STANDARD CONJECTURES FOR THE ARITHMETIC GRASSMANNIAN $G(2,N)$ AND RACAH POLYNOMIALS

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Abstract. We prove the arithmetic Hodge index and hard Lefschetz conjectures for the Grassmannian $G = G(2,N)$ parametrizing lines in projective space, for the natural arithmetic Lefschetz operator defined via the Plücker embedding of G in projective space. The analysis of the Hodge index inequality involves estimates on values of certain Racah polynomials.

0. Introduction

Let X be an arithmetic variety, by which we mean a regular, projective and flat scheme over $\operatorname{Spec}\mathbb{Z}$, of absolute dimension $d+1$. Assume that $\overline{M} = (M, \|\cdot\|)$ is a hermitian line bundle on X which is arithmetically ample, in the sense of [Z] and [So, §5.2]. For each $p \geqslant 0$ the line bundle $\overline{M}$ defines an arithmetic Lefschetz operator

$$\begin{array}{rccc} \widehat{L}\,: & \widehat{CH}^p(X)_{\mathbb{R}} & \longrightarrow & \widehat{CH}^{p+1}(X)_{\mathbb{R}} \\ & \alpha & \longmapsto & \alpha\cdot\widehat{c}_1(\overline{M}). \end{array}$$

Here $\widehat{CH}^*(X)_{\mathbb{R}}$ is the real arithmetic Chow ring of [GS] and $\widehat{c}_1(\overline{M})$ is the arithmetic first Chern class of $\overline{M}$.

In this setting, Gillet and Soulé [GS] proposed arithmetic analogues of Grothendieck's standard conjectures [Gr] on algebraic cycles. A more precise version of the conjectures was formulated in [So, §5.3]; assuming $2p \leqslant d+1$, the statement is

Conjecture 1. (a) *(Hard Lefschetz) The map*

$$\widehat{L}^{d+1-2p} : \widehat{CH}^p(X)_{\mathbb{R}} \longrightarrow \widehat{CH}^{d+1-p}(X)_{\mathbb{R}}$$

is an isomorphism;

(b) *(Hodge index) If the nonzero $x \in \widehat{CH}^p(X)_{\mathbb{R}}$ satisfies $\widehat{L}^{d+2-2p}(x) = 0$, then*

$$(-1)^p\,\widehat{\deg}(x\,\widehat{L}^{d+1-2p}(x)) > 0.$$

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Notice that Conjecture 1 (for all p) implies that the intersection pairing

$$\widehat{CH}^p(X)_{\mathbb{R}} \otimes \widehat{CH}^{d+1-p}(X)_{\mathbb{R}} \longrightarrow \mathbb{R}$$

is nondegenerate. When $d = 1$, the Conjecture follows from the Hodge index theorem for arithmetic surfaces due to Faltings [Fa] and Hriljac [Hr].

We study these conjectures when $X = G(r, N)$ is the arithmetic Grassmannian, parametrizing r-dimensional subspaces of an $(r+N)$-dimensional vector space, over any field, and $\overline{M} = \overline{\mathcal{O}}(1)$ is the very ample line bundle giving the Plücker embedding, equipped with its natural hermitian metric. The latter is the metric induced from the standard metric on complex affine space, so that the first Chern form $c_1(\overline{M})$ on $X(\mathbb{C})$ is $U(r + N)$-invariant and dual to the hyperplane class.

Our main result is that, for $G(r, N)$ and $\overline{\mathcal{O}}(1)$, Conjecture 1 holds when $r = 2$. For projective space ($r = 1$) this was shown by Künnemann [Ku]. Moreover, it is proved in [KM] and [Ta] that Conjecture 1 holds for $G(r, N)$ after a suitable scaling of the metric on $\overline{\mathcal{O}}(1)$. To obtain the precise result for $G(2, N)$ we use the arithmetic Schubert calculus of [T] and linear algebra to reduce the problem to combinatorial estimates. In this case the inequality in part (b) asserts the positivity of a linear combination of harmonic numbers with coefficients certain Racah polynomials. The latter are a system of orthogonal polynomials in a discrete variable introduced by Wilson [Wi] [AW] which generalize the classical Racah coefficients or 6-j symbols [Ra] of quantum physics.

The results of Künnemann [Ku] show that each statement in Conjecture 1 (for given X, p and $\overline{M}$) is true if and only if it holds when $\widehat{CH}^p(X)_{\mathbb{R}}$ is replaced by the Arakelov subgroup $CH^p(\overline{X})_{\mathbb{R}}$ associated to the Kähler form $c_1(\overline{M})$. We therefore restrict attention to this subgroup throughout the paper. In Section 1 we study arithmetic Lefschetz theory for varieties which admit a cellular decomposition and derive a cohomological criterion (Corollary 1) which we use to check Conjecture 1. This criterion does not suffice to check the Hodge index inequality on more general Grassmannians. In Section 2 we apply classical and arithmetic Schubert calculus to reduce the conjecture for $G(2, N)$ to estimates for a class of Racah polynomials. The required bounds for these polynomials are established in Section 3.

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1. Arithmetic standard conjectures on cellular spaces

We study Conjecture 1 for arithmetic varieties X which have a cellular decomposition over $\operatorname{Spec}\mathbb{Z}$, in the sense of [Fu, Exam. 1.9.1]; the Grassmannian $G(r,N)$ is a typical example. See [KM] for more information on these spaces and an approach to a weaker version of the conjecture. Recall that for each p the class map

$$\mathrm{cl}:\ CH^p(X)_{\mathbb{R}} \longrightarrow H^{p,p}(X_{\mathbb{R}})$$

is an isomorphism of the real Chow ring $CH^p(X)_{\mathbb{R}} = CH^p(X)\otimes_{\mathbb{Z}}\mathbb{R}$ with the space $H^{p,p}(X_{\mathbb{R}})$ of real harmonic differential (p,p)-forms on $X(\mathbb{C})$. We denote by

$$\begin{array}{rccc} L: & CH^p(X)_{\mathbb{R}} & \longrightarrow & CH^{p+1}(X)_{\mathbb{R}} \\ & \alpha & \longmapsto & \alpha\cdot c_1(M) \end{array}$$

the classical Lefschetz operator associated to an ample line bundle M over X.

