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Exponentiable motivic measures

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Abstract

In this note we establish some properties of exponentiable motivic measures. As a first application, we show that the rationality of Kapranov's zeta function is stable under products. As a second application, we give an elementary proof of a special case of a result of Totaro.

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1 Motivic measures

Let k be an arbitrary base field and $\mathrm{Var}(k)$ the category of *varieties*, *i.e.* reduced separated k-schemes of finite type. The *Grothendieck ring of varieties* $K_0\mathrm{Var}(k)$ is defined as the quotient of the free abelian group on the set of isomorphism classes of varieties $[X]$ by the relations $[X] = [Y] + [X\backslash Y]$, where Y is a closed subvariety of X. The multiplication is induced by the product of varieties. When k is of positive characteristic, one needs also to impose the relation $[X] = [Y]$ for every surjective radicial morphism $X \to Y$; see Mustaţă [19, Page 78]. Let $\mathbf{L} := [\mathbb{A}^1]$.

The structure of the Grothendieck ring of varieties is quite mysterious; see Poonen [21] for instance. In order to capture some of its flavor several *motivic measures*, *i.e.* ring homomorphisms $\mu : K_0\mathrm{Var}(k) \to R$, have been built. Here are some classical examples:

(i) When k is finite, the assignment $X \mapsto \#X(k)$ gives rise to the counting measure $\mu_\# : \mathrm{Var}(k) \to \mathbb{Z}$; see [19, Ex. 7.7].

(ii) When $k = \mathbb{C}$, the assignment $X \mapsto \chi_c(X) := \sum_i (-1)^i \dim_\mathbb{Q} H^i_c(X^{\mathrm{an}}, \mathbb{Q})$ gives rise to the Euler characteristic measure $\chi_c : \mathrm{Var}(k) \to \mathbb{Z}$; see [19, Ex. 7.8].

(iii) When k is of characteristic zero, the assignment $X \mapsto \mathrm{H}_X(u,v) := \sum_{p,q\geq 0} h^{p,q}(X)u^p v^q$, with X smooth projective, gives rise to the Hodge characteristic measure $\mu_\mathrm{H} : \mathrm{Var}(k) \to \mathbb{Z}[u,v]$; see [14, §4.1].

(iv) When k is of characteristic zero, the assignment $X \mapsto \mathrm{P}_X(u) := \sum_i \dim_k H^i_{dR}(X)u^n$, with X smooth projective, gives rise to the Poincaré characteristic measure $\mu_{\mathrm{P}} : \mathrm{Var}(k) \to \mathbb{Z}[u]$; see [14, §4.1].

Other motivic measures include the Larsen-Lunts "exotic" measure μ_{LL} (see [13]); the Albanese measure μ_{Alb} with values in the semigroup ring of isogeny classes of abelian varieties (see [19, Thm. 7.21]); the Gillet-Soulé measure μ_{GS} with values in the Grothendieck ring $K_0(\mathrm{Chow}(k)_{\mathbb{Q}})$ of Chow motives (see [6]); and the measure μ_{NC} with values in the Grothendieck ring of noncommutative motives (see [23]). There exist several relations between the above motivic measures. For example, $\chi_c, \mu_{\mathrm{H}}, \mu_{\mathrm{P}}, \mu_{\mathrm{NC}}$ factor through μ_{GS}.

2 Kapranov's zeta function

As explained in [19, Prop. 7.27], in the construction of the Grothendieck ring of varieties we can restrict ourselves to quasi-projective varieties. Given a motivic measure μ, Kapranov introduced in [11] the associated zeta function

$$\zeta_\mu(X;t) := \sum_{n=0}^{\infty} \mu([S^n(X)])t^n \in (1 + tR[\![t]\!]), \tag{1}$$

where $S^n(X)$ stands for the n^{th} symmetric product of the quasi-projective variety X. In the particular case of the counting measure, (1) agrees with the classical Weil zeta function. Here are some other computations (with X smooth projective)

$$\zeta_{\chi_c}(X;t) = (1-t)^{-\chi_c(X)} \quad \zeta_{\mathrm{P}}(X;t) = \textstyle\prod_{r\geq 0}(\frac{1}{1-u^r t})^{(-1)^{b_r}} \quad \zeta_{\mathrm{Alb}}(X;t) = \frac{[\mathrm{Alb}(X)]t}{1-t},$$

where $b_r := \dim_{\mathbb{C}} H^r_{dR}(X)$ and $\mathrm{Alb}(X)$ is the Albanese variety of X; see [22, §3].

