A continuum of periodic solutions to the planar four-body problem with various choices of masses

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Planet Dancing With a Pair of Stars

Planet Dancing With a Pair of Stars

Sometimes the orange sun rises first. Sometimes it is the red one, although they are never far apart in the sky and you can see them moving each other, casting double shadows across the firmament and periodically crossing right in front of each other.

Such is life, now known to exist outside the bounds of our own solar system. It is the first planet, astronomers say, that has been definitely shown to be orbiting two stars at once, circling at a distance of some 65 million miles a pair of stars that are themselves circling each other much more closely.

Why does such system exist? What are the Mathematical Model and Simulation?


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Outline of the presentation

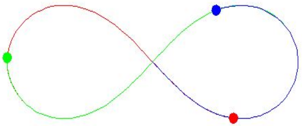
- Introduction to the N-body problem
- The variational method with structural prescribed boundary conditions (SPBC)
- Periodic solutions to the planar four-body problem with various choices of masses
- Some Simulations of orbits discovered by the variational method

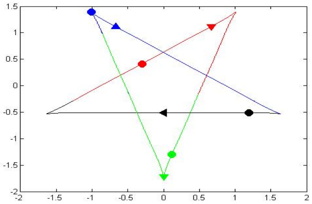
N-body Problem

The Newtonian N-body problem of celestial mechanics concerns the motion of N point masses governed by a second order differential equation system.

$$m_i\ddot{q}_i = -\sum_{j=1, j\neq i}^{N} \frac{m_i m_j}{|q_i - q_j|^2}\frac{(q_i - q_j)}{|q_i - q_j|}$$

where $q_i, i = 1, \cdots, N$ is the position of i-th body with mass m_i. $|q|$ denotes the Euclidean distance.





What we want to do

The original motivation for our work is practical and aesthetic: we want to have a concise and effective method not only to numerically search many different types of new periodic motions for general n-body problem but also to theoretically prove the existence of these solutions.

Two type of problems: (1) Design or recover a periodic orbit. (2) Explore a new periodic orbit.

The Variational method

Let $L(\mathbf{q}, \dot{\mathbf{q}}) = K(\dot{\mathbf{q}}) + U(\mathbf{q})$ be the Lagrangian function, where $K = \sum_{i=1}^{N} \frac{1}{2} m_i |\dot{q}_i|^2$, $U = \sum_{1 \le i < j \le N} \frac{m_i m_j}{\|q_i - q_j\|}$. Then Newton's equations are Euler-Lagrangian equations of the action functional:

$$\mathcal{A}(\mathbf{q}) = \int_0^T L(\mathbf{q}, \dot{\mathbf{q}}) dt.$$

We are looking for an minimizer of $\mathcal{A}(\mathbf{q})$.

$$\mathcal{A}(\mathbf{q}_0) = \min_{\mathbf{q} \in \mathcal{P}} \mathcal{A}(\mathbf{q}).$$

Critical points of $\mathcal{A}$ in $\mathcal{P} = H^1([0, T], (\mathbf{R}^d)^N)$ are solutions of the n-body problem (w/ or w/o collisions).

A brief History

In the 1740s it was constituted as the search for solutions (or at least approximate solutions) of a system of ordinary differential equations by the works of Euler, Clairaut and d'Alembert . Much were developed by Lagrange, Laplace and their followers, the mathematical theory entered a new era at the end of the 19th century with the works of Poincare and others.

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Gordon (Am. J. Math. 1977): Keplerian orbits with the same masses

and least periods have the same action values.

