On the construction of normal mixed difference matrices *

Shanqi Pang[1](email:pangshanqi@263.net), Yingshan Zhang[2]

1. Department of Mathematics, Henan Normal University, Xinxiang, 453007, P. R. China

2. Department of Statistics, East China Normal University, Shanghai, 200062, P. R. China

Abstract: By exploring the relationship between difference matrices and orthogonal decomposition of projection matrices, this paper presents a general method for constructing smaller normal mixed difference matrices. Moreover, given a normal mixed difference matrix of order $r+1$ and a difference matrix of order r, this paper also presents a general method for constructing a lager normal mixed difference matrix of order $r(r+1)$. Furthermore, if the difference matrix of order r does not exist but an orthogonal array of runs r^2 exists, we can also construct a lager normal mixed difference matrix from the normal mixed difference matrix of order $r+1$ and the orthogonal array of runs r^2. Furthermore, if the difference matrix of order r does not exist but an orthogonal array of runs r^2 exists, we can also construct a lager normal mixed difference matrix from the normal mixed difference matrix of order $r+1$ and the orthogonal array of runs r^2. As an application of these methods, some normal mixed difference matrices are constructed.

Key words: Normal mixed difference matrix, Kronecker sum, orthogonal array (OA), projection matrix.

MSC: 62K15, 05B15.

1. Introduction

A $\lambda s \times c$ array $D(\lambda s, c; s)$ with entries from a Galois field of order s, $GF(s)$, is a difference scheme (difference matrix) if for all i and j with $1 \le i \le c$, $1 \le j \le c$, $i \ne j$, the vector difference between the ith and jth column contains every elements of $GF(s)$ λ times. Difference matrix was first defined by Bose and Bush (1952), and it is a simple but powerful tool for the construction of orthogonal arrays of strength 2 (Hedayat et al. 1999). Mixed difference matrices have also been used for construction of orthogonal arrays (see Wang, 1996, Wang and Wu 1991, Wu et al. 1992, Pang et al. 2004a, Seun and Kuhfeld, 2005, etc). Normal mixed difference matrices have also been introduced and used for construction of mixed-level orthogonal arrays by Pang et al. (2004b). However, it has not received much attention to construct mixed difference matrices, especially normal mixed difference matrix. By exploring the relationship between

*Corresponding author. Department of Mathematics, Henan Normal University, Xinxiang, 453007, China. E-mail: pangshanqi@263.net

difference matrices and orthogonal decomposition of projection matrices, this paper presents a general method for constructing smaller normal mixed difference matrices. Moreover, given a normal mixed difference matrix of order $r+1$ and a difference matrix of order r, this paper also presents a general method for constructing lager normal mixed difference matrix of order $r(r+1)$. Furthermore, if the difference matrix of order r does not exist but an orthogonal array of runs r^2 exists, we can also construct a lager normal mixed difference matrix from the normal mixed difference matrix of order $r+1$ and the orthogonal array of runs r^2. As applications of these methods, some normal mixed difference matrices are constructed. Some symbols can be referenced to those by Zhang et al. (2001).

2. Basic concepts and main theorems

The following definitions, notations and results are needed in the sequel.

Definition 1. Let $L_q = L_q(s_1 \cdots s_m) = (C_1, \ldots, C_m)$ be an orthogonal array where C_l is a vector with entries from an additive group G_l of order s_l for any l $(l = 1, \ldots, m)$. The array L_q is said to be **normal** if the set $G = \{A_1, \ldots, A_q : A_i$ is the i-th row of $L_q\}$ constitutes an additive subgroup of order q, where $G \subset G_1 \times \cdots \times G_m := \{(x_1, \ldots, x_m); x_l \in G_l,\ l = 1, 2, \ldots, m,\}$ with the usual addition, i.e., for any $x, y \in G, x = (x_1, x_2, \ldots, x_m), y = (y_1, y_2, \ldots, y_m)$, we have

$$x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_m + y_m).$$

Proposition 1 There is a map $\phi(A_i) = i - 1$ which is a group isomorphism between G and $G_0 = \{0, 1, \ldots, q-1\}$. And the map

$$\phi_t(i-1) = a_{it} \tag{1}$$

is a group homomorphism from $G_0 = \{0, 1, \ldots, q-1\}$ to G_t for $t = 1, 2, \ldots, m$.