3 Big Witt ring

Given a commutative ring R, recall from Bloch [2, Page 192] the construction of the big Witt ring $W(R)$. As an additive group, $W(R)$ is equal to $(1 + tR[\![t]\!], \times)$. Let us write $+_W$ for the addition in $W(R)$ and $1 = 1 + 0t + \cdots$ for the zero element. The multiplication $*$ in $W(R)$ is uniquely determined by the following requirements:

(i) The equality $(1-at)^{-1} * (1-bt)^{-1} = (1-abt)^{-1}$ holds for every $a, b \in R$;

(ii) The assignment $R \mapsto W(R)$ is an endofunctor of commutative rings.

The unit element is $(1-t)^{-1}$. We have also a (multiplicative) Teichmüller map

$$R \longrightarrow W(R) \qquad a \mapsto [a] := (1-at)^{-1}$$

such that $g(t) * [a] = g(at)$ for every $a \in R$ and $g(t) \in W(R)$; see [2, Page 193].

Definition 3.1. *Elements of the form* $p(t) -_W q(t) \in W(R)$, *with* $p(t), q(t) \in R[t]$ *and* $p(0) = q(0) = 1 \in R$, *are called* rational functions.

Let $W_{\text{rat}}(R)$ be the subset of rational elements. As proved by Naumann in [20, Prop. 6], $W_{\text{rat}}(R)$ is a subring of $W(R)$. Moreover, $R \mapsto W_{\text{rat}}(R)$ is an endofunctor of commutative rings. Recall also the construction of the commutative ring $\Lambda(R)$. As an additive group, $\Lambda(R)$ is equal to $W(R)$. The multiplication is uniquely determined by the requirement that the involution group isomorphism $\iota : \Lambda(R) \to W(R), g(t) \mapsto g(-t)^{-1}$, is a ring isomorphism. The unit element is $1 + t$.

4 Exponentiation

Let μ be a motivic measure. As explained by Mustaţă in [19, Prop. 7.28], the assignment $X \mapsto \zeta_\mu(X; t)$ gives rise to a group homomorphism

$$\zeta_\mu(-; t) : K_0\mathrm{Var}(k) \longrightarrow W(R) . \tag{2}$$

Definition 4.1. *([22, §3]) A motivic measure* μ *is (uniquely)* exponentiable[1] *if the above group homomorphism (2) is a ring homomorphism.*

Corollary 4.2. *Given an exponentiable measure, the following holds:*

(i) The ring homomorphism (2) is a new motivic measure;

(ii) Any motivic measure which factors through μ *is also exponentiable.*

This class of motivic measures is well-behaved with respect with rationality:

Proposition 4.3. *Let* μ *be an exponentiable motivic measure. If* $\zeta_\mu(X; t)$ *and* $\zeta_\mu(Y; t)$ *are rational functions, then* $\zeta_\mu(X \times Y; t)$ *is also a rational function.*

Proof. It follows automatically from the fact that $W_{\text{rat}}(R)$ is a subring of $W(R)$. □

As proved by Naumann in [20, Prop. 8] (see also [22, Thm. 2.1]), the counting measure $\mu_\#$ is exponentiable. On the other hand, Larsen-Lunts "exotic" measure μ_{LL} is *not* exponentiable! This would imply, in particular, that

$$\zeta_{\mu_{\text{LL}}}(C_1 \times C_2; t) = \zeta_{\mu_{\text{LL}}}(C_1; t) * \zeta_{\mu_{\text{LL}}}(C_2; t) \tag{3}$$

for any two smooth projective curves C_1 and C_2. As proved by Kapranov in [11] (see also [19, Thm. 7.33]), $\zeta_\mu(C; t)$ is a rational function for every smooth projective curve C

