

FINITE TWO-DISTANCE TIGHT FRAMES

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ABSTRACT. A finite collection of unit vectors $S \subset \mathbb{R}^n$ is called a spherical two-distance set if there are two numbers a and b such that the inner products of distinct vectors from S are either a or b . We prove that if $a \neq -b$, then a two-distance set that forms a tight frame for \mathbb{R}^n is a spherical embedding of a strongly regular graph. We also describe all two-distance tight frames obtained from a given graph. Together with an earlier work by S. Waldron on the equiangular case (*Linear Alg. Appl.*, vol. 41, pp. 2228-2242, 2009) this completely characterizes two-distance tight frames. As an intermediate result, we obtain a classification of all two-distance 2-designs.

1. INTRODUCTION

A finite collection of unit vectors $S \subset \mathbb{R}^n$ is called a spherical two-distance set if there are two numbers a and b such that the inner products of distinct vectors from S are either a or b . If in addition $a = -b$, then S defines a set of equiangular lines through the origin in \mathbb{R}^n . Equiangular lines form a classical subject in discrete geometry following foundational papers of Van Lint, Seidel, and Lemmens [17, 16]. Equiangular line sets are closely related to strongly regular graphs and two-graphs [10, 11] which form the main source of their constructions. Another group of results is concerned with bounding the maximum size $g(n)$ of spherical two-distance sets in n dimensions. We refer to [3, 4] for the latest results on upper bounds on $g(n)$ as well as an overview of the relevant literature.

A finite collection of vectors $S = \{x_1, \dots, x_N\} \subset \mathbb{R}^n$ is called a *finite frame* for the Euclidean space \mathbb{R}^n if there are constants $0 < A \leq B < \infty$ such that for all $x \in \mathbb{R}^n$

$$A\|x\|^2 \leq \sum_{i=1}^N \langle x, x_i \rangle^2 \leq B\|x\|^2. \quad (1.1)$$

If $A = B$, then S is called an A -tight frame, in which case

$$A = \frac{1}{n} \sum_i \|x_i\|^2. \quad (1.2)$$

It is trivially seen that a finite collection of vectors $S = \{x_i : i = 1, \dots, N\} \subset \mathbb{R}^n$ is an A -tight frame if and only if for any $x \in \mathbb{R}^n$,

Date: February 13, 2015.

2000 Mathematics Subject Classification. Primary 42C15; Secondary 15B48.

Key words and phrases. Spherical two-distance sets, finite tight frames, strongly regular graphs, spherical 2-designs, spherical designs of harmonic index 2.

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$$Ax = \sum_{i=1}^N \langle x, x_i \rangle x_i. \quad (1.3)$$

If in addition $\|x_i\| = 1$ for all $i \in I$, then S is a finite unit-norm tight frame or FUNTF. If at the same time S is a spherical two-distance set, we call it a *two-distance tight frame*. In particular, if the two inner products in S satisfy the condition $a = -b$, then it is an *equiangular tight frame* or ETF. All frames in this paper will be assumed unit-norm.

The Gram matrix G of S is defined by $G_{ij} = \langle x_i, x_j \rangle$, $1 \leq i, j \leq N$, where $N = |S|$. If S is a FUNTF for \mathbb{R}^n , then it is straightforward to show [12] that G has one nonzero eigenvalue $\lambda = N/n$ of multiplicity n and eigenvalue 0 of multiplicity $N - n$.

Frames have been used in signal processing and have a large number of applications in sampling theory, wavelet theory, data transmission, and filter banks [8, 14]. The study of ETFs was initiated by Strohmer and Heath [23] and Holmes and Paulsen [13]. In particular, [13] shows that equiangular tight frames give error correcting codes that are robust against two erasures. Bodmann et al. [6] show that ETFs are useful for signal reconstruction when all the phase information is lost. Sustik et al. [22] derived necessary conditions on the existence of ETFs as well as bounds on their maximum cardinality.

Benedetto and Fickus [5] introduced a useful parameter of the frame, called the *frame potential*. For our purposes it suffices to define it as $FP(S) = \sum_{i,j=1}^N \langle x_i, x_j \rangle^2$. For a two-distance frame we obtain

$$\sum_{i,j=1}^N \langle x_i, x_j \rangle^2 = N + 2\nu_a a^2 + (N(N-1) - 2\nu_a) b^2, \quad (1.4)$$

where $\nu_a = |\{(i, j) : i < j : \langle x_i, x_j \rangle = a\}|$.

Theorem 1.1. [5, Theorem.6.2] *Let $N > n$. If $S = \{x_i : 1 \leq i \leq N\}$ is any set of unit-norm vectors, then*

$$FP(S) \geq \frac{N^2}{n} \quad (1.5)$$

with equality if and only if S is a tight frame.

