&\left[\frac{1}{2}\rho|u+\nabla\varphi(\rho)|^2+\frac{1}{\gamma-1}\rho^\gamma\right]dx\\&+\int\nabla\varphi(\rho)\cdot\nabla\rho^\gamma\,dx+\frac{1}{2}\int h(\rho)\left|\nabla u-{}^t\nabla u\right|^2dx=0, \quad (21)\end{aligned}$$

with φ such that

$$\varphi' = 2\frac{h'}{\rho}. \tag{22}$$

This lemma was first proved by Bresch and Desjardin in [BD04]. We recall the proof in Appendix A.

Since the viscosity coefficient $h(\rho)$ is an increasing function of ρ, we immediately see that when the initial data has finite energy and satisfies

$$\int_\Omega \rho_0 |\nabla\varphi(\rho_0)|^2\,dx < +\infty,$$

the equality (21) yields:

$$\frac{1}{2}||\sqrt{\rho}\nabla\varphi(\rho)||_{L^\infty(0,T;L^2(\Omega))} = 2||h'(\rho)\nabla\sqrt{\rho}||_{L^\infty(0,T;L^2(\Omega))} \leq C, \tag{23}$$

and

$$||\sqrt{h'(\rho)\rho^{\gamma-2}}\nabla\rho||_{L^2(0,T;L^2(\Omega))} \leq C. \tag{24}$$

Under assumption (9) on h, those estimates give additional control on the density ρ and on the pressure ρ^γ, which will be enough to prove the stability of weak solution.

Furthermore, (21) gives some control on the antisymmetric part of ∇u. Together with (20), it implies

$$||\sqrt{h(\rho)}\nabla u||_{L^2(0,T;L^2(\Omega))} \leq C. \tag{25}$$

Finally, one of the key tool of the proof will be the following result:

Lemma 3.2 *Assume*

$$2h(\rho) + Ng(\rho) \geq \nu h(\rho)$$

for some $\nu \in (0,1)$ (which is a part of (11)). Then smooth solutions of (5)-(6) satisfy the following inequality:

$$\frac{d}{dt}\int_\Omega \rho\frac{1+|u|^2}{2}\ln(1+|u|^2)\,dx + \frac{\nu}{2}\int_\Omega h(\rho)[1+\ln(1+|u|^2)]|D(u)|^2\,dx$$

$$\leq C\left(\int_\Omega \left(\frac{\rho^{2\gamma-\delta/2}}{h(\rho)}\right)^{2/(2-\delta)} dx\right)^{(2-\delta)/2}\left(\int_\Omega \rho[2+\ln(1+|u|^2)]^{2/\delta}\,dx\right)^{\delta/2}$$

$$+C\int_\Omega h(\rho)|\nabla u|^2\,dx \tag{26}$$

for any $\delta \in (0,2)$*, and with* $|\nabla u|^2 = \sum_i \sum_j |\partial_i u_j|^2$.

This inequality is quite simple to establish and will be essential in the proof of Theorem 2.1 to prove that $\sqrt{\rho_n} u_n$ converges strongly in $L^\infty(0,T;L^2(\Omega))$ (see Lemma 4.3). Note, however, that to derive further estimates from this inequality, we need to control the right hand side of (26). The last term is bounded thanks to (25) and usual a priori estimates provide a bound on $\int \rho |u|^2\,dx$ and $\int \rho\,dx$ and thus on $\int \rho[1+\ln(1+|u|^2)]^{2/\delta}\,dx$. So the difficulty is to control enough power of ρ to get a bound on $\int \left(\frac{\rho^{2\gamma-\delta/2}}{h(\rho)}\right)^{2/(2-\delta)} dx$. This will be achieved using (24). Of course, we also need to assume that the initial condition satisfies

$$\int_\Omega \rho_0 \frac{1+|u_0|^2}{2} \ln(1+|u_0^2|)\,dx < C.$$

Proof of Lemma 3.2. Multiplying (6) by $(1+\ln(1+|u|^2))u$, we get:

$$\begin{aligned}\int \rho\partial_t \left[\frac{1+|u|^2}{2}\ln(1+|u|^2)\right] dx + \int \rho u\cdot\nabla\left[\frac{1+|u|^2}{2}\ln(1+|u|^2)\right] dx \\ + \int 2h(\rho)[1+\ln(1+|u|^2)]|D(u)|^2 dx + \int 2h(\rho)\frac{2u_i u_k}{1+|u|^2}\partial_j u_k D_{ij}(u) dx \\ + \int g(\rho)[1+\ln(1+|u|^2)](\operatorname{div} u)^2\,dx + \int g(\rho)\frac{2u_i u_k}{1+|u|^2}\partial_i u_k \operatorname{div} u\,dx \\ + \int [1+\ln(1+|u|^2)]u\cdot\nabla\rho^\gamma\,dx = 0.\end{aligned}$$

Since

$$(\operatorname{div} u)^2 = \sum_i\sum_j \partial_i u_i \partial_j u_j \le \sum_i\sum_j \frac{1}{2}(\partial_i u_i^2 + \partial_j u_j^2) \le N|D(u)|^2,$$

condition (11) and Remark 2.2 yields:

$$\begin{aligned}\int \rho\partial_t \left[\frac{1+|u|^2}{2}\ln(1+|u|^2)\right] dx + \int \rho u\cdot\nabla\left[\frac{1+|u|^2}{2}\ln(1+|u|^2)\right] dx \\ +\nu \int h(\rho)[1+\ln(1+|u|^2)]|D(u)|^2\,dx \\ \le -\int [1+\ln(1+|u|^2)]u\cdot\nabla\rho^\gamma\,dx + C\int h(\rho)|\nabla u|^2\,dx.\end{aligned}$$

Moreover, multiplying (5) by $\frac{1+|u|^2}{2}\ln(1+|u|^2)$ and integrating by parts, we have

$$\int \frac{1+|u|^2}{2}\ln(1+|u|^2)\partial_t\rho\,dx - \int \rho u\cdot\nabla\left[\frac{1+|u|^2}{2}\ln(1+|u|^2)\right] dx = 0.$$