Let us equip the holomorphic line bundle $M(\mathbb{C})$ with a smooth positive hermitian metric, invariant under complex conjugation, to obtain a hermitian line bundle $\overline{M}$. As we have indicated, to check Conjecture 1 for the operator $\widehat{L}(\alpha)=\alpha\cdot\widehat{c}_1(\overline{M})$ it suffices to work with the Arakelov Chow group $CH^p(\overline{X})_{\mathbb{R}}$ defined using the Kähler form $c_1(\overline{M})$. Since X has a cellular decomposition, we have an exact sequence

$$0\longrightarrow CH^{p-1}(X)_{\mathbb{R}} \xrightarrow{\widetilde{a}} CH^p(\overline{X})_{\mathbb{R}} \xrightarrow{\zeta} CH^p(X)_{\mathbb{R}} \longrightarrow 0 \tag{1}$$

(see [KM, Prop. 6]). Here $\widetilde{a}=a\circ\mathrm{cl}$ is the composite of the class map with the natural inclusion $a: H^{p-1,p-1}(X_{\mathbb{R}})\hookrightarrow CH^p(\overline{X})_{\mathbb{R}}$ and ζ is the projection defined in [GS, §1]. We choose a splitting

$$s_p\colon CH^p(X)_{\mathbb{R}} \longrightarrow CH^p(\overline{X})_{\mathbb{R}}$$

for the sequence (1) and thus arrive at a direct sum decomposition

$$CH^p(\overline{X})_{\mathbb{R}} \cong CH^p(X)_{\mathbb{R}} \oplus CH^{p-1}(X)_{\mathbb{R}}. \tag{2}$$

for every p.

Summing (1) over all p produces a sequence

$$0\longrightarrow CH^{*-1}(X)_{\mathbb{R}} \xrightarrow{\widetilde{a}} CH^*(\overline{X})_{\mathbb{R}} \xrightarrow{\zeta} CH^*(X)_{\mathbb{R}} \longrightarrow 0 \tag{3}$$

which is compatible with the actions of L and $\widehat{L}$. The splitting $s:=\oplus_p s_p$ of (3) does not commute with $\widehat{L}$ in general. Rather, the image of

$\widehat{L} \circ s - s \circ L$ is contained in $\mathrm{Ker}(\zeta)$, hence

$$\widehat{L} \circ s - s \circ L = \widetilde{a} \circ U \tag{4}$$

for a uniquely defined degree-preserving linear operator U on $CH^*(X)_{\mathbb{R}}$.

We now give some conditions equivalent to the arithmetic hard Lefschetz theorem (Theorem 1). When checking these for $G(2,N)$, we obtain something stronger, which establishes the arithmetic Hodge index theorem as well; this is quantified in Theorem 2. Recall the classical Lefschetz decomposition on $CH^m(X)_{\mathbb{R}} \simeq H^{2m}(X(\mathbb{C}),\mathbb{R})$:

$$CH^m(X)_{\mathbb{R}} = \bigoplus_{p \geqslant 0} L^{m-p} CH^p_{\mathrm{prim}}(X)_{\mathbb{R}},$$

where the group of primitive codimension p classes is

$$CH^p_{\mathrm{prim}}(X)_{\mathbb{R}} = \mathrm{Ker}(L^{d+1-2p} : CH^p(X)_{\mathbb{R}} \to CH^{d+1-p}(X)_{\mathbb{R}}).$$

For $m = d-p$ this decomposition induces a projection map

$$\pi_p : CH^{d-p}(X)_{\mathbb{R}} \longrightarrow CH^{d-p}_{\mathrm{coprim}}(X)_{\mathbb{R}}$$

where $CH^{d-p}_{\mathrm{coprim}}(X)_{\mathbb{R}} = L^{d-2p} CH^p_{\mathrm{prim}}(X)_{\mathbb{R}}$.

Theorem 1. *Let X be an arithmetic variety of dimension $d+1$ which admits a cellular decomposition. Let $\widehat{L}$ be the arithmetic Lefschetz operator associated to an ample hermitian line bundle on X. Then the following statements are equivalent:*

(i) $\widehat{L}^{d+1-2p} : CH^p(\overline{X})_{\mathbb{R}} \longrightarrow CH^{d+1-p}(\overline{X})_{\mathbb{R}}$ *is an isomorphism for all p.*

(ii) *There exists a linear map $\widehat{\Lambda}\colon CH^*(\overline{X})_{\mathbb{R}} \longrightarrow CH^{*-1}(\overline{X})_{\mathbb{R}}$ such that for every p and $\alpha \in CH^p(\overline{X})_{\mathbb{R}}$ we have $[\widehat{\Lambda}, \widehat{L}]\alpha = (d+1-2p)\alpha$.*

(iii) *For some (equivalently, any) choice of splitting s of* (3)*, with U as in* (4)*, the map*

$$\delta_p := \pi_p \sum_{i=0}^{d-2p} L^{d-2p-i} U L^i \, : \, CH^p_{\mathrm{prim}}(X)_{\mathbb{R}} \longrightarrow CH^{d-p}_{\mathrm{coprim}}(X)_{\mathbb{R}}$$

is an isomorphism for all p.