[1] Note that Kapranov's zeta function is similar to the exponential function $e^x = \sum_{n=0}^\infty \frac{x^n}{n!}$. The product X^n corresponds to x^n and the symmetric product $S^n(X)$ corresponds to $\frac{x^n}{n!}$ since $n!$ is the size of the symmetric group on n letters.

and motivic measure μ. Using Proposition 4.3, this hence implies that the right-hand side of (3) is also a rational function. On the other hand, as proved by Larsen-Lunts in [13, Thm. 7.6], the left-hand side of (3) is not a rational function whenever C_1 and C_2 have positive genus. We hence obtain a contradiction.

At this point, it is natural to ask which motivic measures are exponentiable? We now provide a general answer to this question using the notion of λ-ring. Recall that a *λ-ring* R consists of a commutative ring equipped with a sequence of maps $\lambda^n : R \to R, n \geq 0$, such that $\lambda^0(a) = 1$, $\lambda^1(a) = a$, and $\lambda^n(a+b) = \sum_{i+j=n} \lambda^i(a)\lambda^j(b)$ for every $a, b \in R$. In other words, the map

$$\lambda_t : R \longrightarrow \Lambda(R) \qquad a \mapsto \lambda_t(a) := \sum_n \lambda^n(a)t^n$$

is a group homomorphism. Equivalently, the composed map

$$\sigma_t : R \xrightarrow{\lambda_t} \Lambda(R) \xrightarrow{\iota} W(R) \qquad a \mapsto \sigma_t(a) = \sum_n \sigma^n(a)t^n := \lambda_{-t}(a)^{-1} \tag{4}$$

is a group homomorphism. This homomorphism is called the *opposite* λ-structure.

Proposition 4.4. *Let μ be a motivic measure and R a λ-ring such that:*

(i) The above group homomorphism (4) is a ring homomorphism;

(ii) We have $\mu([S^n(X)]) = \sigma^n(\mu([X]))$ for every quasi-projective variety X.

Under these conditions, the motivic measure μ is exponentiable.

Proof. Consider the following composed ring homomorphism

$$K_0\mathrm{Var}(k) \xrightarrow{\mu} R \xrightarrow{\sigma_t} W(R). \tag{5}$$

The equalities $\mu([S^n(X)]) = \sigma^n(\mu([X]))$ allow us to conclude that (5) agrees with the group homomorphism $\zeta_\mu(-;t)$. This achieves the proof. □

Remark 4.5. *Let $\mathcal{C}$ be a $\mathbb{Q}$-linear additive idempotent complete symmetric monoidal category. As proved by Heinloth in [9, Lem. 4.1], the exterior powers give rise to a special λ-structure on the Grothendieck ring $K_0(\mathcal{C})$, with opposite λ-structure given by the symmetric powers Sym^n. In this case, (4) is a ring homomorphism.*

Remark 4.6. *Let $\mathcal{T}'$ be a $\mathbb{Q}$-linear thick triangulated monoidal subcategory of compact objects in the homotopy category $\mathcal{T} = \mathrm{Ho}(\mathcal{C})$ of a simplicial symmetric monoidal model category $\mathcal{C}$. As proved by Guletskii in [8, Thm. 1], the exterior powers give rise to a special λ-structure on $K_0(\mathcal{T}')$, with opposite λ-structure given by the symmetric powers Sym^n. In this case, (4) is a ring homomorphism.*

Remark 4.7. *Assume that k is of characteristic zero. Thanks to Heinloth's presentation of the Grothendieck ring of varieties (see [10, Thm. 3.1]), it suffices to verify the equality $\mu([S^n(X)]) = \sigma^n(\mu([X]))$ for every smooth projective variety X.*

As an application of the above Proposition 4.4, we obtain the following result:

Proposition 4.8. *The Gillet-Soulé motivic measure μ_{GS} is exponentiable.*

Proof. Recall from [6] that μ_{GS} is induced by the symmetric monoidal functor

$$\mathfrak{h} : \mathrm{SmProj}(k) \longrightarrow \mathrm{Chow}(k)_{\mathbb{Q}} \tag{6}$$

from the category of smooth projective varieties to the category of Chow motives. Since the latter category is $\mathbb{Q}$-linear, additive, idempotent complete, and symmetric monoidal, Remark 4.5 implies that the Grothendieck ring $K_0(\mathrm{Chow}(k)_{\mathbb{Q}})$ satisfies condition (i) of Proposition 4.4. As proved by del Baño-Aznar in [4, Cor. 2.4], we have $\mathfrak{h}(S^n(X)) \simeq \mathrm{Sym}^n\mathfrak{h}(X)$ for every smooth projective variety X. Using Remark 4.7, this hence implies that condition (ii) of Proposition 4.4 is also satisfied. □

Remark 4.9. *Thanks to Corollary 4.2(ii), all the motivic measures which factor through μ_{GS} (e.g. $\chi_c, \mu_{\mathrm{H}}, \mu_{\mathrm{P}}, \mu_{\mathrm{NC}}$) are also exponentiable.*

5 Application I: rationality of zeta functions

By combining Propositions 4.3 and 4.8, we obtain the following result:

Corollary 5.1. *Let X, Y be two varieties. If $\zeta_{\mu_{\mathrm{GS}}}(X;t)$ and $\zeta_{\mu_{\mathrm{GS}}}(Y;t)$ are rational functions, then $\zeta_{\mu_{\mathrm{GS}}}(X \times Y;t)$ is also a rational function.*

Remark 5.2. *Corollary 5.1 was independently obtained by Heinloth [9, Prop. 6.1] in the particular case of smooth projective varieties and under the extra assumption that $\zeta_{\mu_{\mathrm{GS}}}(X;t)$ and $\zeta_{\mu_{\mathrm{GS}}}(Y;t)$ satisfy a certain functional equation.*

Example 5.3. *Let X, Y be smooth projective varieties (e.g. abelian varieties) for which $\mathfrak{h}(X), \mathfrak{h}(Y)$ are Kimura-finite; see [12, §3]. Consider the ring homomorphism*

$$\sigma_t : K_0(\mathrm{Chow}(k)_{\mathbb{Q}}) \longrightarrow W(K_0(\mathrm{Chow}(k)_{\mathbb{Q}}))\,. \tag{7}$$

As proved by André in [1, Prop. 4.6], $\sigma_t([\mathfrak{h}(X)])$ and $\sigma_t([\mathfrak{h}(Y)])$ are rational functions. Since $\zeta_{\mu_{\mathrm{GS}}}(-;t)$ agrees with the composition of μ_{GS} with (7), these latter functions are equal to $\zeta_{\mu_{\mathrm{GS}}}(X;t)$ and $\zeta_{\mu_{\mathrm{GS}}}(Y;t)$, respectively. Using Corollary 5.1, we hence conclude that $\zeta_{\mu_{\mathrm{GS}}}(X \times Y;t)$ is also a rational function.

When k is of characteristic zero, Voevodsky constructed in [24, §2.2] a functor

$$M^c : \mathrm{Var}(k)^p \longrightarrow \mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}} \tag{8}$$

from the category of varieties and proper morphisms to the triangulated category of geometric motives. As proved in [24, Prop. 4.1.7], the functor (8) is symmetric monoidal. Moreover, given a variety X and a closed subvariety $Y \subset X$, we have a triangle

$$M^c(Y) \longrightarrow M^c(X) \longrightarrow M^c(X \backslash Y) \longrightarrow M^c(Y)[1]$$

in $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$; see [24, Prop. 4.1.5]. Consequently, we obtain the motivic measure:

$$K_0\mathrm{Var}(k) \longrightarrow K_0(\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}) \qquad [X] \mapsto [M^c(X)]\,. \tag{9}$$