A brief History

Chenciner-Venturelli (Cel. Mech. Dyn. Ast. 2000)
Chenciner-Montgomery (Ann. Math. 2000)
K.C. Chen (ARch. Rat. Mech. Ana. 2001,2003, 2006)
Ferrario-Terracini (Invent. Math. 2004)
Venturelli-Terracini (Arch. Rat. Mech. Ana. 2007)
Barutello-Terracini (Nonlinearity 2005)
Ferrario (Arch. Rat. Mech. Ana. 2006, Adv. Math. 2008)
Barutello-Ferrario-Terracini (Arch. Rat. Mech. Ana. 2008)
Deng, Zhang and Zhou (Science in China 2010)
Fusco-Gronchi-Negrini (Invent. Math. 2011)
Chen-Ouyang-Xia (Mathematical Research Letters, 2012)
$\cdots, \quad \cdots$
There are many other references which use variational method to find new periodic solutions.

The Method We Used

Variational Method with Structural Prescribed Boundary Conditions (SPBC): Consider Boundary Value Problem

$$m_i \ddot{\mathbf{q}}_i = - \sum_{1 \leq i < j \leq N} \frac{m_i m_j}{|\mathbf{q}_i - \mathbf{q}_j|^3} (\mathbf{q}_i - \mathbf{q}_j) = \frac{\partial U}{\partial \mathbf{q}_i},$$

$$\mathbf{q}(0) = \mathbf{A}, \quad \mathbf{q}(T) = \mathbf{B},$$

where $\mathbf{A} = (a_{ij}) \in M^{(n \times d)}$, $\mathbf{B} = (b_{ij}) \in M^{(n \times d)}$, are the matrices of $n \times d$.
Let $\mathcal{P}(\mathbf{A}, \mathbf{B})$ be the set in function space of $H^1([0, T] \to \mathbb{R}^n)$,

$$\mathcal{P}(\mathbf{A}, \mathbf{B}) = \{\mathbf{q} \in H^1, \mid \mathbf{q}(0) = \mathbf{A};\ \mathbf{q}(T) = \mathbf{B}\}.$$

Now consider the minimizing problem

$$\mathcal{A}(\mathbf{A},\mathbf{B}) = \min_{\mathbf{q}\in\mathcal{P}(\mathbf{A},\mathbf{B})} \mathcal{A}(\mathbf{q}).$$

By theorems of Marchal and Chenciner, the corresponding minimizer $\mathbf{q}_0(\cdot,\mathbf{A},\mathbf{B})$ of above variational problem is a collision free solution in the interior $(0,T)$ of the Newton's equations:

$$m_i\ddot{\mathbf{q}}_i = -\sum_{j\neq i} \frac{m_i m_j}{|\mathbf{q}_i-\mathbf{q}_j|^3}(\mathbf{q}_i-\mathbf{q}_j),$$

with the boundary condition

$$\mathbf{q}(0)=\mathbf{A}; \quad \mathbf{q}(T)=\mathbf{B}.$$

$\mathcal{A}(\mathbf{A},\mathbf{B})$ is a function of $\mathbf{A}$ and $\mathbf{B}$.

Can the minimizer $\mathbf{q}_0(\cdot, \mathbf{A}, \mathbf{B})$ be extended to a periodic solution of the N-body problem or to an orbit we want?

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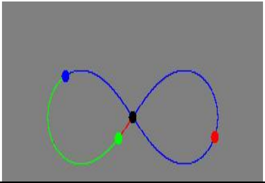
Can the minimizer $\mathbf{q}_0(\cdot, \mathbf{A}, \mathbf{B})$ be extended to a periodic solution of the N-body problem or to an orbit we want?
Yes! How to find such boundary condition $(\mathbf{A}, \mathbf{B})$?
The method to find such boundary condition is to do a second minimizing process under appropriate SPBC $G(\mathbf{A}, \mathbf{B}) = 0$:

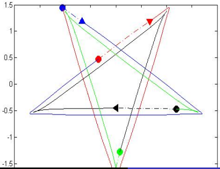
$$\min_{\{G(\mathbf{A},\mathbf{B})=0\}} \mathcal{A}(\mathbf{A}, \mathbf{B}) = \min_{\{G(\mathbf{A},\mathbf{B})=0\}} \min_{\mathbf{q}\in\mathcal{P}(\mathbf{A},\mathbf{B})} \mathcal{A}(\mathbf{q})$$