Proof. The equivalence of (i) and (ii) follows from standard Lefschetz theory; this is described e.g. in [Kl, §4]. Now consider the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & CH^{p-1}(X)_{\mathbb{R}} \arrow[r, "\widetilde{a}"] \arrow[d, "L^{d+1-2p}"] & CH^p(\overline{X})_{\mathbb{R}} \arrow[r, "\zeta"] \arrow[d, "\widehat{L}^{d+1-2p}"] & CH^p(X)_{\mathbb{R}} \arrow[r] \arrow[d, "L^{d+1-2p}"] & 0 \\
0 \arrow[r] & CH^{d-p}(X)_{\mathbb{R}} \arrow[r, "\widetilde{a}"] & CH^{d+1-p}(\overline{X})_{\mathbb{R}} \arrow[r, "\zeta"] & CH^{d+1-p}(X)_{\mathbb{R}} \arrow[r] & 0
\end{tikzcd}$$

By classical Lefschetz theory the left vertical map is injective and the right vertical map is surjective. The snake lemma gives an exact sequence

$$0 \to \mathrm{Ker}(\widehat{L}^{d+1-2p}) \to CH^p_{\mathrm{prim}}(X)_{\mathbb{R}} \xrightarrow{\delta} CH^{d-p}_{\mathrm{coprim}}(X)_{\mathbb{R}} \to \mathrm{Coker}(\widehat{L}^{d+1-2p}) \to 0.$$

The connecting homomorphism δ is characterized by the property that $\widetilde{a} \circ \delta = \widehat{L}^{d+1-2p} \circ s$ modulo the subspace $\widetilde{a}(\mathrm{Ker}(\pi_p))$. Note also that (4) implies

$$\widehat{L}^k \circ s - s \circ L^k = \widetilde{a} \sum_{i=0}^{k-1} L^{k-1-i} U L^i$$

for all k. We deduce that δ coincides with the map δ_p in (iii), and hence that statements (i) and (iii) are equivalent. This also shows that δ_p does not depend on the splitting s of (3) and the associated linear operator U on $CH^*(X)_{\mathbb{R}}$. □

Remark. Assuming statement (iii), it is possible to give an explicit construction of the map $\widehat{\Lambda}$ in (ii), as follows. We first claim that there exists a splitting s' of (3), with associated operator U', such that for any p and $\alpha \in CH^p_{\mathrm{prim}}(X)_{\mathbb{R}}$ we have

$$(5) \qquad U'L^i\alpha = 0 \ \text{ for all } \ i < d-2p \quad \text{and} \quad U'L^{d-2p}\alpha \in CH^{d-p}_{\mathrm{coprim}}(X)_{\mathbb{R}}.$$

Indeed, if we let D be the linear transformation such that $s' - s = \widetilde{a} \circ D$, then

$$U' = U + [L, D],$$

and it is an exercise to check that the space of transformations $[L, D]$ is equal to the set of operators V on $CH^*(X)_{\mathbb{R}}$ satisfying $\pi_p \sum_{i=0}^{d-2p} L^{d-2p-i} V L^i(\alpha) = 0$ for all $\alpha \in CH^p_{\mathrm{prim}}(X)_{\mathbb{R}}$ and every p.

Now choose a primitive basis for $CH^*(X)_{\mathbb{R}}$ and apply s' to get half of a basis for $CH^*(\overline{X})_{\mathbb{R}}$. By (iii), we may apply $(L^{d-2p})^{-1}\pi_p U' L^{d-2p}$ to the basis elements in $CH^p_{\mathrm{prim}}(X)_{\mathbb{R}}$ for each p to obtain another basis for $CH^*(X)_{\mathbb{R}}$, which we view (via $\widetilde{a}$) as the other half of our basis for $CH^*(\overline{X})_{\mathbb{R}}$. Let $v \in CH^p_{\mathrm{prim}}(X)_{\mathbb{R}}$ be one of the basis elements, and let $r = d - 2p$. By our conditions on s', a subset of our basis for $CH^*(\overline{X})_{\mathbb{R}}$ consists of $\widehat{v} := s'(v)$, the iterates $\widehat{L}^i(\widehat{v}) = s'(L^i v)$ of $\widehat{L}$ applied to $\widehat{v}$, the primitive element w satisfying $L^r(w) = \pi_p(U'L^r(v))$, and the iterates of $\widehat{L}$ applied to w:

$$\widehat{v},\ \widehat{L}\widehat{v},\ \ldots,\ \widehat{L}^r\widehat{v},\ w,\ Lw,\ \ldots,\ L^r w. \tag{6}$$

The action of $\widehat{L}$ is to send each element in (6) to the element on its right, except that $\widehat{L}^r\widehat{v}$ is sent to $L^r w$, and $L^r w$ to 0. We now define $\widehat{\Lambda}$ by

$$\widehat{\Lambda}(\widehat{L}^i\widehat{v}) = i(r+2-i)\widehat{L}^{i-1}\widehat{v},$$
$$\widehat{\Lambda}(L^i w) = (r+1)\widehat{L}^i\widehat{v} + i(r-i)L^{i-1}w.$$

Then $\widehat{\Lambda}$ (defined this way for every basis element v) satisfies the condition of (ii).

Theorem 2. *Suppose the arithmetic variety X and p are such that $CH^{p-1}_{\mathrm{prim}}(X)_{\mathbb{R}} = 0$. If, for each nonzero $\alpha \in CH^p_{\mathrm{prim}}(X)_{\mathbb{R}}$, we have*

$$(-1)^p \sum_{i=0}^{d-2p} \int_X L^{d-2p-i}\alpha \wedge UL^i\alpha > 0 \tag{7}$$

then the statements in the arithmetic hard Lefschetz and Hodge index conjectures are true for that X, p and $\overline{M}$.

Proof. Let $(\alpha,\beta) \in CH^p(\overline{X})_{\mathbb{R}}$ be a nonzero element of the kernel of $\widehat{L}^{d+2-2p}$; the notation (α,β) refers to the direct sum decomposition (2), with respect to some splitting. We claim that α must be in $CH^p_{\mathrm{prim}}(X)_{\mathbb{R}}$. Indeed, $\widehat{L}^{d+2-2p}(\alpha,\beta) = (L^{d+2-2p}\alpha,\gamma)$ for some γ and $L^{d+2-2p}\alpha = 0$ implies $L^{d+1-2p}\alpha = 0$ since $CH^{p-1}_{\mathrm{prim}}(X)_{\mathbb{R}}$ vanishes. Also, by the classical hard Lefschetz theorem, $\alpha \neq 0$. Now, if

$$\langle\ ,\ \rangle : CH^*(\overline{X})_{\mathbb{R}} \otimes CH^*(\overline{X})_{\mathbb{R}} \longrightarrow \mathbb{R}$$

denotes the arithmetic intersection pairing, then we have

$$\begin{aligned}\langle\, (\alpha,\beta)\,,\, \widehat{L}^{d+1-2p}(\alpha,\beta)\,\rangle &= \langle\, (\alpha,\beta)\,, (0, \sum_i L^{d-2p-i}UL^i\alpha + L^{d+1-2p}\beta)\,\rangle \\ &= \frac{1}{2}\sum_i \int_X L^{d-2p-i}\alpha \wedge UL^i\alpha.\end{aligned}$$