Proposition 5.4. *The above motivic measure (9) agrees with μ_{GS}.*

Proof. As proved by Voevodsky in [24, Prop. 2.1.4], there exists a $\mathbb{Q}$-linear additive fully-faithful symmetric monoidal functor

$$\mathrm{Chow}(k)_{\mathbb{Q}} \longrightarrow \mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}} \tag{10}$$

such that $(10) \circ \mathfrak{h}(X) \simeq M^c(X)$ for every smooth projective variety. Thanks to the work of Bondarko [3, Cor. 6.4.3 and Rk. 6.4.4], the above functor (10) induces a ring isomorphism $K_0(\mathrm{Chow}(k)_{\mathbb{Q}}) \simeq K_0(\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}})$. Therefore, the proof follows from Heinloth's presentation of the Grothendieck ring of varieties in terms of smooth projective varieties; see [10, Thm. 3.1]. □

Thanks to Proposition 5.4, Example 5.3 admits the following generalization:

Example 5.5. *Let X, Y be varieties for which $M^c(X), M^c(Y)$ are Kimura-finite. Similarly to Example 5.3, $\zeta_{\mu_{\mathrm{GS}}}(X \times Y; t)$ is then a rational function.*

In the above Examples 5.3 and 5.5, the rationality of $\zeta_{\mu_{\mathrm{GS}}}(X \times Y; t)$ can alternatively be deduced from the stability of Kimura-finiteness under tensor products; see [12, §5]. Thanks to the work of O'Sullivan-Mazza [18, §5.1] and Guletskii [8], the above Corollary 5.1 can also be applied to non Kimura-finite situations.

Proposition 5.6. *Let X_0 be a connected smooth projective surface, over an algebraically closed field k_0, with geometric genus $p_g > 0$ and irregularity $q = 0$. Let $k := k_0(X_0)$ the function field of X_0, x_0 a k_0-point of X_0, z the zero-cycle which is the pull-back of the cycle $\Delta(X_0) - (x_0 \times X)$ along $X_0 \times k \to X_0 \times X_0$, Z the support of z, and finally U the complement of Z in $X = X_0 \times k$. Under these notations, the following holds:*

(i) The geometric motive $M^c(U)$ is not Kimura-finite;

(ii) Kapranov's zeta function $\zeta_{\mu_{\mathrm{GS}}}(U; t)$ is rational.

Proof. As proved by O'Sullivan-Mazza in [18, Thm. 5.18], $M(U)$ is not Kimura-finite. Since the surface U is smooth, we have $M^c(U) \simeq M(U)^\vee(2)[4]$ where $(-)^\vee$ stands for the dual; see [24, Thm. 4.3.7]. Using the fact that $-(2)[4]$ is an auto-equivalence and that $M(U)^\vee$ is Kimura-finite if and only if $M(U)$ is Kimura-finite (see Deligne [5, Prop. 1.18]), we conclude that $M^c(U)$ also is not Kimura-finite.

We now prove item (ii). As proved by Guletskii in [8, §3], the category $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ satisfies the conditions of Remark 4.6. Consequently, we have a ring homomorphism

$$\sigma_t : K_0(\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}) \longrightarrow W(K_0(\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}))\,. \tag{11}$$

As explained by Guletskii in [8, Ex. 5], $\sigma_t([M(U)])$ is a rational function. Equivalently, $\sigma_t([M(U)])^\vee$ (obtained from $\sigma_t([M(U)])$ by applying $(-)^\vee$ to each term) is a rational function. Thanks to Lemma 5.7 below, we hence conclude that $\sigma_t([M^c(U)])$ is also a rational function. The proof follows now from the fact that $\zeta_{\mu_{\mathrm{GS}}}(-;t)$ agrees with the composition of the ring homomorphisms (9) and (11). □

Lemma 5.7. *Given a smooth variety X of dimension d, we have the equality*

$$\sigma_t([M^c(X)]) = \sigma_{\mu_{\mathrm{GS}}(\mathbf{L})^d t}([M(X)])^\vee\,.$$