A finite collection of unit vectors $S = \{x_i : i = 1, \dots, N\}$ in \mathbb{R}^n is called a *spherical 2-design* [10] if

$$\sum_{i=1}^N x_i = 0, \quad \sum_{i,j=1}^N \langle x_i, x_j \rangle^2 = \frac{N^2}{n}. \quad (1.6)$$

In other words, a spherical 2-design is a FUNTF with the center of mass at the origin.

Remark 1.1. In [2] spherical sets that satisfy only the tight frame condition (the second condition in (1.6)) are called spherical designs of harmonic index 2. In the sequel we will refer to such spherical designs as shifted 2-designs.

To state our main result we need several definitions. A regular graph of degree k on v vertices is called *strongly regular* if every two adjacent vertices have λ common neighbors and every two non-adjacent vertices have μ common neighbors. Below we use the notation $\text{SRG}(v, k, \lambda, \mu)$ to denote such a graph. Note that the complement of a strongly regular graph $\text{SRG}(v, k, \lambda, \mu)$, is also strongly regular, namely $\text{SRG}(v, v - k - 1, v - 2k + \mu - 2, v - 2k + \lambda)$. The theory of strongly regular graphs is presented, for instance, in [11, 9]. Below we use a classical construction of spherical embeddings of strongly regular graphs introduced by Delsarte, Goethals, and Seidel [10, Example 9.1]; see also [19, 1]. Roughly speaking, a spherical embedding of $\Gamma = \text{SRG}(v, k, \lambda, \mu)$ is obtained by associating a basis of \mathbb{R}^v with the vertices of Γ and projecting these vectors on an eigenspace of the adjacency matrix of Γ . A more detailed description is given in Sect. 3 after we develop all the necessary pieces of notation.

In this paper we characterize two-distance FUNTFs by linking them to spherical 2-designs and strongly regular graphs. Our main result is as follows:

Theorem 1.2. *Let $S = \{x_i : i \in I\}$ be a non-equiangular two-distance FUNTF in \mathbb{R}^n . Then S forms a spherical two-distance 2-design or a shifted 2-design. In either case S can be obtained as a spherical embedding of a strongly regular graph. Under spherical embedding, every strongly regular graph gives rise to three different two-distance 2-designs and therefore, to six different two-distance tight frames, two of which are regular simplices.*

The proof is given in Sections 2, 3. As an intermediate result (see Theorem 3.4), we fully characterize spherical 2-designs that form spherical two-distance sets.

Strongly regular graphs form examples of classical objects in algebraic combinatorics called association schemes [9]. Although we do not use the language of schemes in this paper, we note that our results contribute to the study of the general problem of characterizing spherical designs that can be obtained from association schemes.

Note that the connection between equiangular line sets and strongly regular graphs is well known (Seidel et al. [21, 10]; see also [11]). It has been recently addressed in the context of frame theory, particularly in the study of ETFs [23, 13]. A recent paper by Waldron [24] proves that an ETF in \mathbb{R}^n with $N \geq n$ vectors exists if and only if there exists an SRG($N - 1, k, (3k - N)/2, k/2$), where k is a certain function of n and N . Furthermore, [24] also contains many examples of ETFs in $\mathbb{R}^n, n \leq 50$. Together with Theorem 1.2 this result completes the description of two-distance tight frames, equiangular or not.

2. BASIC PROPERTIES

We begin with an easy example of 2-distance FUNTFs which is given by the following construction. We will need the following theorem.

Theorem 2.1 (Larman, Rogers, and Seidel, [15]). *Let S be a spherical two-distance set in \mathbb{R}^n . If $|S| > 2n + 1$ then the inner products a, b are related by the equation $b = (ka - 1)/(k - 1)$ where $k \in \{2, \dots, \lfloor (1 + \sqrt{2n})/2 \rfloor\}$ is an integer.*

The original proof of [15] had $2n + 3$ in place of $2n + 1$, while the above improvement is due to Neumaier [19]. Given the value of a , we denote by $b_k(a)$ the corresponding value of b .

Proposition 2.2. *Let e_1, \dots, e_{n+1} be the standard basis in \mathbb{R}^{n+1} . The projection of the set*

$$S = \{e_i + e_j, 1 \leq i < j \leq n + 1\} \quad (2.1)$$

on the hyperplane $x_1 + \dots + x_{n+1} = 2$ forms a two-distance tight frame for \mathbb{R}^n .

Proof. Note that the inner products of distinct vectors in S are either 1 or 0. Let

$$\nu_{1,1} = |\{(i, j) : i < j, \langle e_1 + e_2, e_i + e_j \rangle = 1\}|$$

Observe that (i, j) is contained in this set if and only if $i = 1$ or $i = 2$, and we obtain $\nu_{1,1} = 2(n - 1)$. By symmetry, the value $\nu_{1,1}$ does not depend on the choice of the fixed vector $e_1 + e_2$, so the total number of unordered pairs of vectors in S with inner product 1 equals

$$\nu_1 = \frac{1}{2} \binom{n+1}{2} \nu_{1,1} = \frac{1}{2} (n-1)n(n+1).$$

The pairs of distinct vectors not counted in ν_1 are orthogonal, and their number is