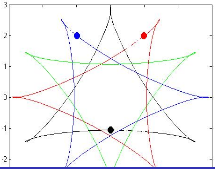
The conditions $G(\mathbf{A}, \mathbf{B}) = 0$, represent the geometric and topological structure of the preassigned orbits of the N-body problems. Intuitively, the second order nonlinear ODE system has unique solution for the initial value problem, i.e. the initial position and initial velocity contains all of the information for its future motion. Now we replace them with special boundary value with $G(\mathbf{A}, \mathbf{B}) = 0$.
Find the right condition $G(\mathbf{A}, \mathbf{B}) = 0$ for our preassigned orbit will be very important.

Periodic Solutions

A *simple choreographic solution* (for short, choreographic solution) is a periodic solution that all bodies chase one another along a single closed orbit.







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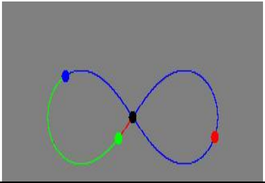

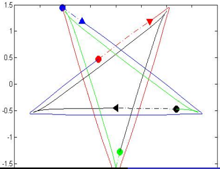

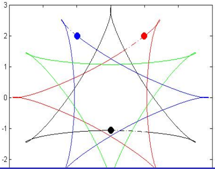

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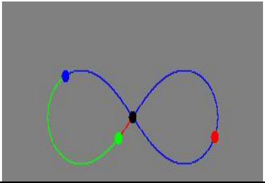

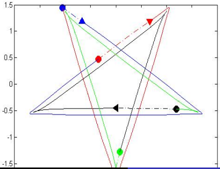

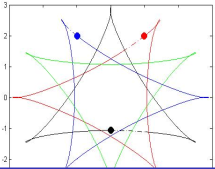

Periodic Solutions

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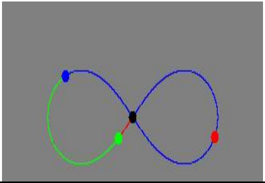

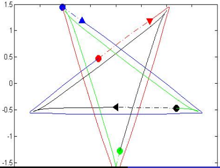

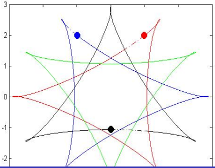

Main Theorem (Ouyang and Xie)

Assumptions or settings: We assume $m_1 = m_3$ and $m_2 = m_4$ and let $\mu = \frac{m_2}{m_1}$. Let $\Gamma = \mathbf{R}^6$ and the rotation matrix $R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$.

SPBC: Given $\vec{a} = (a_1, a_2, \cdots, a_6) \in \Gamma$, two fixed configurations are defined by $Qstart = \begin{pmatrix} 0 & -a_3 \\ -a_1 & a_2 \\ 0 & \frac{-m_2a_2 - m_4a_2 + m_1a_3}{m_3} \\ a_1 & a_2 \end{pmatrix} R(\theta)$, and

$Qend = \begin{pmatrix} a_4 & a_5 \\ 0 & -a_6 \\ -a_4 & a_5 \\ 0 & \frac{-m_1a_5 - m_3a_5 + m_2a_6}{m_4} \end{pmatrix}$. So the configuration of the bodies changes from an isosceles triangle with one on the axis of symmetry to another isosceles triangle for some positive $\vec{a}$.