Proof. The proof is given by the following identifications

$$\begin{aligned} \sigma_t([M^c(X)]) &= \sigma_t([M(X)^\vee(d)[2d]]) && (12)\\ &= \sigma_t([M(X)^\vee]\mu_{\mathrm{GS}}(\mathbf{L}^d)) \\ &= \sigma_t([M(X)^\vee]) * \zeta_{\mu_{\mathrm{GS}}}(\mathbf{L}^d;t) \\ &= \sigma_t([M(X)])^\vee * \zeta_{\mu_{\mathrm{GS}}}(\mathbf{L}^d;t) && (13)\\ &= \sigma_{\mu_{\mathrm{GS}}(\mathbf{L})^d t}([M(X)])^\vee\,, && (14) \end{aligned}$$

where (12) follows from [24, Thm. 4.3.7], (13) from [5, Prop. 1.18], and (14) from Remark 6.2 below with $\mu := \mu_{\mathrm{GS}}$ and $g(t) := \sigma_t([M(X)])^\vee$. □

Example 5.8. *Let U_1, U_2 be two surfaces as in Proposition 5.6. Thanks to the above Corollary 5.1, we hence conclude that $\zeta_{\mu_{\mathrm{GS}}}(U_1 \times U_2;t)$ is a rational function. Note that the geometric motive $M^c(U_1 \times U_2)$ is not Kimura-finite! Choose a rational point x_1 of U_1 and consider the associated morphism $x_1 \times \mathrm{id} : U_2 \to U_1 \times U_2$. Using the projection $U_1 \times U_2 \to U_2$ we observe that $M(U_2)$ is a direct summand of $M(U_1 \times U_2)$. As explained in the proof of Proposition 5.6, $M^c(U_2)$ (resp. $M^c(U_1 \times U_2)$) is Kimura-finite if and only if $M(U_2)$ (resp. $M(U_1 \times U_2)$) is Kimura-finite. Consequently, if $M^c(U_1 \times U_2)$ were Kimura-finite, $M^c(U_2)$ also would be Kimura-finite. This contradicts Proposition 5.6. Finally, note that self-products $U_1 \times \cdots \times U_1$ are examples of arbitrarily high dimension.*

Remark 5.9. *Thanks to Corollary 4.2(ii), the above Examples 5.3, 5.5, and 5.8, hold* mutatis mutandis *for any motivic measure which factors through μ_{GS}.*

6 Application II: Totaro's result

The following result plays a central role in the study of the zeta functions.

Proposition 6.1 (Totaro)**.** *The equality* $\zeta_\mu(X \times \mathbb{A}^n; t) = \zeta_\mu(X; \mu(\mathbf{L})^n t)$ *holds for every variety* X *and motivic measure* μ *.*

Its proof (see [7, Lem. 4.4][19, Prop. 7.32]) is non-trivial and based on a stratification of the symmetric products of $X \times \mathbb{A}^n$. In all the cases where the motivic measure μ is exponentiable, this result admits the following elementary proof:

Proof. Since $[X \times \mathbb{A}^n] = [X][\mathbb{A}^n]$ in the Grothendieck ring of varieties and the motivic measure μ is exponentiable, the proof is given by the identifications

$$\begin{aligned}
\zeta_\mu(X \times \mathbb{A}^n; t) &= \zeta_\mu(X;t) * \zeta_\mu(\mathbf{L}^n; t) \\
&= \zeta_\mu(X;t) * \zeta_\mu(\mathbf{L}; t)^{*n} \\
&= \zeta_\mu(X;t) * (1 + \mu(\mathbf{L})t + \mu(\mathbf{L})t^2 + \cdots)^{*n} \qquad (15) \\
&= \zeta_\mu(X;t) * ((1 - \mu(\mathbf{L})t)^{-1})^{*n} \\
&= \zeta_\mu(X;t) * [\mu(\mathbf{L})]^{*n} \\
&= \zeta_\mu(X;t) * [\mu(\mathbf{L})^n] \\
&= \zeta_\mu(X; \mu(\mathbf{L})^n t) ,
\end{aligned}$$

where (15) follows from [19, Ex. 7.23] and $[\mu(\mathbf{L})]$ stands for the image of $\mu(\mathbf{L}) \in R$ under the multiplicative Teichmüller map $R \to W(R)$. □

Remark 6.2. *The above proof shows more generally that* $g(t) * \zeta_\mu(\mathbf{L}^n; t) = g(\mu(\mathbf{L})^n t)$ *for every* $g(t) \in W(R)$ *and exponentiable motivic measure* μ *.*