We deduce:

$$\frac{d}{dt}\int \rho \frac{1+|u|^2}{2}\ln(1+|u|^2)\,dx + \frac{\nu}{2}\int h(\rho)[1+\ln(1+|u|^2)]|D(u)|^2\,dx$$
$$\leq -\int [1+\ln(1+|u|^2)]u\cdot\nabla\rho^\gamma\,dx + C\int h(\rho)|\nabla u|^2\,dx.$$

It remains to bound the right hand side:

$$\begin{aligned}
&\left|\int [1+\ln(1+|u|^2)]u\cdot\nabla\rho^\gamma\,dx\right| \\
&\quad\leq \left|\int \frac{2u_iu_k}{1+|u|^2}\partial_i u_k\,\rho^\gamma\,dx\right| + \left|\int [1+\ln(1+|u|^2)](\operatorname{div} u)\,\rho^\gamma\,dx\right| \\
&\quad\leq 2\left(\int h(\rho)|\nabla u|^2\,dx\right)^{1/2}\left(\int \frac{\rho^{2\gamma}}{h(\rho)}\,dx\right)^{1/2} \\
&\qquad + \left|\int [1+\ln(1+|u|^2)](\operatorname{div} u)\,\rho^\gamma\,dx\right|
\end{aligned}$$

where

$$\begin{aligned}
&\left|\int [1+\ln(1+|u|^2)](\operatorname{div} u)\,\rho^\gamma\,dx\right| \\
&\qquad\leq \left(\int [1+\ln(1+|u|^2)]h(\rho)(\operatorname{div} u)^2\,dx\right)^{1/2} \\
&\qquad\qquad \times\left(\int [1+\ln(1+|u|^2)]\frac{\rho^{2\gamma}}{h(\rho)}\,dx\right)^{1/2} \\
&\qquad\leq \frac{\nu}{2}\int [1+\ln(1+|u|^2)]h(\rho)|D(u)|^2\,dx \\
&\qquad\qquad + C_\nu\int [1+\ln(1+|u|^2)]\frac{\rho^{2\gamma}}{h(\rho)}\,dx.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\left|\int [1+\ln(1+|u|^2)]u\cdot\nabla\rho^\gamma\,dx\right| \\
&\quad\leq \int h(\rho)|\nabla u|^2\,dx + \frac{\nu}{2}\int [1+\ln(1+|u|^2)]h(\rho)|D(u)|^2\,dx \\
&\qquad + C_\nu\int [2+\ln(1+|u|^2)]\frac{\rho^{2\gamma}}{h(\rho)}\,dx.
\end{aligned}$$

Finally, we notice that the last term satisfies (for any $\delta \in (0,2)$):

$$\int [2+\ln(1+|u|^2)]\frac{\rho^{2\gamma}}{h(\rho)}\,dx \;\leq\; \left(\int \left(\frac{\rho^{2\gamma-\delta/2}}{h(\rho)}\right)^{2/(2-\delta)} dx\right)^{(2-\delta)/2} \times \left(\int \rho[2+\ln(1+|u|^2)]^{2/\delta}\,dx\right)^{\delta/2},$$

which gives the lemma.

Remark 3.1 *A similar estimate can be obtained when considering a viscosity term of the form* $div(2h(\rho)\nabla u)$*. Actually, in that case, it is possible to derive an estimate on* $\int \rho|u|^{2+\delta}\,dx$ *for small* δ *(by multiplying (6) by* $u|u|^\delta$*).*

We now have all the necessary tools to prove Theorem 2.1.

4 Proof of Theorem 2.1

We recall that the initial data must satisfy (16), and (17) to make use of all the inequalities presented in the previous section. More precisely, we have

$$\begin{aligned} &\rho_0^n \text{ is bounded in } L^1\cap L^\gamma(\Omega), \qquad \rho_0^n \geq 0 \text{ a.e. in } \Omega \\ &\rho_0^n|u_0^n|^2 = |m_0^n|^2/\rho_0^n \text{ is bounded in } L^1(\Omega) \\ &\sqrt{\rho_0^n}\nabla\varphi(\rho_0^n) = 2\nabla h(\rho_0^n)/\sqrt{\rho_0^n} \text{ is bounded in } L^2(\Omega), \\ &\int \rho_0^n\frac{|u_0^n|^2}{2}\ln(1+|u_0^n|^2)\,dx < C. \end{aligned} \tag{27}$$

Using inequalities (18) and (21) (and (25)), we deduce the following estimates, which we shall use throughout the proof of Theorem 2.1:

$$\begin{aligned} &||\sqrt{\rho}_n u_n||_{L^\infty(0,T);L^2(\Omega))} \leq C \\ &||\rho_n||_{L^\infty(0,T;L^1\cap L^\gamma(\Omega))} \leq C \\ &||\sqrt{h(\rho_n)}\nabla u_n||_{L^2(0,T;L^2(\Omega))} \leq C \end{aligned} \tag{28}$$

and

$$\begin{aligned} &||h'(\rho_n)\nabla\sqrt{\rho}_n||_{L^\infty(0,T;L^2(\Omega))} \leq C \\ &||\sqrt{h'(\rho_n)\rho_n^{\gamma-2}}\nabla\rho_n||_{L^2(0,T;L^2(\Omega))} \leq C \end{aligned} \tag{29}$$

In view of our hypothesis on the viscosity coefficient (9), the bounds (28) and (29) yields:

$$\begin{aligned} &||\sqrt{\rho_n}\nabla u_n||_{L^2(0,T;L^2(\Omega))} \leq C \\ &||\nabla\sqrt{\rho}_n||_{L^\infty(0,T;L^2(\Omega))} \leq C \\ &||\nabla \rho_n^{\gamma/2}||_{L^2(0,T;L^2(\Omega))} \leq C \end{aligned} \tag{30}$$

The proof of Theorem 2.1 will be divided in 6 steps. In the first two steps, we show the convergence of the density and the pressure (note that the convergence of the pressure is straighforward here). The key argument of the proof is presented in the third step: We prove that $\rho_n u_n^2$ is bounded in a space better than $L^\infty(0,T;L^1(\Omega))$. We then show the convergence of the momentum (step 4) and finally the strong convergence of $\sqrt{\rho_n}u_n$ in $L^2_{loc}((0,T)\times\Omega)$ (step 5). The last step adresses the convergence of the diffusion terms; It is mainly technical and of minor interest.