Main Theorem (Ouyang and Xie)(Continued)

Approach: Two step minimizing process:
Step 1: For a given $\vec{a} = (a_1, a_2, \cdots, a_6) \in \Gamma = \mathbf{R}^6$. Then the set $S(\vec{a})$ of minimizers is defined by

$$S(\vec{a}) = \{q(t) = (q_1, q_2, q_3, q_4)(t) \in C^2((0,T), (\mathbf{R^2})^4) \quad \Big| \quad q(0) = \mathbf{A}, q(T) = \mathbf{B},$$

$$q(t) \text{ is a minimizer of the action functional } \mathcal{A} \text{ over } \mathcal{P}(\mathbf{A}, \mathbf{B})\}.$$

Main Theorem (Ouyang and Xie)(Continued)

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$$q(t) \text{ is a minimizer of the action functional } \mathcal{A} \text{ over } \mathcal{P}(\mathbf{A}, \mathbf{B})\}.$$

Step 2: Then the real value function $\tilde{\mathcal{A}}(\vec{a}) : \Gamma \to \mathbf{R}$ is well defined by

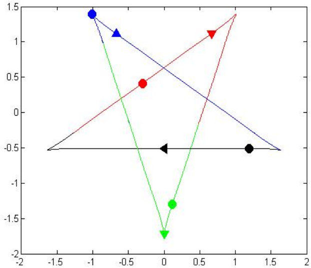
$$\tilde{\mathcal{A}}(\vec{a}) = \int_0^T \frac{1}{2} \sum_{i=1}^{n} m_i \|\dot{q}_i(t, \vec{a})\|^2 + U(q(t, \vec{a}))dt.$$

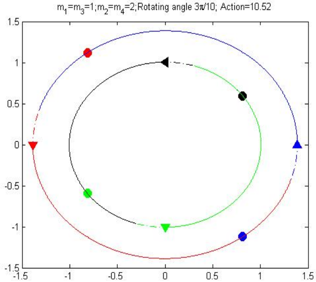
Let $\vec{a}_0 = (a_{10}, a_{20}, \cdots, a_{60}) \in \Gamma$ be a minimizer of $\tilde{\mathcal{A}}(\vec{a})$ over the space Γ and the corresponding path $q^*(t) = q^*(t, \vec{a}_0) \in S(\vec{a}_0)$, i.e.

$$\tilde{\mathcal{A}}(\vec{a}_0) = \min_{\vec{a} \in \Gamma} \tilde{\mathcal{A}}(\vec{a}) = \min_{\vec{a} \in \Gamma} \left\{ \inf_{q(t) \in \mathcal{P}(Q_{start}, Q_{end})} \mathcal{A}(q(t)) \right\}$$

Main Theorem (Ouyang and Xie)(Continued)

Results: For any given $(\theta, \mu) \in \Omega$ and $\theta \neq \pi$, there exists a minimizer $\vec{a} \in \Gamma$ of $\tilde{\mathcal{A}}$ over the space Γ, such that, the corresponding minimizing path $q^*(t)$ on $[0, T]$ connecting $q(0)$ and $q(T)$ can be extended to a non-homographic solution $q(t; \theta, \mu)$ (for short $q(t)$) of the Newton's equation by the extension formula. Each curve $q_i(t), t \in [4kT, (4k+4)T]$ is called a side of the orbit since the orbit of the solution is assembled out the sides $q_i(t), t \in [0, 4T]$ by rotation only. The non-homographic solution $q(t; \theta, \mu)$ can be classified as follows.

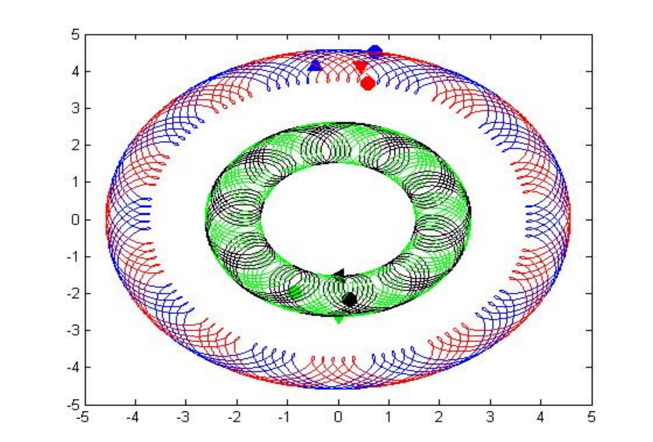




Theorem (Classification)

The non-circular solution $q(t;\theta,\mu)$ can be classified based on the rotation angle θ and mass ratio μ as follows.