Remark 6.3. *(Fiber bundles) Given a fiber bundle* $E \to X$ *of rank* n *, we have* $[E] = [X][\mathbb{A}^n]$ *in the Grothendieck ring of varieties; see [19, Prop. 7.4]. Therefore, the above proof, with* X *replaced by* E *, shows that* $\zeta_\mu(E;t) = \zeta_\mu(X; \mu(\mathbf{L})^n t)$ *.*

Remark 6.4. *(*$\mathbb{P}^n$ *-bundles) Given a* $\mathbb{P}^n$ *-bundle* $E \to X$ *, we have* $[E] = [X][\mathbb{P}^n]$ *in the Grothendieck ring of varieties; see [19, Ex. 7.5]. Therefore, by combining the equality* $[\mathbb{P}^n] = 1 + \mathbf{L} + \cdots + \mathbf{L}^n$ *with the above proof, we conclude that*

$$\zeta_\mu(E;t) = \zeta_\mu(X;t) +_W \zeta_\mu(X; \mu(\mathbf{L})t) +_W \cdots +_W \zeta_\mu(X; \mu(\mathbf{L})^n t) .$$

7 G-varieties

Let G be a finite group and $\mathrm{Var}^G(k)$ the category of G*-varieties*, *i.e.* varieties X equipped with a G-action $\lambda : G \times X \to X$ such that every orbit is contained in an affine open set. The *Grothendieck ring of G-varieties* $K_0\mathrm{Var}^G(k)$ is defined as the quotient of the free abelian group on the set of isomorphism classes of G-varieties $[X,\lambda]$ by the relations $[X,\lambda]=[Y,\tau]+[X\backslash Y,\lambda]$, where (Y,τ) is a closed G-invariant subvariety of (X,λ). The multiplication is induced by the product of varieties. A motivic measure is a ring homomorphism $\mu^G: K_0\mathrm{Var}^G(k)\to R$. As mentioned in [15, §5], the above measures $\chi_c, \mu_{\mathrm{H}}, \mu_{\mathrm{P}}$ admit G-extensions $\chi_c^G, \mu_{\mathrm{H}}^G, \mu_{\mathrm{P}}^G$.

Notation 7.1. *Let* $\mathrm{Chow}^G(k)_{\mathbb{Q}}$ *be the category of functors from the group* G *(considered as a category with a single object) to the category* $\mathrm{Chow}(k)_{\mathbb{Q}}$.

Note that $\mathrm{Chow}^G(k)_{\mathbb{Q}}$ is still a $\mathbb{Q}$-linear additive idempotent complete symmetric monoidal category and that (6) extends to a symmetric monoidal functor

$$\mathfrak{h}^G: \mathrm{SmProj}^G(k) \longrightarrow \mathrm{Chow}^G(k)_{\mathbb{Q}}. \tag{16}$$

Note also that the n^{th} symmetric product of a G-variety is still a G-variety. Therefore, the notion of exponentiation makes sense in this generality. Gillet-Soulé's motivic measure μ_{GS} admits the following G-extension:

Proposition 7.2. *The above functor* (16) *gives rise to an exponentiable motivic measure:*

$$\mu_{\mathrm{GS}}^G: K_0\mathrm{Var}^G(k) \longrightarrow K_0(\mathrm{Chow}^G(k)_{\mathbb{Q}}).$$

Proof. Given a smooth projective variety X and a closed subvariety Y, let us denote by $\mathrm{Bl}_Y(X)$ the blow-up of X along Y and by E the associated exceptional divisor. As proved by Manin in [16, §9], we have a natural isomorphism $\mathfrak{h}(\mathrm{Bl}_Y(X))\oplus\mathfrak{h}(Y)\simeq\mathfrak{h}(X)\oplus\mathfrak{h}(E)$ in $\mathrm{Chow}(k)_{\mathbb{Q}}$. Since this isomorphism is natural, it also holds in $\mathrm{Chow}^G(k)_{\mathbb{Q}}$ when X is replaced by a smooth projective G-variety (X,λ) and Y by a closed G-invariant subvariety (Y,τ). Therefore, thanks to Heinloth's presentation of the Grothendieck ring of G-varieties in terms of smooth projective G-varieties (see [10, Lem. 7.1]), the assignment $X\mapsto\mathfrak{h}^G(X)$ gives rise to a (unique) motivic measure μ_{GS}^G. The proof of Proposition 4.8, with (6) replaced by (16), shows that this motivic measure μ_{GS}^G is exponentiable. □