$$\nu_0 = \binom{n(n+1)/2}{2} - \nu_1 = \frac{1}{8} (n-2)(n-1)n(n+1).$$

Now let us project the vectors of S on the plane $x_1 + \dots + x_{n+1} = 2$ and scale the result to place them on the unit sphere around the point $z_0 = \frac{2}{n+1}(1, 1, \dots, 1)$. Since each vector $z_{i,j} = e_i + e_j$ already belongs to the plane, the resulting vector will be $z'_{i,j} = c(z_{i,j} - z_0)$ where $c = \frac{1}{\sqrt{2 - \frac{4}{n+1}}}$. By a series of computations we see that

$$\langle z'_{i,j}, z'_{k,l} \rangle = c^2 (\langle z_{i,j}, z_{k,l} \rangle - \frac{4}{n+1}).$$

And since $\langle z_{i,j}, z_{k,l} \rangle \in \{0, 1\}$ we arrive that the fact that $\langle z'_{i,j}, z'_{k,l} \rangle \in \{a, b\}$ with $a = (n-3)/(2(n-1))$ and $b = -2/(n-1)$.

This information suffices to compute the frame potential, and we obtain

$$FP(S) = N + 2\nu_1 a^2 + 2\nu_0 b^2 = \frac{N^2}{n}$$

The frame potential meets the lower bound (1.5) with equality, which implies that S forms a FUNTF for \mathbb{R}^n . \square

We next give a characterization result for two-distance FUNTFs.

Definition 2.3. Let $S \subset \mathbb{R}^n$ be a spherical two-distance set with inner products a and b , $b < a$, let $x_i \in S$, and let

$$N_{a,i} = |\{j : x_j \in S, \langle x_i, x_j \rangle = a\}|.$$

S is called *regular* if $N_{a,i}$ does not depend on i . For regular sets we denote this quantity simply by N_a .

Theorem 2.4. Let $S \subset \mathbb{R}^n$, $|S| = N$ be a two-distance FUNTF with inner products a and b such that $a^2 - b^2 \neq 0$. Then S is regular and

$$N_a = \frac{(N/n) - 1 - (N-1)b^2}{a^2 - b^2} \quad (2.2)$$

$$-n(a+b) - nab(N-1) = N-n \quad \text{or} \quad (N-n)(a+b) - nab(N-1) = N-n. \quad (2.3)$$

Proof. G is similar to a diagonal matrix of order N with n nonzero entries $\lambda = N/n$ on the diagonal. Therefore, $G^2 - \lambda G = 0$, so $G^2 = \lambda G$ and $(G^2)_{ii} = \lambda$ for all i since $G_{ii} = 1$. We also have $(G^2)_{ii} = \sum_{j=1}^N G_{ij}^2$, so the norm of every row and of every column is the same and equals $\sqrt{\lambda}$.

Now let N_a be the number of entries a in any fixed column. Then

$$1 + a^2 N_a + b^2 (N-1 - N_a) = \frac{N}{n}.$$

This implies (2.2).

Thus, $\mathbf{1} = (11 \dots 1)$ is an eigenvector of the Gram matrix G with eigenvalue 0 or N/n . Suppose it is the former, then $G \cdot \mathbf{1} = 0$, so the sum of entries in every row is 0. This implies that $1 + a N_a + (N-1 - N_a)b = 0$. By substituting N_a given by (2.2) into this last equation, and after some simplifications we obtain the first of the two options for b in (2.3).

Now suppose that $G \cdot \mathbf{1} = \frac{N}{n} \mathbf{1}$, so the sum of entries of G in any given row equals N/n . Repeating the calculation performed for the first case, we obtain the second of the two possibilities for b . \square

Remark 2.1. Another way to express the alternative in (2.3) is as follows. The sum of squared entries of every row of G equals N/n and the sum of the entries is either 0 or N/n . These two equations translate into the two conditions for a and b .

In the next section we characterize FUNTF for each of the two cases in (2.3).

Remark 2.2. If $a = -b$, then the statement of Theorem 2.4 does not hold. Indeed, consider the set $S = \{x_1, \dots, x_{28}\}$ of 28 vectors in \mathbb{R}^7 constructed as in (2.1). By Theorem 2.1 the inner products between distinct vectors in S are $\pm 1/3$, so they form a set of equiangular lines. For any given vector $x \in S$ we have $|\{y \in S : \langle x, y \rangle = 1/3\}| = 12$ and $|\{y \in S : \langle x, y \rangle = -1/3\}| = 15$. Now consider the set $S' = \{-x_1, x_2, \dots, x_{28}\}$ which is also a FUNTF with inner products $\pm 1/3$, but the first column of G contains 12 entries equal to $-1/3$, which is different from all the other columns.