Hence, assuming $CH^{p-1}_{\mathrm{prim}}(X)_{\mathbb{R}} = 0$, we have $\widehat{L}^{d+1-2p}(\alpha,\beta) \neq 0$ for every nonzero $(\alpha,\beta) \in CH^p(\overline{X})_{\mathbb{R}}$. Moreover, if (α,β) is primitive, then the pairing of (α,β) with $\widehat{L}^{d+1-2p}(\alpha,\beta)$ has the required sign. □

Corollary 1. *Suppose X is such that, for every p,*

$$CH^p_{\mathrm{prim}}(X)_{\mathbb{R}} \neq 0 \qquad \textit{implies} \qquad CH^{p-1}_{\mathrm{prim}}(X)_{\mathbb{R}} = 0. \tag{8}$$

If condition (7) holds for every p and each nonzero $\alpha \in CH^p_{\mathrm{prim}}(X)_{\mathbb{R}}$, then both the arithmetic hard Lefschetz and Hodge index conjectures are true for X, $\overline{M}$.

Proof. If $CH^{p-1}_{\mathrm{prim}}(X)_{\mathbb{R}} = 0$ then the conjectures hold for X, $\overline{M}$ and p by Theorem 2. Assume now that $CH^{p-1}_{\mathrm{prim}}(X)_{\mathbb{R}} \neq 0$. It suffices to prove the Hodge index inequality for a nonzero $x \in CH^p(\overline{X})_{\mathbb{R}}$ in the kernel of $\widehat{L}^{d+2-2p}$. Since, by hypothesis, we have $CH^p_{\mathrm{prim}}(X)_{\mathbb{R}} = 0$, it follows that $x = \widehat{L}(y) + \widetilde{a}(\eta)$ for some $\eta \in CH^{p-1}(X)_{\mathbb{R}}$ and $y \in CH^{p-1}(\overline{X})_{\mathbb{R}}$, with $y \neq 0$ and $\widehat{L}^{d+4-2p}(y) = 0$. Moreover, the condition $\widehat{L}^{d+2-2p}(x) = 0$ implies $\widehat{L}^{d+2-2p}(\widetilde{a}(\eta)) = -\widehat{L}^{d+3-2p}(y)$. Now we find

$$\langle\, x,\, \widehat{L}^{d+1-2p}(x)\,\rangle = -\langle\, y,\, \widehat{L}^{d+3-2p}(y)\,\rangle,$$

and the required Hodge index inequality is a consequence of the degree $p-1$ case of Theorem 2. $\square$

Example. We illustrate the previous results for projective space $\mathbb{P}^n$ and the canonical hermitian line bundle $\overline{\mathcal{O}}(1)$ (compare [Ku, §4]). In this case we choose the splitting

$$CH^*(\overline{\mathbb{P}^n})_{\mathbb{R}} = \bigoplus_{i=0}^{n} \mathbb{R}\cdot\widehat{\omega}^i \;\oplus\; \bigoplus_{i=0}^{n} \mathbb{R}\cdot\omega^i$$

where $\widehat{\omega}^i = \widehat{c}_1(\overline{\mathcal{O}}(1))^i$ and $\omega^i = \widetilde{a}(c_1(\mathcal{O}(1))^i)$. Then the sequence (6) is given by

$$\widehat{1},\, \widehat{\omega},\, \ldots,\, \widehat{\omega}^n,\, \tau_n,\, \tau_n\omega,\, \ldots,\, \tau_n\omega^n.$$

Here $\tau_n = \sum_{k=1}^n \mathcal{H}_k$, where each $\mathcal{H}_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$ is a harmonic number. The Remark after Theorem 1 exhibits an explicit adjoint map $\widehat{\Lambda}$ for the arithmetic Lefschetz operator $\widehat{L}(x) = \widehat{\omega}\cdot x$; in our example it is given by

$$\begin{aligned} \widehat{\Lambda}(\widehat{\omega}^i) &= i(n+2-i)\,\widehat{\omega}^{i-1}, \\ \widehat{\Lambda}(\omega^i) &= \frac{n+1}{\tau_n}\,\widehat{\omega}^i + i(n-i)\,\omega^{i-1}. \end{aligned}$$

Observe that the nonzero primitive elements of $CH^*(\mathbb{P}^n)_{\mathbb{R}}$ are multiples of $1 \in CH^0(\mathbb{P}^n)_{\mathbb{R}}$; hence $\mathbb{P}^n$ satisfies (8). The operator U is given by $U(\omega^i) = \delta_{i,n}\tau_n\omega^n$, and condition (7) for $p=0$, $\alpha = 1$ becomes

$$\sum_{i=0}^{n} \int_{\mathbb{P}^n} \omega^{n-i} \wedge U(\omega^i) = \tau_n \int_{\mathbb{P}^n} \omega^n = \sum_{k=1}^{n} \mathcal{H}_k > 0. \tag{9}$$

The arithmetic Hodge index conjecture for $\mathbb{P}^n$ and $\overline{\mathcal{O}}(1)$ follows by applying Corollary 1.

A hermitian line bundle $\overline{M} = (M, \|\cdot\|)$ on an arithmetic variety X is *arithmetically ample* if M is an ample invertible sheaf on X such that the first Chern form $c_1(\overline{M(\mathbb{C})})$ is nonnegative on $X(\mathbb{C})$ and for all nonempty irreducible closed subsets $Y \subset X$ the height $h_{\overline{M}}(Y)$ is positive [So, §5.2]. We say that $\overline{M}$ is a *limit for arithmetic ampleness* if (i) $(M, t\|\cdot\|)$ is arithmetically ample for all positive scalars $t < 1$, and (ii) $h_{\overline{M}}(Y) = 0$ for some (nonempty) irreducible closed $Y \subset X$.