Definition 2. Suppose that the array L_q is a normal orthogonal array, and that the maps $\phi_1, \ldots, \phi_m$ are defined in (1). Let $D_0 = D(n, k_0; q)$ be a difference matrix based on G_0, and let D_l be a difference matrix based on G_l, $l = 1, 2, \ldots, m$. If the matrices

$$[\phi_1(D_0), D_1], [\phi_2(D_0), D_2], \ldots, [\phi_m(D_0), D_m]$$

are difference matrices based on $G_1, G_2, \ldots, G_m$, respectively, then the matrix

$$[D_0, D_1, D_2, \ldots, D_m]$$

is called a normal mixed difference matrix.

Definitions 1 and 2 can be found in Pang et al (2004b).

Lemma 1. Suppose that $D(\lambda p, m; p)$ is a $\lambda p \times m$ matrix with entries from a Galois field of order p, $GF(p)$, and that γ is a column of orthogonal array $L_n(p^s)$. If $D(\lambda p, m_1; p) \oplus \gamma$ is also an orthogonal array, then $D(\lambda p, m; p)$ is a difference matrix.

Proof. We suppose without lose of generality that $\alpha = (a_1, \ldots, a_{\lambda p})^T$ and $\beta = (b_1, \ldots, b_{\lambda p})^T$ are any two columns in $D(\lambda p, m; p)$, and that $b_1 - a_1 = x_1, b_2 - a_2 = x_2, \ldots, b_{\lambda p} - a_{\lambda p} = x_{\lambda p}$, and that $\gamma = (c_1, c_2, \ldots, c_n)^T$ is a column in the array $L_n(p^s)$. Then we have

$$\alpha \oplus \gamma = (a_1 + c_1, \ldots, a_1 + c_n, a_2 + c_1, \ldots, a_2 + c_n, \ldots, a_{\lambda p} + c_1, \ldots, a_{\lambda p} + c_n)^T,$$

$$\beta \oplus \gamma = (a_1 + c_1 + x_1, \ldots, a_1 + c_n + x_1, a_2 + c_1 + x_2, \ldots, a_2 + c_n + x_2, \ldots, a_{\lambda p} + c_1, \ldots, a_{\lambda p} + c_n + x_{\lambda p})^T.$$

Since γ is a column of orthogonal array $L_n(p^s)$, $c_1, c_2, \ldots, c_n$ contain every element of $GF(p)$ n/p times. So $a_i + c_1, a_i + c_2, \ldots, a_i + c_n$ also contain all elements of $GF(p)$, each of which occurs n/p times. Therefore, by some permutations,

$$(a_i + c_1, a_i + c_1 + x_i), (a_i + c_2, a_i + c_2 + x_i), \ldots, (a_i + c_n, a_i + c_n + x_i)$$

can be written as

$$\underbrace{(0, x_i), \ldots, (0, x_i)}_{n/p}, \underbrace{(1, 1 + x_i), \ldots, (1, 1 + x_i)}_{n/p}, \cdots, \underbrace{(p-1, p-1+x_i), \ldots, (p-1, p-1+x_i)}_{n/p},$$

where $x_i \in \{x_1, \ldots, x_{\lambda p}\}$.

We consider these pairs just like $\underbrace{(0, x_i), \ldots, (0, x_i)}_{n/p}$ in two columns $\alpha \oplus \gamma$ and $\beta \oplus \gamma$. Since these two columns are orthogonal, each element of $GF(p)$ occurs in $\{x_1, \ldots, x_{\lambda p}\}$ with the same frequency. Therefore, $D(\lambda p, m; p)$ is a difference matrix. This completes the proof.

Lemma 2. If $D(m, m; p)$ is a difference matrix, then both $(p) \oplus D(m, m; p)$ and $D(m, m; p) \oplus (p)$ are OAs, and their matrix images satisfy $m((p) \oplus D(m, m; p)) \leq \tau_p \otimes I_m$ and $m(D(m, m; p) \oplus (p)) \leq I_m \otimes \tau_p$, respectively.

Proof. From Wang and Wu (1991), $L = [1_p \otimes (m), (p) \oplus D(m, m; p)]$ is an OA. From Zhang et al. (2001), we have $m(L) \leq \tau_{mp}$ and $m(L) = P_p \otimes \tau_m + m((p) \oplus D(m, m; p))$. Since $\tau_{mp} = P_p \otimes \tau_m + \tau_p \otimes I_m$, it follows that $m((p) \oplus D(m, m; p)) \leq \tau_p \otimes I_m$. Similarly, we can prove the remaining part.