Step 1: Convergence of $\sqrt{\rho_n}$.

Lemma 4.1 *If h satisfies (9), then*

$$\begin{aligned} &\sqrt{\rho_n} \text{ is bounded in } L^\infty(0,T;H^1(\Omega)) \\ &\partial_t\sqrt{\rho_n} \text{ is bounded in } L^2(0,T;H^{-1}(\Omega)). \end{aligned}$$

As a consequence, up to a subsequence, $\sqrt{\rho_n}$ converges almost everywhere and strongly in $L^2(0,T;L^2_{loc}(\Omega))$. We write

$$\sqrt{\rho_n} \longrightarrow \sqrt{\rho} \qquad a.e \text{ and } L^2_{loc}((0,T)\times\Omega) \text{ strong.}$$

Moreover, ρ_n converges to ρ in $C^0(0,T;L^{3/2}_{loc}(\Omega))$.

Proof. The second estimate in (30), together with the conservation of mass $||\rho_n(t)||_{L^1(\Omega)} = ||\rho_{n,o}||_{L^1(\Omega)}$ gives the $L^\infty(0,T;H^1(\Omega))$ bound. Next, we notice that

$$\begin{aligned} \partial_t\sqrt{\rho_n} &= -\frac{1}{2}\sqrt{\rho_n}\mathrm{div}\, u_n - u_n\cdot\nabla\sqrt{\rho_n} \\ &= \frac{1}{2}\sqrt{\rho_n}\mathrm{div}\, u_n - \mathrm{div}\,(u_n\sqrt{\rho_n}) \end{aligned}$$

which yields the second estimate and, thanks to Aubin's Lemma, gives the strong convergence in $L^2_{loc}((0,T)\times\Omega)$.

Sobolev imbedding implies that $\sqrt{\rho_n}$ is bounded in $L^\infty(0,T;L^q(\Omega))$ for $q \in [2,+\infty[$ if $N=2$ and $q \in [2,6]$ if $N=3$. In either cases we deduce that ρ_n is bounded in $L^\infty(0,T;L^3(\Omega))$, and therefore

$$\rho_n u_n = \sqrt{\rho_n}\sqrt{\rho_n} u_n \text{ is bounded in } L^\infty(0,T;L^{3/2}(\Omega)).$$

The continuity equation thus yields $\partial_t \rho_n$ bounded in $L^\infty(0,T,W^{-1,3/2}(\Omega))$. Moreover, since $\nabla\rho_n = 2\sqrt{\rho_n}\nabla\sqrt{\rho_n}$, we also have that $\nabla\rho_n$ is bounded in $L^\infty(0,T;L^{3/2}(\Omega))$, hence the compactness of ρ_n in $C([0,T];L^{3/2}_{loc}(\Omega))$.

Step 2: Convergence of the pressure

Lemma 4.2 *The pressure ρ_n^γ is bounded in $L^{5/3}((0,T)\times\Omega)$ when $N=3$ and $L^r((0,T)\times\Omega)$ for all $r\in[1,2[$ when $N=2$. In particular, ρ_n^γ converges to ρ^γ strongly in $L^1_{loc}((0,T)\times\Omega)$.*

Proof. Inequalities (30) and (28) yield $\rho_n^{\gamma/2} \in L^2(0,T;H^1(\Omega))$.

When $N=2$, we deduce $\rho_n^{\gamma/2} \in L^2(0,T;L^q(\Omega))$ for all $q\in[2,\infty[$. So ρ_n^γ is bounded in $L^1(0,T;L^p(\Omega))\cap L^\infty(0,T;L^1(\Omega))$ for all $p\in[1,\infty[$, hence ρ_n^γ is bounded in $L^r((0,T)\times\Omega)$ for all $r\in[1,2[$.

When $N=3$, we only get $\rho_n^{\gamma/2}\in L^2(0,T;L^6(\Omega))$, or

$$\rho_n^\gamma \in L^1(0,T;L^3(\Omega)).$$

Since ρ_n^γ is bounded in $L^\infty(0,T;L^1(\Omega))$, Hölder inequality gives

$$||\rho_n^\gamma||_{L^{5/3}((0,T)\times\Omega)} \leq ||\rho_n^\gamma||^{2/5}_{L^\infty(0,T;L^1(\Omega))} ||\rho_n^\gamma||^{3/5}_{L^1(0,T;L^3(\Omega))} \leq C.$$

hence ρ_n^γ is bounded in $L^{5/3}((0,T)\times\Omega)$.

Since we already know that ρ_n^γ converges almost everywhere to ρ^γ, those bounds yield the strong convergence of ρ_n^γ in $L^1_{loc}((0,T)\times\Omega)$.

Step 3: Bounds for $\sqrt{\rho_n}u_n$

Lemma 4.3 *If $\gamma<3$, or if $N=3$, $\gamma\geq 3$ and (12) holds, then*

$$\rho_n|u_n|^2\ln(1+|u_n|^2) \text{ is bounded in } L^\infty(0,T;L^1(\Omega)).$$

This Lemma is really the corner stone of the stability result. As a matter of fact, at this point, the main difficulty is to prove the strong convergence of $\sqrt{\rho_n}u_n$ in $L^1(0,T;L^2_{loc}(\Omega))$. Thanks to Lemma 4.3, it will be enough to prove the convergence almost everywhere. Moreover, since we are only able to prove the convergence of the momentum $\rho_n u_n$ (see Step 4), we need to control $\sqrt{\rho_n}u_n$ on the vacuum set $\{\rho(t,x)=0\}$ (and prove that it converges to zero almost everywhere on the vacuum). This fact will also be a consequence of Lemma 4.3 (see Step 5).