3. TWO-DISTANCE FUNTFs AND STRONGLY REGULAR GRAPHS

Connections between equiangular line sets and ETFs on the one side and strongly regular graphs on the other are well known and have been used in the literature to characterize the sets of parameters of ETFs [11, Ch. 11], [24]. In this section we extend this connection by relating two-distance (non equiangular) FUNTFs, 2 designs, and strongly regular graphs.

We begin with a necessary condition for the existence of two-distance FUNTFs. Let S be such a frame. The Gram matrix of any two-distance set with inner products a, b can be written as

$$G = I + a\Phi_1 + b\Phi_2, \quad (3.1)$$

where Φ_1 and Φ_2 are the corresponding indicator matrices. We also denote by Γ_1 and Γ_2 the graphs with adjacency matrices Φ_1 and Φ_2 , respectively.

Proposition 3.1. *If S is a 2-distance FUNTF in \mathbb{R}^n with inner products a, b , then S is either an n -dimensional spherical 2-design, or is similar to an $(n - 1)$ -dimensional spherical 2-design contained in a subsphere of radius $\sqrt{1 - 1/n}$. In the former (resp., latter) case a and b satisfy the first (resp., second) equality in (2.3).*

Proof. Let $S = \{x_i : 1 \leq i \leq N\}$ and let $s = \sum_{i=1}^N x_i$. Then for each $i, 1 \leq i \leq N$ the value $t := \langle x_i, s \rangle$ does not depend on i and is equal to $t = N_a a + (N - N_a)b + 1$, where N_a is given in (2.2).

Applying (1.3) for $x = s$, we obtain

$$\frac{N}{n}s = \sum_{i=1}^N tx_i = ts.$$

Hence either $s = 0$ and S is a spherical 2-design, or $t = \frac{N}{n}$ and then $\langle s, s \rangle = Nt = \frac{N^2}{n}$.

Suppose that $s \neq 0$ (equivalently $t = N/n$). For each $i, 1 \leq i \leq N$, denote $y_i = \frac{x_i - s/N}{\sqrt{1 - 1/n}}$. We will show that the set $S' = \{y_i : i = 1, \dots, N\}$, which is similar to the set S , forms a spherical 2-design in \mathbb{R}^{n-1} . This will imply that S lies on a sphere of radius $\sqrt{1 - 1/n}$ in \mathbb{R}^n .

First we check that $\langle y_i, s \rangle = 0$ for all i . Indeed,

$$\langle y_i, s \rangle = \frac{\langle x_i, s \rangle - \langle s, s \rangle/N}{\sqrt{1 - 1/n}} = \frac{N/n - \frac{N^2/n}{N}}{\sqrt{1 - 1/n}} = 0. \quad (3.2)$$

This establishes that S' is an $(n - 1)$ -dimensional set. Moreover, S' lies on the unit sphere. Indeed, using that $\langle y_i, s \rangle = 0$, we obtain

$$\|y_i\|^2 = \frac{\|x_i\|^2 - \|s/N\|^2}{1 - 1/n} = \frac{1 - \frac{N^2/n}{N^2}}{1 - 1/n} = 1.$$

Clearly S' is a two-distance set. It remains to show that S' forms a 2-design (1.6). The center-of-masses condition is clearly satisfied. To check the tight frame condition let us compute the frame potential of S' and use Theorem 1.1. We

have

$$\begin{aligned}
\frac{N^2}{n} &= \sum_{i,j=1}^N |\langle x_i, x_j \rangle|^2 = \sum_{i,j=1}^N \left| \left\langle \sqrt{1 - \frac{1}{n}} y_i + \frac{s}{N}, \sqrt{1 - \frac{1}{n}} y_j + \frac{s}{N} \right\rangle \right|^2 \\
&= \sum_{i,j=1}^N \left(\left(1 - \frac{1}{n}\right) \langle y_i, y_j \rangle + \frac{\|s\|^2}{N^2} \right)^2 \\
&= \left(1 - \frac{1}{n}\right)^2 FP(S') + \frac{2}{n} \sum_{i,j=1}^N \langle y_i, y_j \rangle + \frac{N^2}{n^2} \\
&= \left(1 - \frac{1}{n}\right)^2 FP(S') + \frac{N^2}{n^2}
\end{aligned}$$

where the last step uses the condition $\sum_i y_i = 0$. Thus, $FP(S') = \frac{N^2}{n-1}$ and therefore, S' is an $(n-1)$ -dimensional 2-design.

Finally, note that t is an eigenvalue of G , namely, $G \cdot \mathbf{1} = t\mathbf{1}$. Recalling that the two cases in (2.3) correspond to $t = 0$ and $t = N/n$, we obtain the final claim of the proposition. \square

Observe that a related result was proved in [20]. Namely, Theorem 4.7 in that paper states (in our terms) that a spherical set $S \subset \mathbb{R}^n$ is a 2-design if and only if $G \cdot \mathbf{1} = 0$ and $G^2 = \frac{\sum_{x \in S} \|x\|^2}{n} G$.