Remark 7.3. *Similarly to Remark 4.9, all the motivic measures which factor through* μ_{GS}^G *(e.g.* $\chi_c^G, \mu_{\mathrm{H}}^G, \mu_{\mathrm{P}}^G$ *) are also exponentiable.*

Proposition 4.3 admits the following G-extension:

Proposition 7.4. *Let* μ^G *be an exponentiable motivic measure and* $(X,\lambda),(Y,\tau)$ *two* G*-varieties. If* $\zeta_{\mu^G}((X,\lambda);t)$ *and* $\zeta_{\mu^G}((Y,\tau);t)$ *are rational functions, then* $\zeta_{\mu^G}((X\times Y,\lambda\times\tau);t)$ *is also a rational function.*

Example 7.5. *Assume that the group* G *(of order* r *) is abelian and that the base field* k *is algebraically closed of characteristic zero or of positive characteristic* p *with* $p\nmid r$. *Under these assumptions, Mazur proved in [17, Thm. 1.1] that* $\zeta_{\mu^G}((C,\lambda);t)$ *is a rational function for every smooth projective* G*-curve* (C,λ) *and motivic measure* μ^G. *Thanks* *to Proposition 7.4, we hence conclude that* $\zeta_{\mu^G}((C_1 \times C_2, \lambda_1 \times \lambda_2); t)$ *is still a rational function for every exponentiable motivic measure* μ^G *and for any two smooth projective* G*-curves* (C_1, λ_1) *and* (C_2, λ_2).

Finally, Totaro's result admits the following G-extension:

Proposition 7.6. *Let* μ^G *be an exponentiable motivic measure and* $(X.\lambda), (\mathbb{A}^n, \tau)$ *two* G*-varieties. When* G *(of order* r *) is abelian and* k *is algebraically closed, Kapranov's zeta function* $\zeta_{\mu^G}((X \times \mathbb{A}^n, \lambda \times \tau); t)$ *agrees with*

$$\zeta_{\mu^G}\left((X,\lambda); \mu^G(S^r(\mathbb{A}^n,\tau))t\right) +_W \zeta_{\mu^G}((X,\lambda);t) * \left(\sum_{l=0}^{r-1}\prod_{i=1}^{n}\mu^G([\mathbb{A}^1,\tau_i]\cdots[\mathbb{A}^1,\tau_i^l])t^l\right),$$

where $[\mathbb{A}^n, \tau] = [\mathbb{A}^1, \tau_1] \cdots [\mathbb{A}^1, \tau_n]$.

Proof. Since $[X \times \mathbb{A}^n, \lambda \times \tau] = [X, \lambda][\mathbb{A}^n, \tau]$ in the Grothendieck ring of G-varieties and the motivic measure μ^G is exponentiable, we have the equality

$$\zeta_{\mu^G}((X \times \mathbb{A}^n, \lambda \times \tau); t) = \zeta_{\mu^G}((X, \lambda); t) * \zeta_{\mu^G}((\mathbb{A}^n, \tau); t).$$

Moreover, as explained in [17, Page 1338], we have the following computation

$$\zeta_{\mu^G}((\mathbb{A}^n,\tau);t) = \frac{1}{1-\mu^G(S^r(\mathbb{A}^n,\tau))t}\left(\sum_{l=0}^{r-1}\prod_{i=1}^{n}\mu^G([\mathbb{A}^1,\tau_i]\cdots[\mathbb{A}^1,\tau_i^l])t^l\right).$$

Therefore, since $(1-\mu^G(S^r(\mathbb{A}^n,\tau))t)^{-1}$ is the Teichmüller class $[\mu^G(S^r(\mathbb{A}^n,\tau))]$, the proof follows from the combination of the above equalities. □

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