Proof. The proof of Lemma 4.3 relies on Lemma 3.2: for any $\delta\in(0,2)$, we have:

$$\frac{d}{dt}\int \rho_n \frac{1+|u_n|^2}{2}\ln(1+|u_n|^2)\,dx + \frac{\nu}{2}\int h(\rho_n)[1+\ln(1+|u_n|^2)]|D(u_n)|^2 dx$$

$$\leq C\left(\int\left(\frac{\rho_n^{2\gamma-\delta/2}}{h(\rho_n)}\right)^{2/(2-\delta)} dx\right)^{(2-\delta)/2}\left(\int \rho_n[2+\ln(1+|u_n|^2)]^{2/\delta}\,dx\right)^{\delta/2}$$

$$+C\int h(\rho_n)|\nabla u_n|^2\,dx$$

Using (28) and the natural bounds on $\int \rho_n\,dx$ and $\int \rho_n u_n^2\,dx$, we deduce:

$$\frac{d}{dt}\int \rho_n \frac{1+|u_n|^2}{2}\ln(1+|u_n|^2)\,dx \leq C\left(\int\left(\frac{\rho_n^{2\gamma-\delta/2}}{h(\rho_n)}\right)^{2/(2-\delta)} dx\right)^{(2-\delta)/2} + C. \tag{31}$$

Moreover, condition (9) yields $h(\rho)\geq \nu\rho$ and so

$$\frac{d}{dt}\int \rho_n \frac{1+|u_n|^2}{2}\ln(1+|u_n|^2)\,dx \leq C\left(\int\left(\rho_n^{2\gamma-1-\delta/2}\right)^{2/(2-\delta)} dx\right)^{(2-\delta)/2} + C.$$

Finally, using Lemma 4.2, we check that the right hand side is bounded L^1 in time (for small δ), without any condition when $N=2$, and when $N=3$ under the condition that

$$2\gamma-1<\frac{5}{3}\gamma,$$

which gives rise to the restriction $\gamma<3$. In either cases, we deduce

$$\frac{d}{dt}\int \rho_n \frac{1+|u_n|^2}{2}\ln(1+|u_n|^2)\,dx \leq C.$$

and (17) gives the lemma. When $N = 3$ and $\gamma \geq 3$ we need the extra hypothesis (12) to show that the right hand side of (31) is bounded and to achieve the same result.

Step 4: Convergence of the momentum

Lemma 4.4 *Up to a subsequence, the momentum $m_n = \rho_n u_n$ converges strongly in $L^2(0,T;L^p_{loc}(\Omega))$ to some $m(x,t)$ for all $p \in [1, 3/2)$. In particular*

$$\rho_n u_n \longrightarrow m \qquad \textit{almost everywhere } (x,t) \in \Omega \times (0,T).$$

Note that we can already define $u(x,t) = m(x,t)/\rho(x,t)$ outside the vacuum set $\{\rho(x,t) = 0\}$, but we do not know yet whether $m(x,t)$ is zero on the vacuum set.
Proof. We have

$$\rho_n u_n = \sqrt{\rho_n}\sqrt{\rho_n} u_n,$$

where $\sqrt{\rho_n}$ is bounded in $L^\infty(0,T;L^q(\Omega))$ for $q \in [2,+\infty[$ if $N = 2$ and $q \in [2,6]$ if $N = 3$; Since $\sqrt{\rho_n} u_n$ is bounded in $L^\infty(0,T;L^2(\Omega))$, we deduce that

$$\rho_n u_n \text{ is bounded in } L^\infty(0,T,L^q(\Omega)) \text{ for all } q \in [1,3/2].$$

Next, we have

$$\begin{aligned} \partial_i(\rho_n u_{nj}) &= \rho_n \partial_i u_{nj} + u_{nj}\partial_i \rho_n \\ &= \sqrt{\rho_n}\sqrt{\rho_n}\partial_i u_{nj} + 2\sqrt{\rho_n} u_{nj} \partial_i \sqrt{\rho_n}. \end{aligned}$$

Using Lemma 4.3 and (30), it is readily seen that the second term is bounded in $L^\infty(0,T;L^1(\Omega))$, while the first term is bounded in $L^2(0,T,L^q(\Omega))$ for all $q \in [1,3/2]$. Hence

$$\nabla(\rho_n u_n) \text{ is bounded in } L^2(0,T;L^1(\Omega)).$$

In particular, we have

$$\rho_n u_n \text{ bounded in } L^2(0,T;W^{1,1}(\Omega)).$$

It remains to show that for every compact set $K \subset \Omega$, we have

$$\partial_t(\rho_n u_n) \text{ is bounded in } L^2(0,T;W^{-2,4/3}(K)). \tag{32}$$

Indeed, (32) together with Aubin's Lemma (see Lions [Lio69] or Dubinskii [Dub65]), yields the compactness of $\rho_n u_n$ in $L^2(0,T;L^p(K))$ for all $p \in [1,3/2)$.

To prove (32), we use the momentum equation (6), first noticing from Lemma 4.2 and Lemma 4.3 that

$$\begin{aligned} \mathrm{div}\,(\sqrt{\rho_n}u_n \otimes \sqrt{\rho_n}u_n) &\in L^\infty(0,T;W^{-1,1}(K)) \\ \nabla\rho_n^\gamma &\in L^\infty(0,T;W^{-1,1}(K)), \end{aligned}$$

So we only have to check that the terms $\nabla(h(\rho_n)\nabla u_n)$, $\nabla(h(\rho_n)^t\nabla u_n)$ and $\nabla(g(\rho_n)\mathrm{div}\,u_n)$ are bounded in $L^\infty(0,T;W^{-2,4/3}(K))$. To that purpose, we write

$$h(\rho_n)\nabla u_n = \nabla(h(\rho_n)u_n) - u_n\nabla h(\rho_n), \tag{33}$$

(and we proceed similarly with the other two terms). The second term in (33) reads

$$u_n\nabla h(\rho_n) = \sqrt{\rho_n}u_n\frac{\nabla h(\rho_n)}{\sqrt{\rho_n}} = 2\sqrt{\rho_n}u_n h'(\rho_n)\nabla\sqrt{\rho_n}$$

which is bounded in $L^\infty(0,T;L^1(\Omega))$ thanks to (29) and Lemma 4.3. The first term in (33) can be rewritten

$$\nabla[h(\rho_n)u_n] = \nabla\left[\frac{h(\rho_n)}{\sqrt{\rho_n}}\sqrt{\rho_n}u_n\right],$$

which is bounded in $L^\infty(0,T;W^{-1,3/2}(\Omega))$ thanks to the following lemma:

Lemma 4.5 *For all compact set K, $h(\rho_n)/\sqrt{\rho_n}$ and $g(\rho_n)/\sqrt{\rho_n}$ are bounded in $L^\infty(0,T;L^6(K))$.*

The proof of this Lemma is a bit technical in full generality and will be postponed to Appendix B. However, note that, in the particular case $h(\rho) = \nu\rho$, we have $h(\rho_n)/\sqrt{\rho_n} = \sqrt{\rho}_n$ and Lemma 4.5 follows straightforwardly from Lemma 4.1.

We deduce that $h(\rho_n)D(u_n)$ and $g(\rho_n)\mathrm{div}\,u_n$ are bounded in

$$L^\infty(0,T;W^{-1,3/2}(K) + L^1(K)),$$

and since $L^1(K) \subset W^{-1,4/3}(K)$ and $W^{-1,3/2}(K) \subset W^{-1,4/3}(K)$ we conclude that $h(\rho_n)D(u_n)$ and $g(\rho_n)\mathrm{div}\,u_n$ are bounded in $L^\infty(0,T;W^{-1,4/3}(K))$, which conclude the proof of Lemma 4.4.

Step 5: Convergence of $\sqrt{\rho_n}u_n$

Lemma 4.6 *The quantity $\sqrt{\rho_n}u_n$ converges strongly in $L^2_{loc}((0,T)\times\Omega)$ to $m/\sqrt{\rho}$ (defined to be zero when $m=0$).*

In particular, we have $m(x,t)=0$ a.e. on $\{\rho(x,t)=0\}$ and there exists a function $u(x,t)$ such that $m(x,t)=\rho(x,t)u(x,t)$ and

$$\begin{array}{ll} \rho_n\, u_n \longrightarrow \rho\, u & \text{strongly in } L^2(0,T;L^p_{loc}(\Omega))\,, \quad p\in[1,3/2) \\ \sqrt{\rho_n}\, u_n \longrightarrow \sqrt{\rho}\, u & \text{strongly in } L^2_{loc}((0,T)\times\Omega) \end{array}$$

(note that u is not uniquely defined on the vacuum set $\{\rho(x,t)=0\}$).

Proof. First of all, since $m_n/\sqrt{\rho_n}$ is bounded in $L^\infty(0,T;L^2(\Omega))$, Fatou's lemma yields

$$\int \liminf \frac{m_n^2}{\rho_n}\,dx<\infty.$$

In particular, we have $m(x,t)=0$ a.e. in $\{\rho(x,t)=0\}$. So if we define the limit velocity $u(x,t)$ by setting $u(x,t)=m(x,t)/\rho(x,t)$ when $\rho(x,t)\neq 0$ and $u(x,t)=0$ when $\rho(x,t)=0$, we have

$$m(x,t)=\rho(x,t)u(x,t)$$

and

$$\int\frac{m^2}{\rho}\,dx=\int\rho|u|^2\,dx<\infty.$$

Moreover, Fatou's lemma yields

$$\begin{aligned} \int\rho|u|^2\ln(1+|u|^2)\,dx &\leq \int\liminf\rho_n|u_n|^2\ln(1+|u_n|^2)\,dx \\ &\leq \liminf\int\rho_n|u_n|^2\ln(1+|u_n|^2)\,dx \end{aligned}$$

and so $\rho|u^2|\ln(1+|u|^2)$ is in $L^\infty(0,T;L^1(\Omega))$.

Next, since m_n and ρ_n converge almost everywhere, it is readily seen that in $\{\rho(x,t)\neq 0\}$, $\sqrt{\rho_n}u_n=m_n/\sqrt{\rho_n}$ converges almost everywhere to $\sqrt{\rho}u=m/\sqrt{\rho}$. Moreover, we have:

$$\sqrt{\rho_n}u_n 1_{|u_n|\leq M}\longrightarrow\sqrt{\rho}u1_{u\leq M}\quad\text{almost everywhere.}\tag{34}$$

As a matter of fact, the convergence holds almost everywhere in $\{\rho(x,t)\neq 0\}$, and in $\{\rho(x,t)=0\}$, we have $\sqrt{\rho_n}u_n1_{|u_n|\leq M}\leq M\sqrt{\rho_n}\longrightarrow 0$.

We are now in position to complete the proof of Lemma 4.6: For $M > 0$, we cut the L^2 norm as follows:

$$\begin{aligned}\int |\sqrt{\rho_n} u_n - \sqrt{\rho} u|^2 \, dx \, dt \quad \leq \quad & \int |\sqrt{\rho_n} u_n 1_{|u_n| \leq M} - \sqrt{\rho} u 1_{|u| \leq M}|^2 \, dx \, dt \\ & + 2 \int |\sqrt{\rho_n} u_n 1_{|u_n| \geq M}|^2 \, dx \, dt \\ & + 2 \int |\sqrt{\rho} u 1_{|u| \geq M}|^2 \, dx \, dt\end{aligned}$$

It is obvious that $\sqrt{\rho_n} u_n 1_{|u_n| \leq M}$ is bounded uniformly in $L^\infty(0, T; L^3(\Omega))$, so (34) gives the convergence of the first integral:

$$\int |\sqrt{\rho_n} u_n 1_{|u_n| \leq M} - \sqrt{\rho} u 1_{|u| \leq M}|^2 \, dx \, dt \longrightarrow 0. \tag{35}$$

Finally, we write

$$\int |\sqrt{\rho_n} u_n 1_{|u_n| \geq M}|^2 \, dx \, dt \leq \frac{1}{\ln(1 + M^2)} \int \rho_n u_n^2 \ln(1 + |u_n|^2) \, dx \, dt \tag{36}$$

and

$$\int |\sqrt{\rho} u 1_{|u| \geq M}|^2 \, dx \, dt \leq \frac{1}{\ln(1 + M^2)} \int \rho u^2 \ln(1 + |u|^2) \, dx \, dt. \tag{37}$$

Putting together (35), (36) and (37), we deduce

$$\limsup_{n \to \infty} \int |\sqrt{\rho_n} u_n - \sqrt{\rho} u|^2 \, dx \, dt \leq \frac{C}{\ln(1 + M^2)}$$

for all $M > 0$, and the lemma follows by taking $M \to \infty$.