The line bundle $\overline{\mathcal{O}}(1)$ on $\mathbb{P}^n$ considered above is a limit for arithmetic ampleness. Indeed, if $Y \in Z_1(\mathbb{P}^n)$ is the cycle attached to the rational point $[1:0:\cdots:0]$, then $h_{\overline{\mathcal{O}}(1)}(Y) = 0$ (see for instance [BGS, (3.1.6)]). Furthermore, property (i) for $\overline{\mathcal{O}}(1)$ follows from the argument in [BGS, Prop. 3.2.4]. On the other hand, one sees that the arithmetic Hodge index inequality (9) does not fail when the natural metric on $\mathcal{O}(1)$ is scaled by a factor $t \in (1-\epsilon, 1+\epsilon)$ for small $\epsilon > 0$ (see also [BGS, Prop. 3.2.2]). Observe however that (9) becomes sharp at $t = 1$ if we insist that it should hold for *any* positive real constants $\mathcal{H}_k$ (following the point of view in [T, §6]). Further evidence for this statement on more general Grassmannians is given in Section 3.

2. The arithmetic Grassmannian $G(2, N)$

In this section we study Conjecture 1 for the Grassmannian of lines in projective space. For computational purposes we will work with the isomorphic Grassmannian $G = G(N, 2)$ parametrizing N-planes in $(N+2)$-space throughout. Note that $d = \dim_{\mathbb{C}} G(\mathbb{C}) = 2N$. There is a universal exact sequence of vector bundles

$$0 \longrightarrow S \longrightarrow E \longrightarrow Q \longrightarrow 0$$

over G; the complex points of E and Q are metrized by giving the trivial bundle $E(\mathbb{C})$ the trivial hermitian metric and the quotient bundle $Q(\mathbb{C})$ the induced metric. The hermitian vector bundles that result are denoted $\overline{E}$ and $\overline{Q}$.

The real vector space $CH^*(G)_{\mathbb{R}} \cong H^{2*}(G(\mathbb{C}), \mathbb{R})$ decomposes as

$$CH^*(G)_{\mathbb{R}} = \bigoplus_{a,b} \mathbb{R} \cdot s_{a,b}(Q),$$

summed over all partitions $\lambda = (a, b)$ with $a \leqslant N$, i.e., whose Young diagrams are contained in the $2 \times N$ rectangle (N, N). Moreover $s_\lambda(Q) = s_{a,b}(Q)$ is the characteristic class coming from the Schur polynomial $s_{a,b}$ in the Chern roots of Q; this is dual to the class of a codimension $|\lambda| = a+b$ Schubert variety in G. In the following s_1 denotes the Schur polynomial $s_{1,0}$ and is thus just the first elementary symmetric function.

The line bundle $M = \det(Q)$ giving the Plücker embedding has $c_1(M) = s_1(Q)$; let $L : CH^p(G)_{\mathbb{R}} \to CH^{p+1}(G)_{\mathbb{R}}$ be the associated classical Lefschetz operator. Further for all p let $* : CH^p(G)_{\mathbb{R}} \to CH^{2N-p}(G)_{\mathbb{R}}$ denote the Hodge star operator induced by the Kähler form $s_1(\overline{Q})$. We then have

Proposition 1. *The space $CH^p_{\mathrm{prim}}(G)_{\mathbb{R}}$ is nonzero if and only if $p = 2k \leqslant N$. In the latter case it is one dimensional and spanned by the class*

$$\alpha_k = \sum_{j=0}^{k} (-1)^j \binom{N+1-j}{N-2k} \binom{N-2k+j}{N-2k} s_{2k-j,j}(Q).$$

Proof. By computing the Betti numbers for G one sees that

$$\dim CH^p(G)_{\mathbb{R}} - \dim CH^{p-1}(G)_{\mathbb{R}} > 0$$

if and only if $p = 2k \leqslant N$, and in this case the above difference equals 1. For such p we have

(10)

$$\mathrm{Ker}(L : CH^{2N-2k}(G)_{\mathbb{R}} \to CH^{2N-2k+1}(G)_{\mathbb{R}}) = \mathrm{Span}\{\sum_{j=0}^{k}(-1)^j s_{N-j,N-2k+j}(Q)\}.$$

One checks (10) easily using the Pieri rule:

$$L(s_{a,b}(Q)) = s_{a+1,b}(Q) + s_{a,b+1}(Q)$$

where it is understood that $s_{c,c'}(Q) = 0$ if $c < c'$ or $c > N$.

From [KT] we know the action of the Hodge star operator on $CH^*(G)_{\mathbb{R}}$ is given by

$$* \, s_{a,b}(Q) = \frac{(a+1)!\, b!}{(N-a)!\,(N-b+1)!}\, s_{N-b,N-a}(Q). \tag{11}$$

Since

$$CH^{2k}_{\mathrm{prim}}(G)_{\mathbb{R}} = * \, \mathrm{Ker}\left(L : CH^{2N-2k}(G)_{\mathbb{R}} \to CH^{2N-2k+1}(G)_{\mathbb{R}}\right),$$

the proof is completed by applying (11) to (10) and noting that the result is proportional to α_k. □

We now pass to the arithmetic setting, where we use the arithmetic Schubert calculus of [T, §3,4]. The real Arakelov Chow group $CH^p(\overline{G})_{\mathbb{R}}$ decomposes as

$$CH^p(\overline{G})_{\mathbb{R}} = \bigoplus_{a+b=p} \mathbb{R} \cdot \widehat{s}_{a,b}(\overline{Q}) \, \oplus \bigoplus_{a'+b'=p-1} \mathbb{R} \cdot s_{a',b'}(\overline{Q}). \tag{12}$$

Here the indexing sets satisfy $N \geqslant a \geqslant b \geqslant 0$, $\widehat{s}_{a,b}(\overline{Q})$ is an arithmetic characteristic class and we identify the harmonic differential form $s_{a',b'}(\overline{Q})$

with its image in $CH^p(\overline{G})_{\mathbb{R}}$. The decomposition (12) is induced by the splitting map $s_{a,b}(Q) \longmapsto \widehat{s}_{a,b}(\overline{Q})$ which agrees with the one used in [T].