Now we state the following theorem.

Theorem 1. Suppose that $L_p(s_1 \cdots s_n) = (c_1, \ldots, c_n)$ is a normal orthogonal array, and that $L = [D(m, k; p) \oplus (p), D(m, k_1; s_1) \oplus c_1, \ldots, D(m, k_n; s_n) \oplus c_n]$ is also an orthogonal array. Then

$$D = [D(m, k; p), D(m, k_1; s_1), \ldots, D(m, k_n; s_n)]$$

is a normal mixed difference matrix.

Proof. Since $L_p(s_1 \cdots s_n) = (c_1 \cdots c_n)$ is a normal orthogonal array, we have a group homomorphism ϕ_l from $G_0 = \{0, 1, \ldots, p-1\}$ to G_l. Set $c_0 = (p) = (0, 1, \ldots, p-1)^T$, then $\phi_l(c_0) = c_l$. From Proposition 1 and the expansive replacement method in Hedayat et al. (1999), if the levels $0, 1, \ldots, p-1$ in $[D(m, k; p) \oplus (p), D(m_l, k_l; s_l) \oplus c_l]$ are replaced with $A_1, \ldots, A_p$, respectively, where A_i is the ith row of L_p, we can obtain a mixed orthogonal array with the levels $s_1, \cdots, s_n$. Pick out all the s_l-level columns, we can get an s_l-level orthogonal array, which can be written as

$$[\phi_l(D(m, k; p)), D(m, k_l; s_l)] \oplus c_l.$$

It follows from Lemma 1 that $[\phi_l(D(m, k; p)), D(m, k_l; s_l)]$ is a difference matrix, $l = 1, \ldots, n$. We have $D = [D(m, k; p),\ D(m, k_1; s_1), \ldots, D(m, k_n; s_n)]$ is a normal mixed difference matrix.

3. Construction method for smaller normal mixed difference matrices.

The following method is from orthogonal decomposition of projection matrices. We illustrate its applications with examples.

Example 1. Construction of normal mixed difference matrix $D = [D_0(8, 5; 4), D_1(8, 1; 2), D_2(8, 1; 2),\ D_3(8, 1; 2)]$ and $D = [D_0(8, 6; 4), D_1(8, 2; 2), D_2(8, 2; 2),\ D_3(8, 2; 2)]$

Orthogonally decompose $\tau_4 \otimes I_2 \otimes \tau_4$ as follows:

$$\begin{aligned}
& \tau_4 \otimes I_2 \otimes \tau_4 \\
= \ & (\tau_2 \otimes P_2 \otimes P_2 \otimes \tau_2 \otimes P_2 + P_2 \otimes \tau_2 \otimes P_2 \otimes P_2 \otimes \tau_2 + \tau_2 \otimes \tau_2 \otimes P_2 \otimes \tau_2 \otimes \tau_2) \\
+ \ & (\tau_2 \otimes \tau_2 \otimes P_2 \otimes \tau_2 \otimes P_2 + P_2 \otimes \tau_2 \otimes \tau_2 \otimes P_2 \otimes \tau_2 + \tau_2 \otimes P_2 \otimes \tau_2 \otimes \tau_2 \otimes \tau_2) \\
+ \ & (P_2 \otimes \tau_2 \otimes \tau_2 \otimes \tau_2 \otimes P_2 + \tau_2 \otimes \tau_2 \otimes \tau_2 \otimes P_2 \otimes \tau_2 + \tau_2 \otimes P_2 \otimes P_2 \otimes \tau_2 \otimes \tau_2) \\
+ \ & (\tau_2 \otimes \tau_2 \otimes \tau_2 \otimes \tau_2 \otimes P_2 + \tau_2 \otimes P_2 \otimes \tau_2 \otimes P_2 \otimes \tau_2 + P_2 \otimes \tau_2 \otimes P_2 \otimes \tau_2 \otimes \tau_2) \\
+ \ & (\tau_2 \otimes P_2 \otimes \tau_2 \otimes \tau_2 \otimes P_2 + \tau_2 \otimes \tau_2 \otimes P_2 \otimes P_2 \otimes \tau_2 + P_2 \otimes \tau_2 \otimes \tau_2 \otimes \tau_2 \otimes \tau_2) \\
+ \ & P_2 \otimes \tau_2 \otimes P_2 \otimes \tau_2 \otimes P_2 + \tau_2 \otimes P_2 \otimes P_2 \otimes P_2 \otimes \tau_2 + \tau_2 \otimes \tau_2 \otimes \tau_2 \otimes \tau_2 \otimes \tau_2.
\end{aligned}$$