Step 6: Convergence of the diffusion terms

Lemma 4.7 *We have*

$$h(\rho_n) \nabla u_n \longrightarrow h(\rho) \nabla u \text{ in } \mathcal{D}'$$

$$h(\rho_n)^t \nabla u_n \longrightarrow h(\rho)^t \nabla u \text{ in } \mathcal{D}'$$

and

$$g(\rho_n) \, div \, u_n \longrightarrow g(\rho) \, div \, u \text{ in } \mathcal{D}'$$

Proof. Let ϕ be a test function, then

$$\int h(\rho_n)\nabla u_n \phi \, dx\, dt = -\int h(\rho_n) u_n \nabla\phi \, dx\, dt + \int u_n \nabla h(\rho_n)\phi \, dx\, dt$$
$$= -\int \frac{h(\rho_n)}{\sqrt{\rho}_n}\sqrt{\rho_n} u_n \nabla\phi \, dx\, dt + \int \sqrt{\rho_n} u_n \frac{h'(\rho_n)}{\sqrt{\rho_n}}\nabla\rho_n \phi \, dx\, dt$$

Thanks to Lemma 4.5, we know that $\frac{h(\rho_n)}{\sqrt{\rho}_n}$ is bounded in $L^\infty(0,T;L^6_{loc}(\Omega))$. Moreover, since $h(\rho_n)/\sqrt{\rho}_n \leq \nu\sqrt{\rho_n}$, this term converges almost everywhere to $h(\rho)/\sqrt{\rho}$ (defined to be zero on the vacuum set). Therefore, it converges strongly in $L^2_{loc}((0,T)\times\Omega)$; This is enough to prove the convergence of the first term.

Next, we note that

$$\frac{h'(\rho_n)}{\sqrt{\rho_n}}\nabla\rho_n = \nabla\psi(\rho_n)$$

with $\psi'(\rho) = h'(\rho)/\sqrt{\rho} = \sqrt{\rho}\varphi'(\rho)$. Since

$$\int |\nabla\psi(\rho)|^2 \, dx = \int \rho|\nabla\varphi(\rho)|^2 \, dx,$$

we have that $\nabla\psi(\rho_n)$ is bounded in $L^\infty(0,T,L^2(\Omega))$. Moreover, (13) yields

$$h'(\rho) \leq C\rho^{-1/2+\nu/3} \text{ when } \rho \leq 1$$

and so

$$\psi(\rho) \leq C\rho^{\nu/3} \text{ when } \rho \leq 1.$$

Therefore, an argument similar to the proof of Lemma 4.5 shows that $\psi(\rho_n)$ is bounded in $L^\infty(0,T;L^6_{loc}(\Omega))$. Since it converges almost everywhere (ψ is a continuous function), it converges strongly in $L^2_{loc}((0,T)\times\Omega)$. It follows that

$$\nabla\psi(\rho_n) \rightharpoonup \nabla\psi(\rho) \qquad L^2_{loc}((0,T)\times\Omega)\text{-weak} .$$

A similar argument holds for $h(\rho_n)^t\nabla u_n$ and $g(\rho_n)\mathrm{div}\, u_n$ using the fact that $|g(\rho)| \leq Ch(\rho)$ and $|g'(\rho)| \leq Ch'(\rho)$.

A Proof of Lemma 3.1

In this appendix, we briefly recall the proof of the mathematical entropy of Bresch and Desjardins (21). To that purpose, we have to evaluate

$$\frac{d}{dt}\int\left[\frac{1}{2}\rho|u|^2 + \rho u\cdot\nabla\varphi(\rho) + \frac{1}{2}\rho|\nabla\varphi(\rho)|^2\right] dx + \frac{d}{dt}\int \frac{1}{\gamma-1}\rho^\gamma \, dx.$$

Step 1: First of all, we recall the usual entropy equality:

$$\frac{d}{dt}\int\left[\frac{1}{2}\rho|u|^2+\frac{1}{\gamma-1}\rho^\gamma\right]dx=-\int 2h(\rho)|D(u)|^2\,dx-\int g(\rho)|\mathrm{div}\,u|^2\,dx \tag{38}$$

Step 2: Next, (5) gives

$$\begin{aligned}&\int\rho\partial t\frac{|\nabla\varphi(\rho)|^2}{2}\,dx-\int\frac{|\nabla\varphi(\rho)|^2}{2}\mathrm{div}\,\rho u\,dx\\ &\quad=-\int\rho\nabla u:\nabla\varphi(\rho)\otimes\nabla\varphi(\rho)\,dx+\int\rho^2\varphi'(\rho)\Delta\varphi(\rho)\mathrm{div}\,u\,dx\\ &\qquad+\int\rho[\nabla\varphi(\rho)]^2\mathrm{div}\,u\,dx\end{aligned}$$

and so

$$\begin{aligned}\frac{d}{dt}\int\rho\frac{|\nabla\varphi(\rho)|^2}{2}\,dx &= -\int\rho\nabla u:\nabla\varphi(\rho)\otimes\nabla\varphi(\rho)\,dx\\ &\quad+\int\rho^2\varphi'(\rho)\Delta\varphi(\rho)\mathrm{div}\,u\,dx\\ &\quad+\int\rho[\nabla\varphi(\rho)]^2\mathrm{div}\,u\,dx\end{aligned}\tag{39}$$