The hermitian line bundle $\overline{M}$ has $\widehat{c}_1(\overline{M}) = \widehat{s}_1(\overline{Q})$ and is a limit for arithmetic ampleness; the latter property follows as in the remarks at the end of Section 1. We now apply the arithmetic Pieri rule of [T, §4] to compute the action of the arithmetic Lefschetz operator $\widehat{L}(x) = \widehat{s}_1(\overline{Q}) \cdot x$ on the above basis elements. The induced operator $U : CH^*(G)_{\mathbb{R}} \to CH^*(G)_{\mathbb{R}}$ of (4) satisfies $U(s_{a,b}) = 0$ for $a < N$ and

$$U(s_{N,b}) = \Big(\sum_{i=0}^{N+1} \mathcal{H}_i\Big) s_{N,b} - \sum_{i=0}^{\lfloor (N-b)/2 \rfloor} (\mathcal{H}_{N-b+1-i} - \mathcal{H}_i) s_{N-i,b+i}. \tag{13}$$

Here and in the rest of this section $s_{a,b}$ will denote the Schubert class $s_{a,b}(Q) \in CH^{a+b}(G)_{\mathbb{R}}$ and $\mathcal{H}_i$ is a harmonic number; by convention $\mathcal{H}_0 = 0$. Recall that the classical intersection pairing on $CH^*(G)_{\mathbb{R}}$ satisfies

$$\langle s_{a,b},\, s_{a',b'} \rangle = \int_G s_{a,b} \wedge s_{a',b'} = \delta_{(a,b),(N-b',N-a')}.$$

The sequence of Betti numbers for G shows that G satisfies condition (8) of Corollary 1. We proceed to check the inequality (7) for all even $p = 2k$; this will establish Conjecture 1 for $G(N,2)$. In our case (7) may be written as

$$\Sigma(N,k) := \int_G \sum_{b=0}^{N-2k} L^{N-2k-b} \alpha_k \wedge U L^{N-2k+b} \alpha_k > 0.$$

To compute iterates of the classical Lefschetz operator L on the Schubert basis, note that

$$L^r s_\mu = \sum_{\substack{\lambda \supset \mu \\ |\lambda| = |\mu| + r}} f^{\lambda/\mu} s_\lambda. \tag{14}$$

When $\lambda = (\lambda_1, \lambda_2)$ and $\mu = (\mu_1, \mu_2)$ are partitions with at most two parts the skew f-number in (14) satisfies

$$f^{\lambda/\mu} = \binom{|\lambda| - |\mu|}{\lambda_1 - \mu_1} - \binom{|\lambda| - |\mu|}{\lambda_1 - \mu_2 + 1}. \tag{15}$$

This follows from the determinantal formula for $f^{\lambda/\mu}$, given for example in [St, Corollary 7.16.3].

We now apply Proposition 1 and (14), (15) to calculate

$$L^{c-2k}\alpha_k = \sum_{j=0}^{k}(-1)^j\binom{N+1-j}{N-2k}\binom{N-2k+j}{N-2k}L^{c-2k}s_{2k-j,j}$$

$$= \sum_{j=0}^{k}\sum_{i\geqslant j}(-1)^j\binom{N+1-j}{N-2k}\binom{N-2k+j}{N-2k}\left[\binom{c-2k}{i-j}-\binom{c-2k}{i-2k-1+j}\right]s_{c-i,i}.$$

Therefore,

$$L^{c-2k}\alpha_k = \sum_{i,j}(-1)^j\binom{N+1-j}{N-2k}\binom{N-2k+j}{N-2k}\binom{c-2k}{i-j}s_{c-i,i}. \tag{16}$$

We use (16) with $c = N + b$ to identify the coefficient of $s_{N,b}$ in the expansion of $L^{N+b-2k}\alpha_k$ as

$$\begin{aligned}\left\langle L^{N+b-2k}\alpha_k,\, s_{N-b}\right\rangle &= \sum_{j}(-1)^j\binom{N+1-j}{N-2k}\binom{N-2k+j}{j}\binom{N+b-2k}{N-2k+j}\\ &= \binom{N-2k+b}{N-2k}\sum_{j}(-1)^j\binom{b}{j}\binom{N+1-j}{N-2k}\\ &= \binom{N+1-b}{2k+1}\binom{N-2k+b}{N-2k} =: C_b.\end{aligned}$$

It now follows from (13) that

$$UL^{N-2k+b}\alpha_k = C_b\Big(\sum_{i=0}^{N+1}\mathcal{H}_i\Big)s_{N,b} - \sum_{i=0}^{\lfloor (N-b)/2\rfloor} C_b(\mathcal{H}_{N-b+1-i}-\mathcal{H}_i)s_{N-i,b+i}. \tag{17}$$

Note also that we have the identity

$$\sum_{b=0}^{N-2k} C_b = \binom{2N-2k+2}{N+2}. \tag{18}$$

Now we substitute $c = N - b$ in (16), pair with (17) and use (18) to sum over b and obtain

$$\Sigma(N,k) = A_{N,k}\sum_{i=1}^{N+1}\mathcal{H}_i + \sum_{b=0}^{N-2k}\sum_{i=0}^{\lfloor (N-b)/2\rfloor} C_{N,k}^{b,i}, \tag{19}$$

where

$$A_{N,k} = \binom{N+1}{N-2k}\binom{2N-2k+2}{N+2}$$

and

$$C_{N,k}^{b,i} = \sum_j (-1)^j \binom{N+1-j}{N-2k}\binom{N-2k+j}{N-2k}\binom{N-2k-b}{i-j} C_b(\mathcal{H}_i - \mathcal{H}_{N-b+1-i}).$$

By dividing the expression for $C_{N,k}^{b,i}$ into two sums and substituting in (19) one gets

$$\Sigma(N,k) = A_{N,k}\sum_{i=1}^{N+1}\mathcal{H}_i + \sum_{i=1}^{N+1} B_{N,k}^i \mathcal{H}_i, \tag{20}$$

where

$$B_{N,k}^i = \sum_{j,b}(-1)^j \binom{N+1-j}{N-2k}\binom{N-2k+j}{N-2k}\binom{N-2k-b}{i-j} C_b.$$

(Notice that when $N-b = 2r-1$ is odd, there is a missing summand (for $i=r$)

$$\sum_j (-1)^j \binom{N+1-j}{N-2k}\binom{N-2k+j}{N-2k}\binom{2r-2k-1}{r-j} C_b \mathcal{H}_r$$

which vanishes, as can be seen by the change of variable $j \mapsto 2k+1-j$.)