By using the method in Zhang et al. (1999), we can construct an orthogonal array $L_{32}(2^{18})$ as follows:

$$\begin{aligned}
& L_{32}(2^{18}) \\
= \ & [((2) \oplus 0_2 \oplus 0_2 \oplus (2) \oplus 0_2, 0_2 \oplus (2) \oplus 0_2 \oplus 0_2 \oplus (2), (2) \oplus (2) \oplus 0_2 \oplus (2) \oplus (2)), \\
& ((2) \oplus (2) \oplus 0_2 \oplus (2) \oplus 0_2, 0_2 \oplus (2) \oplus (2) \oplus 0_2 \oplus (2), (2) \oplus 0_2 \oplus (2) \oplus (2) \oplus (2)),
\end{aligned}$$

$(0_2 \oplus (2) \oplus (2) \oplus (2) \oplus 0_2, (2) \oplus (2) \oplus (2) \oplus 0_2 \oplus (2), (2) \oplus 0_2 \oplus 0_2 \oplus (2) \oplus (2)),$

$((2) \oplus (2) \oplus (2) \oplus (2) \oplus 0_2, (2) \oplus 0_2 \oplus (2) \oplus 0_2 \oplus (2), 0_2 \oplus (2) \oplus 0_2 \oplus (2) \oplus (2)),$

$((2) \oplus 0_2 \oplus (2) \oplus (2) \oplus 0_2, (2) \oplus (2) \oplus 0_2 \oplus 0_2 \oplus (2), 0_2 \oplus (2) \oplus (2) \oplus (2) \oplus (2)),$

$0_2 \oplus (2) \oplus 0_2 \oplus (2) \oplus 0_2, (2) \oplus 0_2 \oplus 0_2 \oplus 0_2 \oplus (2), (2) \oplus (2) \oplus (2) \oplus (2) \oplus (2)]$

By using the generalized Hadamard product in Zhang et al. (2001), i.e.,$((2)\oplus 0_2)\square(0_2\oplus(2)) = (4)$ and its MI satisfies $m(((2) \oplus 0_2)\square(0_2 \oplus (2))) = \tau_2 \otimes P_2 + P_2 \otimes \tau_2 + \tau_2 \otimes \tau_2 = \tau_4$, we can construct an orthogonal array

$$L_{32}(4^5 2^3) = [L_{32}(4^5), 0_2 \oplus (2) \oplus 0_2 \oplus (2) \oplus 0_2, (2) \oplus 0_2 \oplus 0_2 \oplus 0_2 \oplus (2), (2) \oplus (2) \oplus (2) \oplus (2) \oplus (2)]$$

where $L_{32}(4^5) = D(8, 5; 4) \oplus (4)$ and

$$D(8,5;4) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 3 & 2 \\ 1 & 3 & 3 & 2 & 1 \\ 1 & 2 & 0 & 1 & 3 \\ 2 & 2 & 1 & 3 & 3 \\ 2 & 3 & 2 & 0 & 1 \\ 3 & 1 & 2 & 1 & 2 \\ 3 & 0 & 1 & 2 & 0 \end{pmatrix}.$$

From orthogonal decomposition of $\tau_4 \otimes I_2 \otimes \tau_4$ and the construction of 4-level columns, we can easily construct a normal mixed difference matrix $D = [D_0(8, 5; 4), D_1(8, 1; 2), D_2(8, 1; 2), D_3(8, 1; 2)]$ (see table 1).