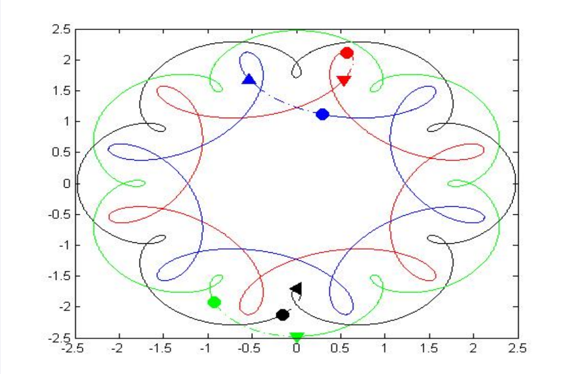
(1) **[Quasi-Periodic Solutions]** $q(t;\theta,\mu)$ is a quasi-periodic solution if θ is not commensurable with π. $(\theta,\mu) = (1.1\pi, 2)$



Theorem (Classification)

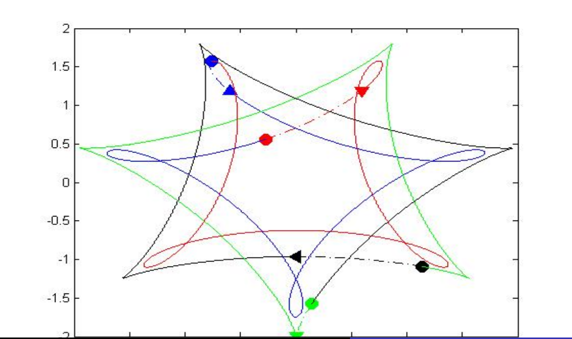
(2) **[Periodic Solutions]** $q(t;\theta,\mu)$is a periodic solution if $\theta = \frac{P}{Q}\pi$, where the positive integers P and Q are relatively prime.

- When Q is even, the periodic solution $q(t;\theta,\mu)$ is a non-choreographic solution. Each closed curve has $\frac{Q}{2}$ sides. The minimum period is $\mathcal{T} = 2QT$. $(\theta,\mu) = (\frac{13\pi}{12}, 0.8)$



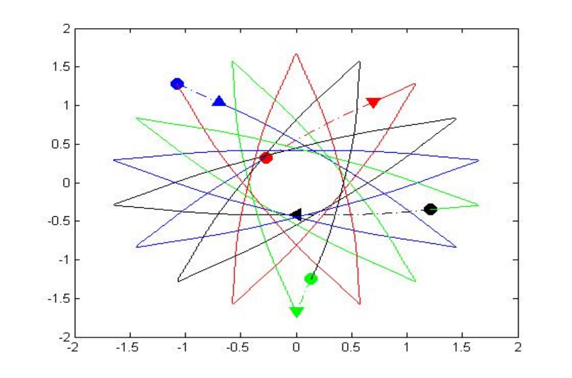
Theorem (Classification) Continued

- When Q is odd, there are four cases.
  Case 1: If $\mu \neq 1$, the periodic solution $q(t;\theta,\mu)$ is a double-choreographic solution. Each closed curve has Q sides. The minimum period is $\mathcal{T} = 4QT$. Body q_1 chases body q_3 on a closed curve and body q_2 chases body q_4 on another closed curve. $q_1(t+2QT) = q_3(t)$ and $q_3(t+2QT) = q_1(t)$. $q_4(t+2QT) = q_2(t)$ and $q_2(t+2QT) = q_4(t)$. $(\theta,\mu) = (\frac{6\pi}{7}, 0.8)$