At this point it is convenient to introduce the variable change

$$n = N - 2k \qquad \text{and} \qquad T = N+2$$

and write equation (20) in the new coordinates as

$$\Sigma(n,T) = A_{n,T}\sum_{i=1}^{T-1}\mathcal{H}_i + \sum_{i=1}^{T-1} B_{n,T}^i \mathcal{H}_i. \tag{21}$$

Observe that

$$\begin{aligned} B_{n,T}^i &= \sum_j (-1)^j \binom{n+j}{n}\binom{T-1-j}{n}\sum_b \binom{n-b}{i-j}\binom{T-1-b}{n-b}\binom{n+b}{n} \\ &= \sum_j (-1)^j \binom{n+j}{n}\binom{T-1-j}{n}\binom{T-1-n+i-j}{i-j}\binom{T+n}{n-i+j}. \end{aligned}$$

We now substitute $r = i-j$ and write the resulting sum in hypergeometric notation [Ro] [VK, Chap. 3]:

$$\begin{aligned} (-1)^i B_{n,T}^i &= \sum_r (-1)^r \binom{n+i-r}{n}\binom{T-1+r-i}{n}\binom{T-1-n+r}{r}\binom{T+n}{n-r} \\ &= \binom{n+i}{n}\binom{T+n}{n}\binom{T-1-i}{n} {}_4F_3\left(\left.\begin{matrix} -n,\, -i,\, T-n,\, T-i \\ -n-i,\, T+1,\, T-n-i \end{matrix}\,\right|\, 1\right). \end{aligned}$$

The Whipple transformation [Wh, §10] applied to the above ${}_4F_3$ gives

$$(-1)^i \frac{B^i_{n,T}}{A_{n,T}} = {}_4F_3\left(\begin{matrix} -n,\ n+1,\ -i,\ i+1 \\ 1,\ 1+T,\ 1-T \end{matrix} \,\middle|\, 1\right). \tag{22}$$

The hypergeometric term (22) belongs to a class of orthogonal polynomials called Racah polynomials, which are studied in the next section.

3. Bounds for Racah polynomials

The Racah coefficients [Ra] or 6-j symbols have long been used by physicists as the transformation coefficients between two different coupling schemes of three angular momenta; see [BL] for an exposition. In mathematical language they are the entries of a change of basis matrix for the tensor product of three irreducible representations of SU(2); the two bases involved come from the associativity relation for this product (see [VK, §8.4]). It was recognized later by Wilson [Wi] that these coefficients are special cases of a class of orthogonal polynomials $R_n(x;\alpha,\beta,\gamma,\delta)$, called Racah polynomials [VK, §8.5]:

$$R_n(s(s+\gamma+\delta+1);\alpha,\beta,\gamma,\delta) = {}_4F_3\left(\begin{matrix} -n,\ n+\alpha+\beta+1,\ -s,\ s+\gamma+\delta+1 \\ \alpha+1,\ \beta+\delta+1,\ \gamma+1 \end{matrix} \,\middle|\, 1\right).$$

The Racah polynomials in (22) have $\alpha=\beta=\gamma+\delta=0$, with $\gamma=T$, a positive integer. We let

$$R_n(s,T) = R_n(s(s+1);0,0,T,-T).$$

Observe that $R_n(s,T)$ is symmetric in n and s. The orthogonality condition ([VK] or [AW]) reads:

$$\sum_{s=0}^{T-1}(2s+1)R_n(s,T)R_m(s,T) = \frac{T^2}{(2n+1)}\delta_{nm}. \tag{23}$$

The arithmetic Hodge index inequality $\Sigma(n,T)>0$ can be rephrased using (21) and (22) as

$$\sum_{s=1}^{T-1}(-1)^{s+1}R_n(s,T)\mathcal{H}_s < \sum_{s=1}^{T-1}\mathcal{H}_s. \tag{24}$$

We give a proof of (24) which does not depend on the precise values of the harmonic numbers. Let us say that a sequence $\{\mathcal{H}_k\}_{k\geqslant 1}$ of positive real numbers (with $\mathcal{H}_0=0$) is *concave increasing* if $\mathcal{H}_k=\sum_{i=1}^k h_i$ for some monotone decreasing sequence $\{h_i\}$ of positive reals.

Theorem 3. *Let $\{\mathcal{H}_k\}$ be any concave increasing sequence of real numbers and n, T integers with $0\leqslant n\leqslant T-1$ and $T\geqslant 3$. Then inequality (24) holds.*

We believe that, in fact, (24) holds for an arbitrary sequence of positive real numbers $\mathcal{H}_k$, that is, the arithmetic standard conjectures for $G(2,N)$ do not depend on the relative sizes of the harmonic numbers involved:

Conjecture 2. *For any integers n, s with $0 \leqslant n, s \leqslant T-1$ we have $|R_n(s,T)| \leqslant 1$.*

In Proposition 2 we check this conjecture for some values of n near the endpoints 0 and $T-1$. Computer calculations support the validity of Conjecture 2 for general n.

Proof of Theorem 3. We shall see that (23) implies (24) except when T is exponentially large compared to n. For large T, the Racah polynomials are close approximations of classical orthogonal polynomials, in this case the Legendre polynomials, and we know how to bound these.