Table 1. The mixed difference matrix $D = [D_0(8, 5; 4), D_1(8, 1; 2), D_2(8, 1; 2),\ D_3(8, 1; 2)]$

$D_0(8,5;4)$	$D_1(8,1;2)$	$D_2(8,1;2)$	$D_3(8,1;2)$
0 0 0 0 0	0	0	0
0 1 3 3 2	0	0	1
1 3 3 2 1	1	0	1
1 2 0 1 3	1	0	0
2 2 1 3 3	0	1	1
2 3 2 0 1	0	1	0
3 1 2 1 2	1	1	0
3 0 1 2 0	1	1	1

Furthermore, since $\tau_4 \otimes I_2 \otimes \tau_4$, $P_8 \otimes \tau_4$ and $P_4 \otimes \tau_2 \otimes \tau_4$ are pairwise orthogonal, we can construct orthogonal array $L_{32}(4^6 2^6) = [L_{32}(4^5), 0_8 \oplus (4), (0_2 \oplus (2) \oplus 0_2, 0_2 \oplus 0_2 \oplus (2)) \oplus (2) \oplus 0_2, ((2) \oplus 0_2 \oplus 0_2, 0_2 \oplus 0_2 \oplus (2)) \oplus 0_2 \oplus (2), ((2) \oplus (2) \oplus (2), 0_2 \oplus 0_2 \oplus (2)) \oplus (2) \oplus (2)]$. Then we can obtain a normal mixed difference matrix $D = [D_0(8, 6; 4), D_1(8, 2; 2), D_2(8, 2; 2),\ D_3(8, 2; 2)]$ (see table 2).

Table 2. The mixed difference matrix $D = [D_0(8,6;4), D_1(8,2;2), D_2(8,2;2),\ D_3(8,2;2)]$

$D_0(8,6;4)$	$D_1(8,2;2)$	$D_2(8,2;2)$	$D_3(8,2;2)$
0 0 0 0 0 0	00	00	00
0 0 1 3 3 2	01	01	11
0 1 3 3 2 1	10	00	10
0 1 2 0 1 3	11	01	01
0 2 2 1 3 3	00	10	10
0 2 3 2 0 1	01	11	01
0 3 1 2 1 2	10	10	00
0 3 0 1 2 0	11	11	11

Example 2. Construction of normal mixed difference matrix $[D_0(24,20;4), D_1(24,4;2), D_2(24,4;2),\ D_3(24,4;2)]$

By using the construction method in Zhang et al. (2002), we can obtain a difference matrix $D(12,12;4)$ as follows:

$$D(12,12;4) = \begin{pmatrix} 0&0&0&1&1&1&2&2&2&3&3&3\\ 0&0&0&2&2&2&3&3&3&1&1&1\\ 0&0&0&3&3&3&1&1&1&2&2&2\\ 1&2&3&1&2&3&1&2&3&1&2&3\\ 1&2&3&2&3&1&2&3&1&2&3&1\\ 1&2&3&3&1&2&3&1&2&3&1&2\\ 2&3&1&1&2&3&2&3&1&3&1&2\\ 2&3&1&2&3&1&3&1&2&1&2&3\\ 2&3&1&3&1&2&1&2&3&2&3&1\\ 3&1&2&1&2&3&3&1&2&2&3&1\\ 3&1&2&2&3&1&1&2&3&3&1&2\\ 3&1&2&3&1&2&2&3&1&1&2&3 \end{pmatrix}.$$

From Lemma 2, we have $D(12,12;4) \oplus (4)$ is an orthogonal array and its matrix image satisfies that $m(D(12,12;4) \oplus (4)) = I_{12} \otimes \tau_4$. On the other hand, $D(12,12;4) \oplus (4)$ can be written as

$$D(12,12;4) \oplus (4) =$$

$$[D(4,3;4) \oplus 0_3 \oplus (4), T_1(D(4,3;4) \oplus 0_3 \oplus (4)), T_2(D(4,3;4) \oplus 0_3 \oplus (4)), T_3(D(4,3;4) \oplus 0_3 \oplus (4))],$$

where the difference matrix $D(4,3;4)$, and the operation $\oplus$, and the permutation matrices T_1, T_2 and T_3 are as follows:

$$D(4,3;4) = \begin{pmatrix} 0&0&0\\ 1&2&3\\ 2&3&1\\ 3&1&2 \end{pmatrix},$$

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

$$T_1 = diag(\sigma(1), \sigma(2), \sigma(3), K(3,3) \otimes I_4),$$

$$T_2 = diag(\sigma(2), \sigma(3), \sigma(1), (diag(I_3, N_3, N_3^2)K(3,3)) \otimes I_4),$$
$$T_3 = diag(\sigma(3), \sigma(1), \sigma(2), (diag(I_3, N_3^2, N_3)K(3,3)) \otimes I_4),$$

where $\sigma(1)$, $\sigma(2)$ and $\sigma(3)$ are the following permutation matrices: $\sigma(1) = I_2 \otimes N_2$, $\sigma(2) = N_2 \otimes I_2$ and $\sigma(3) = N_2 \otimes N_2$.