Due to the Delsarte-Goethals-Seidel theorem ([10, Theorem 7.4]), any spherical two-distance 2-design is associated with a strongly regular graph and therefore, due to Proposition 3.1, any two-distance tight frame, too, is associated with a strongly regular graph. To keep our exposition self-contained we give a short direct proof of this fact.

Proposition 3.2. *If S is a two-distance tight frame with inner products a and b , $a^2 - b^2 \neq 0$, then its associated graph Γ_1 (and Γ_2 as the complement of Γ_1) is a strongly regular graph.*

Proof. It follows from (1.3) and (1.2) that for any two vectors x_k, x_l of S ,

$$\frac{N}{n} \langle x_k, x_l \rangle = \sum_{i=1}^N \langle x_k, x_i \rangle \langle x_i, x_l \rangle. \quad (3.3)$$

Fix indices k and l and assume $\langle x_k, x_l \rangle = a$. Let

$$I_{\alpha,\beta} = \{i \in \{1, \dots, N\} : \langle x_k, x_i \rangle = \alpha \text{ and } \langle x_i, x_l \rangle = \beta\},$$

where $\alpha, \beta \in \{a, b\}$, and let $C_a := |I_{a,a}|$. Note that by the symmetry of the Gram matrix G , we have that $|I_{a,b}| = |I_{b,a}|$. Let us find the cardinality of $I_{a,b}$, i.e., the set of indices i with the entry a in row k and entry b in row l . Consider the subset of indices i in row k of G with $\langle x_k, x_i \rangle = a$ except $i = l$ (in this position row l contains 1). There are $N_a - 1$ such indices, where N_a is the number of a 's in the row (see the remark before Theorem 2.4). We then need to subtract the number of those i for which $\langle x_i, x_l \rangle = a$. But those are precisely the indices in the set $I_{a,a}$. Consequently,

$$|I_{a,b}| = |I_{b,a}| = N_a - C_a - 1. \quad (3.4)$$

We next observe that the union of the (disjoint) sets $I_{\alpha,\beta}$ $\alpha, \beta \in \{a, b\}$ gives all the indices in row k of G except the diagonal entry. Therefore, we obtain

$$N - 1 = |I_{a,a}| + 2|I_{a,b}| + |I_{b,b}|. \quad (3.5)$$

Recall that we have $N_b + N_a = N - 1$. Taking this together with (3.4) and (3.5) and performing simplifications, we obtain

$$|I_{b,b}| = N_b - N_a + C_a + 1.$$

We can then rewrite (3.3) as

$$\begin{aligned} \frac{N}{n}a &= 2(N_a - C_a - 1)ab + C_a a^2 + (N_b - N_a + C_a + 1)b^2 \\ &= 2(N_a - 1)ab + (N_b - N_a + 1)b^2 + C_a(a - b)^2. \end{aligned}$$

Since $a \neq b$, there is a unique C_a that satisfies this equality. In other words, any pair of connected vertices of the associated graph Γ_1 has the same number C_a of common neighbors. Similarly, any two non-connected vertices of Γ_1 have the same number C_b of common neighbors. Therefore, Γ_1 is a strongly regular graph. \square

We now set out to describe all two-distance tight frames. Propositions 3.1 and 3.2 imply that we just need to find all spherical two-distance embeddings of strongly regular graphs and check the 2-design conditions for them.

Spherical embeddings of SRGs. Let Γ_1 be an $\text{SRG}(v, k, \lambda, \mu)$ which is not a complete or empty graph and let Φ_1 be its adjacency matrix. As already mentioned, the set of vertices of Γ_1 can be embedded in the sphere by projecting the vectors of the standard basis of the space \mathbb{R}^v on the eigenspaces of Φ_1 .

The spectral structure of the matrix Φ_1 is as follows. It has three mutually orthogonal eigenspaces that correspond to three eigenvalues: the all-one vector $\mathbf{1}$ with eigenvalue k , an eigenspace E_1 of dimension n_1 with eigenvalue r_1 , and an eigenspace E_2 of dimension n_2 with eigenvalue r_2 [9, p.117]. Note that for Turán graphs and their complements it is possible that $r_1 = k$. The values of n_1, r_1, n_2, r_2 can be found explicitly via the parameters (v, k, λ, μ) . Since these values are useful in constructing examples, we quote the expressions for them from [11, pp.219-220]:

$$\begin{aligned} r_{1,2} &= \frac{1}{2}(\lambda - \mu \pm \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}) \\ n_{1,2} &= \frac{1}{2}\left(v - 1 \mp \frac{2k + (v - 1)(\lambda - \mu)}{\sqrt{(\lambda - \mu)^2 + 4(k - \mu)}}\right). \end{aligned}$$

Geometrically the Delsarte-Goethals-Seidel construction amounts to projecting orthogonally the standard basis vectors of \mathbb{R}^v on an eigenspace, for instance E_1 , and normalizing the projections (they all have the same length) to obtain unit lengths. Denote the obtained spherical set by $S_1 = S_1(\Gamma_1)$ and denote its two inner products by a_1 and b_1 , so that Γ_1 is the graph of inner products a_1, b_1 ; cf. (3.1). It is easy to show [10] that this spherical set supports an n_1 -dimensional 2-design.