Step 3: It remains to evaluate the derivative of the cross-product:

$$\begin{aligned}\frac{d}{dt}\int\rho u\cdot\nabla\varphi(\rho)\,dx &= \int\nabla\varphi(\rho)\cdot\partial_t(\rho u)\,dx+\int\rho u\cdot\partial_t\nabla\varphi(\rho)\,dx\\ &= \int\nabla\varphi(\rho)\cdot\partial_t(\rho u)\,dx-\int\mathrm{div}\,(\rho u)\varphi'(\rho)\partial_t\rho\,dx\\ &= \int\nabla\varphi(\rho)\cdot\partial_t(\rho u)\,dx+\int(\mathrm{div}\,(\rho u))^2\varphi'(\rho)\,dx\end{aligned}\tag{40}$$

Multiplying (6) by $\nabla\varphi(\rho)$, we get:

$$\begin{aligned}&\int\nabla\varphi(\rho)\cdot\partial_t(\rho u)\,dx\\ &\quad=-\int(2h(\rho)+g(\rho))\Delta\varphi(\rho)\mathrm{div}\,u\,dx+2\int\nabla u:\nabla\varphi(\rho)\otimes\nabla h(\rho)\,dx\\ &\qquad-2\int\nabla\varphi(\rho)\cdot\nabla h(\rho)\mathrm{div}\,u\,dx-\int\nabla\varphi(\rho)\cdot\nabla\rho^\gamma\,dx\\ &\qquad-\int\nabla\varphi(\rho)\mathrm{div}\,(\rho u\otimes u)\,dx,\end{aligned}$$

where we used the fact that

$$\int \nabla(g(\rho)\mathrm{div}\, u)\cdot\nabla\varphi(\rho)\, dx = -\int g(\rho)\Delta\varphi(\rho)\mathrm{div}\, u\, dx$$

and

$$\begin{aligned}
&\int \mathrm{div}\,(2h(\rho)D(u))\cdot\nabla\varphi(\rho)\, dx\\
&\quad= \int \partial_j(h(\rho)\partial_j u_i)\partial_i\varphi(\rho)\, dx + \int \partial_j(h(\rho)\partial_i u_j)\partial_i\varphi(\rho)\, dx\\
&\quad= \int \partial_i(h(\rho)\partial_j u_i)\partial_j\varphi(\rho)\, dx + \int \partial_j(h(\rho)\partial_i u_j)\partial_i\varphi(\rho)\, dx\\
&\quad= \int \partial_i h(\rho)\partial_j u_i\partial_j\varphi(\rho)\, dx - \int \partial_i u_i\partial_j h(\rho)\partial_j\varphi(\rho)\, dx\\
&\qquad - \int \partial_i u_i h(\rho)\partial_{jj}\varphi(\rho)\, dx\\
&\qquad + \int \partial_j h(\rho)\partial_i u_j\partial_i\varphi(\rho)\, dx - \int \partial_j u_j\partial_i h(\rho)\partial_i\varphi(\rho)\, dx\\
&\qquad - \int \partial_j u_j h(\rho)\partial_{ii}\varphi(\rho)\, dx\\
&\quad= 2\int \nabla u : \nabla h(\rho)\otimes\nabla\varphi(\rho)\, dx - 2\int \nabla h(\rho)\cdot\nabla\varphi(\rho)\mathrm{div}\, u\, dx\\
&\qquad -2\int h(\rho)\Delta\varphi(\rho)\mathrm{div}\, u\, dx
\end{aligned}$$

Step 4: When φ, h and g satisfies (8) and (22), then (39) and (40) yields

$$\begin{aligned}
&\frac{d}{dt}\left\{\int \rho u\cdot\nabla\varphi(\rho) + \rho\frac{|\nabla\varphi(\rho)|^2}{2}\, dx\right\} + \int \nabla\varphi(\rho)\cdot\nabla p\, dx\\
&\qquad = -\int \nabla\varphi(\rho)\mathrm{div}\,(\rho u\otimes u)\, dx + \int \varphi'(\rho)(\mathrm{div}\,(\rho u))^2\, dx.
\end{aligned}$$

Finally, we have

$$\begin{aligned}
&-\int \nabla\varphi(\rho)\mathrm{div}\,(\rho u\otimes u)\, dx + \int \varphi'(\rho)(\mathrm{div}\,(\rho u))^2\, dx\\
&\quad= \int -\varphi'(\rho)u\cdot\nabla\rho\,\mathrm{div}\,(\rho u) - \varphi'(\rho)\nabla\rho(\rho u\cdot\nabla u) + \varphi'(\rho)(\mathrm{div}\,\rho u)^2\, dx\\
&\quad= \int \rho\varphi'(\rho)\,\mathrm{div}\, u\,\mathrm{div}\,(\rho u) - \rho\varphi'(\rho)\nabla\rho(u\cdot\nabla u)\, dx\\
&\quad= \int \rho^2\varphi'(\rho)(\mathrm{div}\, u)^2 + \rho\varphi'(\rho)u\cdot\nabla\rho\,\mathrm{div}\, u - \rho\varphi'(\rho)\nabla\rho(u\cdot\nabla u)\, dx
\end{aligned}$$

so using (22) and (8), we get

$$
\begin{aligned}
-\int \nabla\varphi(\rho)\,\mathrm{div}\,(\rho u\otimes u)\,dx &+ \int \varphi'(\rho)(\mathrm{div}\,(\rho u))^2\,dx \\
&= 2\int \rho h'(\rho)(\mathrm{div}\,u)^2 + \nabla(h(\rho))\cdot u\,\mathrm{div}\,u - \nabla(h(\rho))(u\cdot\nabla u)\,dx \\
&= 2\int \rho h'(\rho)(\mathrm{div}\,u)^2 - h(\rho)(\mathrm{div}\,u)^2 - h(\rho)u\cdot\nabla\mathrm{div}\,u\,dx \\
&\qquad +2\int h(\rho)\partial_i u_j\partial_j u_i + h(\rho)u\cdot\nabla\mathrm{div}\,u\,dx \\
&= \int (2\rho h' - 2h)(\mathrm{div}\,u)^2 + 2h(\rho)\partial_i u_j\partial_j u_i\,dx \\
&= \int g(\rho)\,(\mathrm{div}\,u)^2\,dx + \int 2h(\rho)\partial_j u_i\partial_i u_j\,dx
\end{aligned}
$$

which yields

$$
\begin{aligned}
\frac{d}{dt}\left\{\int \rho u\cdot\nabla\varphi(\rho) + \rho\frac{|\nabla\varphi(\rho)|^2}{2}\,dx\right\} &+ \int \nabla\varphi(\rho)\cdot\nabla\rho^\gamma\,dx \\
&= \int g(\rho)(\mathrm{div}\,u)^2\,dx + \int 2h(\rho)\partial_j u_i\partial_i u_j\,dx.
\end{aligned}
$$