Then we have

$$I_{12} \otimes \tau_4 = \sum_{i=0}^{3} T_i(\tau_4 \otimes P_3 \otimes \tau_4)T_i^T,$$

where $T_0 = I_{48}$. It follows that

$$\begin{aligned} & I_{24} \otimes \tau_4 \\ = \ & (I_{12} \otimes K(2,4))(I_{12} \otimes \tau_4 \otimes I_2)(I_{12} \otimes K(2,4)^T) \\ = \ & (I_{12} \otimes K(2,4))[(\sum_{i=0}^{3} T_i(\tau_4 \otimes P_3 \otimes \tau_4)T_i^T) \otimes I_2](I_{12} \otimes K(2,4)^T) \\ = \ & \sum_{i=0}^{3} M_i(P_3 \otimes \tau_4 \otimes I_2 \otimes \tau_4)M_i^T, \end{aligned}$$

where $M_0 = K(4,3) \otimes I_8$, $M_i = (I_{12} \otimes K(2,4))(T_i \otimes I_2)(I_{12} \otimes K(4,2))(K(4,3) \otimes I_8)$. The decomposition is orthogonal because of the orthogonality in each step.

From Example 1, we can find an orthogonal array $L_{32}(4^5 2^3) = D(8,5;4) \oplus (4), 0_2 \oplus (2) \oplus 0_2 \oplus (2) \oplus 0_2, (2) \oplus 0_2 \oplus 0_2 \oplus 0_2 \oplus (2), (2) \oplus (2) \oplus (2) \oplus (2) \oplus (2)]$ such that its MI is less than or equal to $\tau_4 \otimes I_2 \otimes \tau_4$. From the orthogonal decomposition of $I_{24} \otimes \tau_4$, we can construct an orthogonal array

$$L_{96}(4^{20}2^{12}) = [M_0(0_3 \oplus L_{32}(2^3 4^5)), M_1(0_3 \oplus L_{32}(2^3 4^5)), M_2(0_3 \oplus L_{32}(2^3 4^5)), M_3(0_3 \oplus L_{32}(2^3 4^5))].$$

Now we consider a 4-level subarray L^1 of $L_{96}(4^{20}2^{12})$.

$$L^1 = [M_0(0_3 \oplus L_{32}(4^5)), M_1(0_3 \oplus L_{32}(4^5)), M_2(0_3 \oplus L_{32}(4^5)), M_3(0_3 \oplus L_{32}(4^5))].$$

And

$$\begin{aligned} & M_0(0_3 \oplus L_{32}(4^5)) \\ = \ & (K(4,3) \otimes I_8)(0_3 \oplus D(8,5;4) \oplus (4)) \\ = \ & [(K(4,3) \otimes I_2)(0_3 \oplus D(8,5;4))] \oplus (4). \end{aligned}$$

Set $D_{24} = (K(4,3) \otimes I_2)(0_3 \oplus D(8,5;4))$, we have

$$\begin{aligned} & M_1(0_3 \oplus L_{32}(4^5)) \\ = \ & (I_{12} \otimes K(2,4))(T_1 \otimes I_2)(I_{12} \otimes K(4,2))(K(4,3) \otimes I_8)(0_3 \oplus D(8,5;4) \oplus (4)) \\ = \ & diag(I_2 \otimes \sigma(1), I_2 \otimes \sigma(2), I_2 \otimes \sigma(3), K(3,3) \otimes I_8)[D_{24} \oplus (4)] \\ = \ & D'_{24} \oplus (4), \end{aligned}$$

where

$$D'_{24} = \begin{pmatrix} 1 \oplus D_1(2,5,4) \\ 2 \oplus D_2(2,5,4) \\ 3 \oplus D_3(2,5,4) \\ (K(3,3) \otimes I_2)D_4(18,5,4) \end{pmatrix}$$

and $D_1(2,5,4)$, $D_2(2,5,4)$, $D_3(2,5,4)$ and $D_4(18,5,4)$ are respectively the first two rows, the 3th and 4th rows, the 5th and 6th rows and the last 18 rows of the matrix D_{24}.