Similarly we can obtain an n_2 -dimensional 2-design $S_2 = S_2(\Gamma_1)$ with inner products a_2 and b_2 by projecting \mathbb{R}^N on E_2 and normalizing the projections. Finally, let S_0 denote the trivial one-dimensional embedding and note that $a_0 = b_0 = 1$.

Let Γ_2 be the complement graph of Γ_1 and let Φ_2 be its adjacency matrix. We have

$$\Phi_2 = J - I - \Phi_1.$$

The vector $\mathbf{1}$ is an eigenvector of each of these matrices, and any vector z such that $\langle z, \mathbf{1} \rangle = \sum z_i = 0$ is an eigenvector of J and I . Hence if such vector z is an eigenvector of Φ_1 , it is also an eigenvector of Φ_2 . Thus, the matrices Φ_1 and Φ_2 share the same spectral structure. In particular, Φ_2 also has three eigenvalues and three eigenspaces that coincide with the eigenspaces of Φ_1 : a vector of all ones $\mathbf{1}$ with eigenvalue $v - 1 - k$, an eigenspace E_1 of dimension n_1 with eigenvalue s_1 , an eigenspace E_2 of dimension n_2 with eigenvalue s_2 .

Proposition 3.3. *Let $\Gamma_1(N, k, \lambda, \mu)$ be a strongly regular graph that is not complete or empty. For any two-distance spherical embedding $S = \{x_1, \dots, x_N\}$ of Γ_1 , there are three nonnegative real numbers α, β, γ , $\alpha^2 + \beta^2 + \gamma^2 = 1$ such that for all $i = 1, \dots, N$*

$$x_i = \alpha x_i(0) + \beta x_i(1) + \gamma x_i(2) \tag{3.6}$$

for some $x_i(j), j = 0, 1, 2$. The sets $S_j(\Gamma_1) = \{x_i(j) : 1 \leq i \leq N\}, j = 0, 1, 2$ form the Delsarte-Goethals-Seidel spherical embeddings of Γ_1 and are contained in mutually orthogonal unit spheres of dimensions 1, n_1 , and n_2 , respectively.

Proof. Let S be a two-distance spherical embedding of Γ_1 with distances a and b . Write the Gram matrix G of S as in (3.1). The embedding S exists if and only if G is positive semidefinite. Since the matrices Φ_1 and Φ_2 share the spectral structure, we can find all eigenvalues of G and check their non-negativity. This results in the following inequalities:

$$\begin{aligned} 1 + ak + b(N - 1 - k) &\geq 0 \\ 1 + ar_1 + bs_1 &\geq 0 \\ 1 + ar_2 + bs_2 &\geq 0, \end{aligned} \tag{3.7}$$

(some of these inequalities may trivialize to $1 \geq 0$). The set of all feasible pairs (a, b) is the intersection of at most three half-planes in the plane. Note that this set must belong to the square $[-1, 1]^2$, so it is bounded. Moreover, $G \succeq 0$ if and only if the inequalities (3.7) hold true, so this region is either a triangle or a single point. Since there are always at least two different embeddings, namely S_0 and the $(N - 1)$ -dimensional regular simplex, this set must be a triangle whose vertices are the intersections of any two of the three lines defining the inequalities.

Next we note that these intersection points precisely represent S_0 , S_1 , and S_2 so they are (a_0, b_0) , (a_1, b_1) , and (a_2, b_2) . Indeed, project the basis orthogonally on one of the spaces $\mathbf{1}$, E_1 , E_2 and denote the (normalized) resulting set by X . The eigenvectors of this projection, corresponding to the two other spaces have zero eigenvalues. Subsequently, the eigenvalues of these vectors for the Gram matrix $G = X^t X$ are also zero, which turns two of the inequalities in (3.7) into equalities.

Any other pair (a, b) can be represented as $(a, b) = \alpha^2(a_0, b_0) + \beta^2(a_1, b_1) + \gamma^2(a_2, b_2)$, where $\alpha^2 + \beta^2 + \gamma^2 = 1$ and α, β, γ are non-negative. Now note that the set $\{x_i : 1 \leq i \leq N\}$ such that $x_i = \alpha x_i(0) + \beta x_i(1) + \gamma x_i(2)$, where the set of all vectors $x_i(0)$ forms S_0 , the set of all $x_i(1)$ forms S_1 , and the set of all $x_i(2)$ forms S_2 in mutually orthogonal unit spheres, gives a two-distance spherical embedding of Γ_1 with inner products a and b . Moreover, any such embedding is completely determined by its Gram matrix, and therefore, this gives a description of all spherical two-distance embeddings of Γ_1 . This completes the proof. \square

Proposition 3.3 entails the following description of two-distance 2-designs.