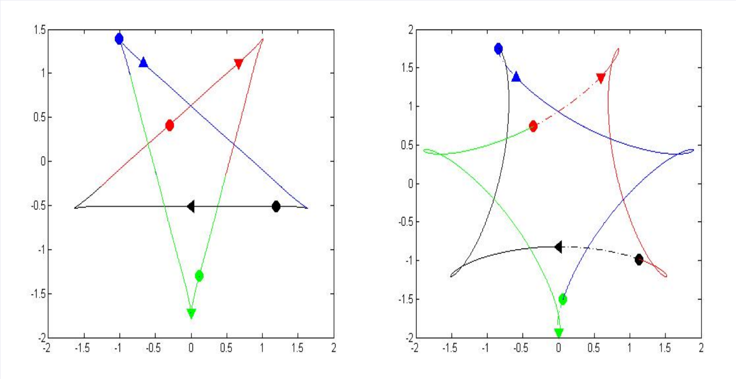
Theorem (Classification) Continued

case 2: If $\mu = 1$ and P is odd , the periodic solution $q(t;\theta,\mu)$ is a double-choreographic solution with minimum period $\mathcal{T} = 4QT$. Body q_1 chases body q_3 on a closed curve and body q_2 chases body q_4 on another closed curve.$(\theta,\mu) = (\frac{7\pi}{9}, 1)$



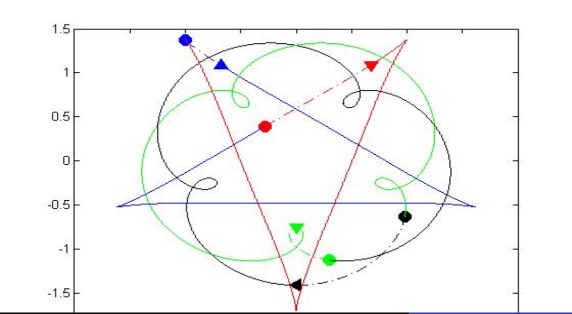
Theorem (Classification) Continued

Case 3: If $\mu = 1$, P is even and the initial configuration $q(0)$ is geometrically same to the ending configuration $q(T)$ then the periodic solution $q(t;\theta,1)$ is a choreographic solution. The closed curve has Q sides. The minimum period is $\mathcal{T} = 4QT.(\theta,\mu) = (\frac{4\pi}{5},1),(\frac{6\pi}{7},1)$



Theorem (Classification) Continued

Case 4: If $\mu = 1$, P is even and the initial configuration $q(0)$ is not geometrically same to the ending configuration $q(T)$, i.e. $(a_{10}, a_{20}, a_{30}) \neq (a_{40}, a_{50}, a_{60})$, then the periodic solution is a double choreographic solution. Each closed curve has Q sides. The minimum period is $\mathcal{T} = 4QT$. Body q_1 chases body q_3 on a closed curve and body q_2 chases body q_4 on another closed curve. $q_1(t + 2QT) = q_3(t)$ and $q_3(t + 2QT) = q_1(t)$. $q_4(t + 2QT) = q_2(t)$ and $q_2(t + 2QT) = q_4(t)$.
$(\theta, \mu) = (\frac{4\pi}{5}, 1)$



Theorem (Linear Stability)

If $\theta = \frac{2P-1}{2P}\pi$ and $\mu = 0.5, 1, 1.5$, the non-choreographic solutions $q(t)$ are linearly stable for $P = 3, 4, 5, \cdots, 15$. If $\theta = \frac{2P}{2P+1}$ and $\mu = 0.5, 1.5$, the double choreographic solutions $q(t)$ are linearly stable for $P = 2, 3, \cdots, 15$. If $\theta = \frac{2P-1}{2P+1}$ and $\mu = 1$, the double choreographic solutions $q(t)$ are linearly stable for $P = 4, 5, 6, \cdots, 15$.