Adding this equality and (38), and using the fact that

$$
\int 2h(\rho)|D(u)|^2\,dx - \int 2h(\rho)\partial_j u_i\partial_i u_j\,dx = \int 2h(\rho)\left(\frac{\partial_i u_j - \partial_j u_i}{2}\right)^2
$$

we easily get (21).

B Proof of Lemma 4.5

We shall only prove the result for $h(\rho_n)/\sqrt{\rho_n}$. Using the fact that

$$
|g(\rho)| \le Ch(\rho), \quad \text{and } |g'(\rho)| \le Ch'(\rho) \quad \text{for all } \rho,
$$

a similar proof follows for $g(\rho_n)/\sqrt{\rho_n}$

Note that In view of (13), we have

$$
\frac{h(\rho)}{\sqrt{\rho}} \le C\rho^\nu \qquad \text{if } \rho \le 1,
$$

so we only need to control $\frac{h(\rho_n)}{\sqrt{\rho_n}}$ for large ρ_n. This will be achieved differently depending on the dimension.

When $N = 2$, the fact that $\sqrt{\rho_n}$ is bounded in $L^\infty(0,T;H^1(\Omega))$ and Sobolev's inequalities implies that ρ_n is bounded in $L^\infty(0,T;L^p(\Omega))$ for all $p \in [1,\infty[$. Moreover, in view of (13), we have

$$\frac{h(\rho)}{\sqrt{\rho}} \leq \left\{ \begin{array}{ll} C\rho^{1/\nu} & \text{if } \rho \geq 1 \\ C\rho^{\nu} & \text{if } \rho \leq 1 \end{array} \right. .$$

So there exists $q_0 > 1$ such that $h(\rho_n)/\sqrt{\rho_n}$ is bounded in $L^\infty(0,T;L^q(\Omega))$ for all $q > q_0$. In particular, $h(\rho_n)/\sqrt{\rho_n}$ is bounded in $L^\infty(0,T;L^p(K))$ for all $p \in [1,\infty[$ for any compact set K.

When $N = 3$, we note that

$$\nabla\left(\frac{h(\rho)}{\sqrt{\rho}}\right) = 2h'(\rho)\nabla\sqrt{\rho} - \frac{h(\rho)}{2\rho^{3/2}}\nabla\rho,$$

and since conditions (8) and (11) yields

$$h'(\rho)\rho = g(\rho) + h(\rho) \geq \frac{3g(\rho) + h(\rho)}{3} \geq \frac{\nu}{3}h(\rho),$$

we have

$$|\nabla\left(\frac{h(\rho)}{\sqrt{\rho}}\right)| \leq C|h'(\rho)\nabla\sqrt{\rho}|.$$

So inequality (23) yields

$$||\nabla\left(\frac{h(\rho_n)}{\sqrt{\rho_n}}\right)||_{L^\infty(0,T;L^2(\Omega))} \leq C \tag{41}$$

When $\Omega = \mathbb{R}^3$, Sobolev's inequalities implies that $h(\rho_n)/\sqrt{\rho_n}$ is bounded in $L^\infty(0,T;L^6(\Omega))$. When Ω is a subset of $\mathbb{R}^3$, we note that (13) gives

$$\frac{h(\rho)}{\sqrt{\rho}} \leq \left\{ \begin{array}{ll} C\rho^{1/6+3/\nu} & \text{if } \rho \geq 1 \\ C\rho^{1/6+\nu/3} & \text{if } \rho \leq 1 \end{array} \right. .$$

So there exists a constant $s \leq 1$ such that

$$\left(\left(\frac{h(\rho_n)}{\sqrt{\rho_n}}\right)^s - 1\right)_+ \in L^\infty(0,T;L^2(\Omega))$$

Moreover

$$\begin{aligned}
\left|\nabla\left(\frac{h(\rho_n)}{\sqrt{\rho_n}}\right)^s 1_{h(\rho_n)/\sqrt{\rho_n}\geq 1}\right| &= \left|\left(\frac{h(\rho_n)}{\sqrt{\rho_n}}\right)^{s-1}\nabla\left(\frac{h(\rho_n)}{\sqrt{\rho_n}}\right)1_{h(\rho_n)/\sqrt{\rho_n}\geq 1}\right| \\
&\leq \left|\nabla\left(\frac{h(\rho_n)}{\sqrt{\rho_n}}\right)\right| \in L^\infty(0,T;L^2(\Omega)),
\end{aligned}$$

using the fact that $s-1\leq 0$. It follows that $(h(\rho_n)/\sqrt{\rho_n})^s 1_{\rho_n\geq 1}$ is bounded in $L^\infty(0,T;H^1(\Omega))$ which in turn gives

$$\left(\frac{h(\rho_n)}{\sqrt{\rho_n}}\right)^{s_1} 1_{\rho_n\geq 1} \in L^\infty(0,T;L^2(\Omega)),$$

for all $s_1\in(s,3s)$. As long as $3s\leq 1$, we can repeat this argument with $3s$ instead of s. Eventually, this will lead to

$$\left(\frac{h(\rho_n)}{\sqrt{\rho_n}}\right) 1_{\rho_n\geq 1} \in L^\infty(0,T;L^2(\Omega)),$$

which, together with (41) implies that $(h(\rho_n)/\sqrt{\rho_n})1_{\rho_n\geq 1}$ is bounded in $L^\infty(0,T;L^6(\Omega))$.

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