Table 3. The normal mixed difference matrix $D = [D_0(24,20;4), D_1(24,4;2), D_2(24,4;2), D_3(24,4;2)]$

$D_0(24,20;4)$	$D_1(24,4;2)$	$D_2(24,4;2)$	$D_3(24,4;2)$
0 0 0 0 0 1 1 1 1 1 2 2 2 2 2 3 3 3 3 3	0 0 1 1	0 1 0 1	0 1 1 0
0 1 3 3 2 1 0 2 2 3 2 3 1 1 0 3 2 0 0 1	0 0 1 1	0 1 0 1	1 0 0 1
0 0 0 0 0 2 2 2 2 2 3 3 3 3 3 1 1 1 1 1	0 1 1 0	0 0 1 1	0 1 0 1
0 1 3 3 2 2 3 1 1 0 3 2 0 0 1 1 0 2 2 3	0 1 1 0	0 0 1 1	1 0 1 0
0 0 0 0 0 3 3 3 3 3 1 1 1 1 1 2 2 2 2 2	0 1 0 1	0 1 1 0	0 0 1 1
0 1 3 3 2 3 2 0 0 1 1 0 2 2 3 2 3 1 1 0	0 1 0 1	0 1 1 0	1 1 0 0
0 2 2 3 0 0 2 2 3 0 0 2 2 3 0 0 2 2 3 0	1 1 1 1	1 1 1 1	0 0 0 0
0 3 1 0 2 0 3 1 0 2 0 3 1 0 2 0 3 1 0 2	1 1 1 1	1 1 1 1	1 1 1 1
0 2 2 3 0 3 3 0 2 2 3 3 0 2 2 3 3 0 2 2	1 0 0 0	1 0 0 0	0 0 0 0
0 3 1 0 2 3 2 3 1 0 3 2 3 1 0 3 2 3 1 0	1 0 0 0	1 0 0 0	1 1 1 1
0 2 2 3 0 2 0 3 0 3 2 0 3 0 3 2 0 3 0 3	1 1 1 1	1 0 0 0	0 1 1 1
0 3 1 0 2 2 1 0 3 1 2 1 0 3 1 2 1 0 3 1	1 1 1 1	1 0 0 0	1 0 0 0
0 0 3 1 1 3 1 1 0 3 0 0 3 1 1 1 3 0 3 0	1 0 1 0	1 0 1 1	0 0 0 1
0 1 0 2 3 3 0 2 3 1 0 1 0 2 3 1 2 3 0 2	1 0 1 0	1 0 1 1	1 1 1 0
0 0 3 1 1 0 0 3 1 1 1 3 0 3 0 3 1 1 0 3	1 1 0 0	1 1 1 0	0 0 1 0
0 1 0 2 3 0 1 0 2 3 1 2 3 0 2 3 0 2 3 1	1 1 0 0	1 1 1 0	1 1 0 1
0 0 3 1 1 1 3 0 3 0 3 1 1 0 3 0 0 3 1 1	1 0 0 1	1 1 0 1	0 1 0 0
0 1 0 2 3 1 2 3 0 2 3 0 2 3 1 0 1 0 2 3	1 0 0 1	1 1 0 1	1 0 1 1
0 2 1 2 1 2 0 0 1 2 0 2 1 2 1 1 1 2 0 0	0 0 0 1	0 1 0 0	0 1 0 1
0 3 2 1 3 2 1 3 2 0 0 3 2 1 3 1 0 1 3 2	0 0 0 1	0 1 0 0	1 0 1 0
0 2 1 2 1 1 1 2 0 0 2 0 0 1 2 0 2 1 2 1	0 1 0 0	0 0 1 0	0 1 1 0
0 3 2 1 3 1 0 1 3 2 2 1 3 2 0 0 3 2 1 3	0 1 0 0	0 0 1 0	1 0 0 1
0 2 1 2 1 0 2 1 2 1 1 1 2 0 0 2 0 0 1 2	0 0 1 0	0 0 0 1	0 0 1 1
0 3 2 1 3 0 3 2 1 3 1 0 1 3 2 2 1 3 2 0	0 0 1 0	0 0 0 1	1 1 0 0