Remark: A periodic solution of the planar n-body problem has eight trivial characteristic multipliers of $+1$. The solution is spectrally stable if the remaining multipliers lie on the unit circle and linearly stable if, in addition, the monodromy matrix restricted to the reduced space is diagonalizable.

Most important to astronomy are stable periodic solutions which means that there is some chance that such periodic solutions might actually be seen in some stellar system.

Conjecture (Linear Stability)

For every $(\theta, \mu) \in \Omega$ in the main theorem, if θ is commeasurable with π, there is a linear stable periodic solution. A minimizer who has smaller action is more likely stable

Proof??? Need to develop some new methods such as index theory...

Outline of the Proof

The proof of the theorem consists of several lemmas.

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Lemma 1 (Existence)

For $\theta \in (0, 2\pi)\backslash\{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$, $\tilde{\mathcal{A}}(\vec{a}) \to +\infty$ if $|\vec{a}| \to +\infty$. There exists a minimizer $\vec{a}_0 = (a_{10}, a_{20}, \cdots, a_{60}) \in \Gamma$ of $\tilde{\mathcal{A}}(\vec{a})$ over the space Γ and the corresponding path $q^*(t) = q^*(t, \vec{a}_0) \in S(\vec{a}_0)$.

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Lemma 2 (Noncollision)

For $\theta \in (0, 2\pi) \backslash \{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$, let $\vec{a}_0$ be a minimizer of $\tilde{A}(\vec{a})$ over the space Γ and the corresponding path $q^*(t) \in S(\vec{a}_0)$. Then q^* satisfying SPBC is a classical collision-free solution of Newton's equation in the whole interval $[0, T]$.

Lemma 3 (Extension)

For any minimizer $\vec{a} \in \Gamma$ of $\tilde{\mathcal{A}}$ over the space Γ, the corresponding minimizing path $q^*(t)$ on $[0, T]$ connecting $q(0)$ and $q(T)$ can be extended to a classical solution $q(t) = (q_1(t),\, q_2(t),\, q_3(t),\, q_4(t))$ of the Newton's equation by the reflection $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, the permutation σ and the rotation $R(\theta)$ as follows: $q(t) = q^*(t)$ on $[0, T]$,

$$q(t) = (q_3^*(2T-t), q_2^*(2T-t), q_1^*(2T-t), q_4^*(2T-t))B \quad \text{on} \quad (T, 2T],$$

and

$$q(t) = \sigma^k(q(t-2kT))R(-2k\theta) \text{ for } t \in (2kT, (2k+2)T] \text{ and } k \in \mathbf{Z}^+,$$

where $\sigma = [3, 4, 1, 2]$ is a permutation such that $\sigma(q(t-2T)) = (q_3(t-2T), q_4(t-2T), q_1(t-2T), q_2(t-2T))$.

Lemma 3 (Extension) Proof 1:

Because $q^*(t)$ is a classic solution of Newton's equation on $[0, T]$, it is easy to check that $q(t)$ is a classical solution in each interval $((n-1)T, nT)$ for any given positive integer n. To prove $q(t)$ is a classical solution for all real t, we need to prove that $q(t)$ is connected very well at $t = nT$ for any integer n, i.e.
$\lim_{t\to(nT)^-} q(t) = \lim_{t\to(nT)^+} q(t)$ and
$\lim_{t\to(nT)^-} \dot{q}(t) = \lim_{t\to(nT)^+} \dot{q}(t)$.
By the structure of the extension equation, we only need prove it for $n = 1$ and $n = 2$.
By the SPBC, $q(t)$ fits well at the end point. As for velocity, it is equivalent to, at $t = T$

$$\dot{q}_{11}(T) = \dot{q}_{31}(T), \dot{q}_{12}(T) = -\dot{q}_{32}(T), \dot{q}_{22}(T) = \dot{q}_{42}(T) = 0,$$