Theorem 3.4. *Any spherical two-distance 2-design $S = \{x_1, \dots, x_N\}$ with graph Γ_1 for one of the distances is either $S_1(\Gamma_1)$ or $S_2(\Gamma_1)$, or a regular $(N - 1)$ -dimensional simplex.*

Proof. We begin with the representation of the vectors x_i given by (3.6). Note that since $\sum_{i=1}^N x_i = 0$, the coefficient α must be 0. If one of β or γ is 0, then S is either S_1 or S_2 . The remaining case is when they are both positive. In this case the set S is $(n_1 + n_2)$ -dimensional, so it must satisfy the tight-frame condition (1.1)-(1.2) for any $x \in \mathbb{R}^{n_1+n_2}$:

$$\frac{N}{n_1 + n_2} \|x\|^2 = \sum_{i=1}^N \langle x, x_i \rangle^2. \tag{3.8}$$

Now we express x as the sum of $x(1)$ and $x(2)$, where $x(1)$ belongs to the space \mathbb{R}^{n_1} that contains all the vectors $x_i(1)$, and $x(2)$ belongs to the space \mathbb{R}^{n_2} containing all $x_i(2)$. Since S_1 and S_2 form 2-designs, they must satisfy the tight-frame condition, namely

$$\frac{N}{n_j} \|x(j)\|^2 = \sum_{i=1}^N \langle x(j), x_i(j) \rangle^2, \quad j = 1, 2.$$

Using (3.6) and (3.8), we obtain:

$$\begin{aligned} \frac{N}{n_1 + n_2} (\|x(1)\|^2 + \|x(2)\|^2) &= \sum_{i=1}^N (\beta \langle x(1), x_i(1) \rangle + \gamma \langle x(2), x_i(2) \rangle)^2 \\ &= \beta^2 \frac{N}{n_1} \|x(1)\|^2 + \gamma^2 \frac{N}{n_2} \|x(2)\|^2 + 2\beta\gamma \sum_{i=1}^N \langle x(1), x_i(1) \rangle \langle x(2), x_i(2) \rangle. \end{aligned}$$

This equality must hold for any $x(1)$ and $x(2)$, so $\beta^2 = \frac{n_1}{n_1+n_2}$ and $\gamma^2 = \frac{n_2}{n_1+n_2}$. To show that with these values of β and γ the set S forms a 2-design we just need to explain why $\sum_{i=1}^N \langle x(1), x_i(1) \rangle \langle x(2), x_i(2) \rangle$ is always 0. Refer to the definition of S_1 and S_2 and let their ambient spaces be E_1 and E_2 . Then the vector with components $\langle x(1), x_i(1) \rangle$ is just $\Phi_1 x(1)$ times a normalizing coefficient, and the vector with components $\langle x(2), x_i(2) \rangle$ is $\Phi_1 x(2)$ with its normalizing coefficient. The first vector belongs to E_1 and the second vector belongs to E_2 so they must be orthogonal.

A regular $(N-1)$ -dimensional simplex is obviously a 2-design and can be considered as a two-distance embedding of Γ_1 with equal distances. Since S_1 and S_2 are not $(N-1)$ -dimensional, the third 2-design that we constructed must be a regular simplex (recall that $n_1 + n_2 = N-1$). This observation finishes the proof of the theorem. \square

Remark 3.1. The regular simplex can be constructed similarly to S_1 and S_2 : it is obtained by finding orthogonal projections of (the basis vectors of) \mathbb{R}^N on $E_1 \cup E_2$ and normalizing to get unit lengths. Another simplex is given by the orthonormal basis itself which represents a trivial projection.

Proof. Proof of Theorem 1.2. We now recap the arguments that lead to the classification of all non-equiangular two-distance tight frames in Theorem 1.2. Let S be such a frame and assume that a and b are the two distinct inner products of the vectors in S . Then, $a^2 - b^2 \neq 0$. First, by Proposition 3.1, S is either a n dimensional spherical 2–design or similar to $(n-1)$ dimensional spherical 2–design. On account of Prop. 3.2 the graphs defined by the Gram matrix of a non-equiangular two-distance FUNTF are strongly regular, so we need to describe all spherical two-distance embeddings of SRGs and check if they satisfy the design condition. We show in Theorem 3.4 that all such embeddings are of the Delsarte-Goethals-Seidel type, and yield spherical two-distance 2-designs. Since each such design gives rise to two FUNTFs, this completes the classification. \square

The results established above enable us to construct large classes of two-distance tight frames. For brevity we write FUNTF(n, N, N_a, a, b) to refer to a two-distance tight frame in n dimensions, with N points, inner products $b < a$, and with N_a entries a in each row of G . We give a few examples of 2-distance frames derived from the table of strongly regular graphs in [9, pp.143ff]. Many more examples can be easily obtained using the described recipe.