By Cauchy's inequality, (23) gives

$$\left(\sum_{s=0}^{T-1} |R_n(s,T)|\,\mathcal{H}_s\right)^2 \leqslant \frac{T^2}{2n+1}\sum_{s=0}^{T-1}\frac{\mathcal{H}_s^2}{2s+1}.$$

So, (24) holds whenever

$$\sum_{s=0}^{T-1}\frac{\mathcal{H}_s^2}{2s+1} < (2n+1)\left(\frac{1}{T}\sum_{s=0}^{T-1}\mathcal{H}_s\right)^2. \tag{25}$$

Since $\{\mathcal{H}_k\}$ is concave increasing, the average value of $\mathcal{H}_0, \ldots, \mathcal{H}_{T-1}$ is at least $\mathcal{H}_{T-1}/2$. As $\sum_{s=1}^{T-1} 2/(2s+1) \leqslant \log T$, the inequality (25) holds whenever

$$\log T < n + \frac{1}{2}. \tag{26}$$

To analyze the case where T is exponentially large compared to n, it is convenient to introduce the change of variable

$$x = s(s+1) = -1/4 + T^2(1+t)/2 \tag{27}$$

and the rescaling

$$p_n(t) = (-1)^n \prod_{i=1}^{n}\frac{T^2-i^2}{T^2} R_n(x;0,0,T,-T).$$

Let $P_n(t) = P_n^{(0,0)}(t)$ denote the n^{th} Legendre polynomial. It is known [NSU, §3.8] that

$$p_n(t) = P_n(t) + O(1/T^2)$$

where the constant in the error term depends on both n and t. For our purposes, we demonstrate

Lemma 1. a) *Let n and T be positive integers such that $1+2n+2n^2 < T^2/10$. Then*

$$|p_n(t) - P_n(t)| \leqslant (3/2) \cdot 4^n/T^2 \tag{28}$$

for all t with $-1 \leqslant t \leqslant 1$.

b) *We have $|p_n(t) - P_n(t)| \leqslant 1/10$ whenever $T \geqslant 90$ and $n < \log T$.*

Proof. We have the following recurrences ([NSU]; for $P_n(t)$ this is classical)

(29)

$$tP_n(t) = \frac{n+1}{2n+1}P_{n+1}(t) + \frac{n}{2n+1}P_{n-1}(t)$$

(30)

$$tp_n(t) = \frac{n+1}{2n+1}p_{n+1}(t) - \frac{2n^2+2n+1}{2T^2}p_n(t) + \left(1 - \frac{n^2}{T^2}\right)^2 \frac{n}{2n+1}p_{n-1}(t)$$

with initial data

$$\begin{aligned} P_0(t) = 1 \quad &; \quad P_1(t) = t \\ p_0(t) = 1 \quad &; \quad p_1(t) = t + 1/(2T^2). \end{aligned} \tag{31}$$

Subtracting (29) from (30) leads to a recurrence in $p_n(t) - P_n(t)$. Then (28) follows by induction on n, using the known bound $|P_n(t)| \leqslant 1$ for all n and all t with $-1 \leqslant t \leqslant 1$. The statement (b) is a corollary of (a). □

Proposition 2. *We have $|R_n(s,T)| \leqslant 1$ when $n \leqslant 3$ or $n = T-1$.*

Proof. For $n = 0$ we have $R_0(s,T) = 1$. When $n = 1$, we see from (31) that $p_1(t)$ is a increasing linear function in t, attaining minimum when $s = 0$, giving $R_1(0,T) = 1$, and maximum when $s = T-1$, giving $R_1(T-1,T) = (1-T)/(1+T)$, hence the inequality holds. For $n = T-1$ the Pfaff-Saalschütz identity [Ro] [VK, §8.3.3] gives

$$\begin{aligned} R_{T-1}(s,T) &= \sum_j (-1)^j \frac{T}{T+j}\binom{s}{j}\binom{s+j}{j} \\ &= {}_3F_2\left(\begin{matrix} -s,\ s+1,\ T \\ 1,\ T+1 \end{matrix} \,\middle|\, 1\right) \\ &= \frac{(1-T)(2-T)\cdots(s-T)}{(1+T)(2+T)\cdots(s+T)} \end{aligned}$$

so the inequality is clear.

When $n = 2$ or $n = 3$, $p_n(t)$ is a quadratic or cubic polynomial, and it is a calculus exercise to check that $|R_n(s,T)| \leqslant 1$ for every integer s with $0 \leqslant s \leqslant T-1$. In fact, the integrality condition on s is required only when $n = 3$, $T = 4$. □

Lemma 2. a) *We have* $|P_n(t)| \leqslant 3/4$ *for* $t \in \mathbb{R}$, $|t| \leqslant 0.9$ *and* $n \geqslant 2$.

b) *For* $T \geqslant 10$, *we have* $|t| \leqslant 0.9$ *in* (27) *whenever* $\sqrt{5}/10 \leqslant s/T \leqslant 4/5$.

c) *Assume* $T \geqslant 90$ *and* $n < \log T$. *Then* $\dfrac{1}{T^{2n}} \displaystyle\prod_{i=1}^{n} (T^2 - i^2) > 40/41$.

Proof. The indicated bound on Legendre polynomials is evident for $n = 2$, and for larger n it follows from the inequality

$$\sqrt{\sin\theta}\, |P_n(\cos\theta)| < \sqrt{\frac{2}{\pi n}}, \qquad 0 \leqslant \theta \leqslant \pi. \tag{32}$$

One obtains (32) by using the transformed differential equation for $\sqrt{\sin\theta}\, P_n(\cos\theta)$ [Sz, (4.24.2)]; this is indicated in [Ho, Chap. 5, Exer. 15–16]. The proofs of (b) and (c) are routine; for the latter, one may use the inequality

$$-\log(T^{-2n} \prod_{i=1}^{n} (T^2 - i^2)) \leqslant \frac{2}{T^2} \sum_{i=1}^{n} i^2.$$

□

To complete the proof of Theorem 3, assume that (26) fails, so that $n < \log T - 1/2$. If $T \leqslant 90$ then $n \leqslant 3$ and (24) follows from Proposition 2 (note that the inequality in the proposition is strict unless $n = 0$ or $s = 0$). When $T \geqslant 90$ we combine Lemma 1(b) with Lemma 2 to deduce the inequality (24). Indeed, $(40/41)|R_n(s, T)|$ is bounded by $1 + 1/10$ for every s, and by $3/4 + 1/10$ over the middle half of the summation range. By pairing terms $\mathcal{H}_s$ with $\mathcal{H}_{T-1-s}$ and using the fact that $\mathcal{H}_s + \mathcal{H}_{T-1-s}$ is monotone increasing for $0 \leqslant s \leqslant (T-1)/2$, we obtain (24). □

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