and at $t = 2T$,

$$\begin{aligned}
\dot{q}_1(0) &= (\dot{q}_{11}(0), -\dot{q}_{12}(0))R(2\theta), & \dot{q}_2(0) &= (\dot{q}_{41}(0), -\dot{q}_{42}(0))R(2\theta), \\
\dot{q}_3(0) &= (\dot{q}_{31}(0), -\dot{q}_{32}(0))R(2\theta), & \dot{q}_4(0) &= (\dot{q}_{21}(0), -\dot{q}_{22}(0))R(2\theta).
\end{aligned}$$

Lemma 3 (Extension) Proof 2:

Consider an admissible variation $\xi \in \mathcal{P}(\mathbf{A}, \mathbf{B})$ with $\xi(0) \in \mathbf{A}$ and $\xi(T) \in \mathbf{B}$, then the first variation $\delta_\xi \mathcal{A}(q)$ is computed as:

$$\delta_\xi \mathcal{A}(q) = \lim_{\delta \to 0} \frac{\mathcal{A}(q + \delta\xi) - \mathcal{A}(q)}{\delta}$$

$$= \sum_{i=1}^{4} m_i < \dot{q}_i, \xi_i >\Big|_{t=0}^{t=T} + \int_0^T < -m_i \ddot{q}_i + \frac{\partial}{\partial q_i}(U(q(t))), \xi_i > dt.$$

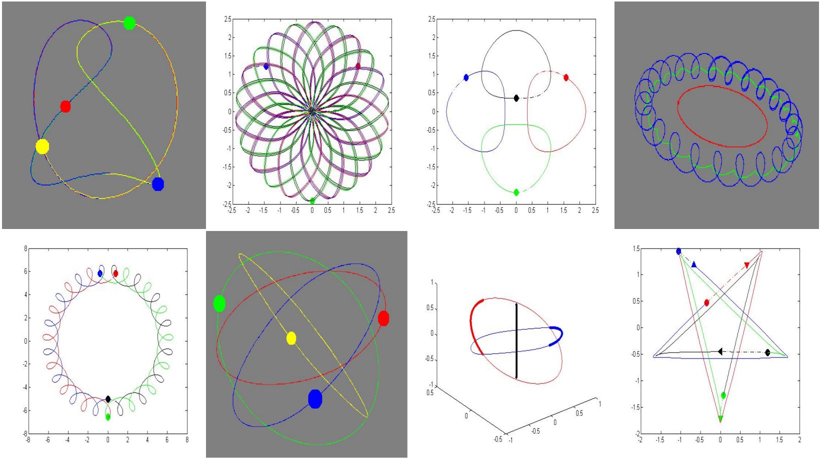
Because the first variation $\delta_\xi \mathcal{A}(q)$ is zero for any ξ, and q satisfies Newton's equation, we have

$$\delta_\xi \mathcal{A}(q) = \sum_{i=1}^{4} (m_i < \dot{q}_i(T), \xi_i(T) >) - \sum_{i=1}^{4} (m_i < \dot{q}_i(0), \xi_i(0) >) = 0.$$

Lemma 3 (Extension) Proof 3:

By choosing appropriate admissible variation $\xi \in \mathcal{P}(\mathbf{A}, \mathbf{B})$, we are able to prove the extension $q(t)$ connects very well at $t = nT$. More details... (see our paper).

Some interesting orbits are discovered by the Variational Method with SPBC. Simulations by giving initial positions and velocities.



If you are interested in the simulations or you would like to run simulations for your own initial data, you can visit my homepage at
http://sest.vsu.edu/~zxie/.
Another online simulation on our paper is at
http://www.princeton.edu/~rvdb/WebGL/OuyangXie.html by Professor Robert Vanderbei at Princeton University.

Thank you very much for your attention!