SRG(N, k, λ, μ)	2-design FUNTF(n, N, N_a, a, b)	shifted 2-design FUNTF(n, N, N_a, a, b)
(10, 6, 3, 4)	(4, 10, 6, 1/6, -2/3), (5, 10, 3, 1/3, -1/3)	(5, 10, 6, 1/3, -1/3), (6, 10, 3, 4/9, -1/9)
(15, 8, 4, 4)	(5, 15, 8, 1/4, -1/2), (9, 15, 8, 1/6, -1/4)	(6, 15, 8, 3/8, -1/4), (10, 15, 6, 1/4, -1/8)
(16, 10, 6, 6)	(5, 16, 10, 1/5, -3/5), (10, 16, 5, 1/5, -1/5)	(6, 16, 10, 1/3, -1/3), (11, 16, 5, 3/11, -1/11)

REFERENCES

- [1] E. Bannai and Et. Bannai, *A note on the spherical embeddings of strongly regular graphs*, European J. Comb., **26** (2005), 1177–1179.
- [2] E. Bannai, T. Okuda, and M. Tagami, *Spherical designs of harmonic index t* , J. Approximation Theory, in press; Preprint available arXiv:1306:5101.
- [3] A. Barg and W.-H. Yu, *New bounds for spherical two-distance sets*, Experimental Mathematics, **22**, no. 2, (2013), 187–194.
- [4] A. Barg and W.-H. Yu, *New bounds on equiangular lines*, in: Discrete Geometry and Algebraic Combinatorics, A. Barg and O. Musin, eds., (Contemporary Mathematics, vol. 625), Amer. Math. Soc., Providence, RI, 2014, pp. 111–121.
- [5] J. J. Benedetto and M. Fickus, *Finite normalized tight frames*, Advances in Computational Math, **Vol 18**, nos. 2-4, (2003), 357–385.
- [6] B. Bodmann, P. Casazza, and R. Balan, *Frames for linear reconstruction without phase*, Proc. 42nd IEEE Annual Conference on Information Sciences and Systems (CISS 2008), Princeton, NJ, March 19-21, 2008, pp. 721–726.
- [7] A. E. Brouwer and W. H. Haemers, *Spectra of graphs*, Springer, New York e.a., (2012).
- [8] P. G. Casazza and G. Kutyniok (Editors), “Finite Frame Theory,” Birkhäuser, Boston (2012).
- [9] A. E. Brouwer and W. H. Haemers, *Spectra of graphs*, Springer, New York e.a., 2012.
- [10] P. Delsarte, J. M. Goethals, and J. J. Seidel, *Spherical codes and designs*, Geometriae Dedicata **6** (1977), 363–388.

- [11] C. Godsil and G. Royle, *Algebraic Graph Theory*, Springer, New York 2001.
- [12] D. Han, K. Kornelson, D. Larson, and E. Weber, *Frames for Undergraduates*, American Mathematical Society, Providence, RI, 2007.
- [13] R. B. Holmes and V.I. Paulsen, *Optimal frames for erasures*, *Linear Alg. and Application*, **377** (2004), 31-51.
- [14] J. Kovačević and A. Chebira, *Life Beyond Bases: The Advent of Frames (Part I-II)*, *Signal Processing Magazine, IEEE*, Volume **24** (2007), 86–104 and 115–125.
- [15] D. G. Larman, C. A. Rogers, and J. J. Seidel, *On two-distance sets in Euclidean space*, *Bull. London Math. Soc.* **9** (1977), 261–267.
- [16] P. W. H. Lemmens and J. J. Seidel, *Equiangular lines*, *Journal of Algebra* **24** (1973), 494–512.
- [17] J. H. van Lint and J.J. Seidel, *Equiangular point sets in elliptic geometry* *Proc. Nedert. Akad. Wetensh. Series* **69** (1966), 335-348.
- [18] O. R. Musin, *Spherical two-distance sets*, *J. Combin. Theory Ser. A* **116**, no. 4 (2009), 988–995.
- [19] A. Neumaier, *Distance matrices, dimension, and conference graphs*, *Indag. Math.*, **43**, no. 4 (1981), 385–391.
- [20] H. Nozaki and M. Shinohara, *A geometrical characterization of strongly regular graphs*, *Linear Alg. Appl.* **437** (2012), 2587–2600.
- [21] J. J. Seidel, *A survey of two-graphs*, *Colloquio Internazionale sulle Teorie Combinatorie (Rome, 1973)*, Tomo I, pp. 481–511. *Atti dei Convegni Lincei*, No. 17, Accad. Naz. Lincei, Rome, 1976.
- [22] M. A. Sustik, J. A. Tropp, I. S. Dhillon and R.W. Heath, Jr., *On the existence of equiangular tight frames*, *Linear Alg. and Applications* **426**, no. 2-3 (2007), 619-635.
- [23] T. Strohmer and R. W. Heath, *Grassmannian frames with applications to coding and communication*, *Appl. Comp. Harmonic Anal.* **14**, no. 3 (2003), 619-635.
- [24] S. Waldron, *On the construction of equiangular frames from graph*, *Linear Alg. and its Applications* **431**, no. 11 (2009), 2228-2242.