Hence L^1 can be written as $L = D(24,20;4) \oplus (4) = [D_{24}, D'_{24}, D''_{24}, D'''_{24}] \oplus (4)$, where

$$D''_{24} = \begin{pmatrix} 2 \oplus D_1(2,5,4) \\ 3 \oplus D_2(2,5,4) \\ 1 \oplus D_3(2,5,4) \\ ((diag(I_3, N_3, N_3^2)K(3,3)) \otimes I_2)D_4(18,5,4) \end{pmatrix}$$

and

$$D_{24}''' = \begin{pmatrix} 3 \oplus D_1(2,5,4) \\ 1 \oplus D_2(2,5,4) \\ 2 \oplus D_3(2,5,4) \\ ((diag(I_3, N_3^2, N_3)K(3,3)) \otimes I_2)D_4(18,5,4) \end{pmatrix}.$$

By Theorem 1, $D(24, 20; 4)$ is a difference matrix, which is listed in Table 3.

And through similar computing, $L_{96}(4^{20}2^{12})$ can be written as

$$[D_1(24,4;2) \oplus ((2) \oplus 0_2), D_2(24,4;2) \oplus (0_2 \oplus (2)), D_3(24,4;2) \oplus ((2) \oplus (2)), D(24,20;4) \oplus (4)].$$

From orthogonal decompositions of $I_{24} \otimes \tau_4$, $\tau_4 \otimes I_2 \otimes \tau_4$ and Theorem 1, it is obvious that $D = [D_0(24, 20; 4),\ D_1(24, 4; 2), D_2(24, 4; 2), D_3(24, 4; 2)]$ is a normal mixed difference matrix (see Table 3).

Theorem 2 Suppose that $L_p(s_1 \cdots s_n) = (c_1, \ldots, c_n)$ is a normal orthogonal array, and that $D(m, k; p)$ is a difference matrix. If we partition $D(m, k; p)$ into $[D(m, k; p) = [D(m, k - r; p), D(m, r; p)]$, then $[D(m, k-r; p), \phi_1(D(m, r; p)), \ldots, \phi_n(D(m, r; p))]$ is a normal mixed difference matrix, where ϕ_i is as in (1).

Proof. It follows from that $\phi_i(D(m, k; p))$ is a s_i-level difference matrix.

4. Construction method for larger normal mixed difference matrices.

The following Lemma is due to Zhang et al.(2002).

Lemma 3 Suppose that both

$$D(r, r; r) = (d_{ij})_{r \times r} = (d_1, \ldots, d_r)$$

and

$$D(r+1, r+1; p) = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & (a_{ij})_{r \times r} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & a_1 & \cdots & a_r \end{pmatrix}$$

are difference matrices with entries from two additive groups $G_r = \{0, 1, \ldots, r-1\}$ and $G_p = \{0, 1, \ldots, p-1\}$, respectively. For any $d_{ij} \in G_r$, define a permutation matrix $\sigma(d_{ij})$ as follows

$$\sigma(d_{ij}) \cdot (r) = d_{ij} + (r). \tag{2}$$

Set $F = (\sigma(d_{ij})A)_{1 \le i \le r, 1 \le j \le r}$. Then the following array

$$D(r(r+1), r(r+1); p) = \begin{pmatrix} 0 & A \oplus 0_r^T \\ A \oplus 0_r & F \end{pmatrix}.$$

is a difference matrix.

Lemma 4 If $D = [D(m,k;p), D(m,k_1;s_1), \ldots, D(m,k_n;s_n)]$ is a normal mixed difference matrix based on a normal orthogonal array $L_p(s_1 \cdots s_n) = (c_1, \ldots, c_n)$, then D is still an normal mixed difference matrix after performing the following operations:
(1) adding an i $(i \in \{0,1,\ldots,p-1\}$ to any column of $D(m,k;p)$ or adding an i_j $i_j \in G_j$ to any column of $D(m,k_j;s_j)$.
(2) adding an i $(i \in \{0,1,\ldots,p-1\}$ to row of $D(m,k;p)$ and adding $\phi_j(i)$ to the same row of $D(m,k_j;s_j)$ for $j = 1,\ldots,n$.

Proof It is obvious from the definition of normal mixed difference matrix.

Theorem 3. Under the conditions of Lemma 3, suppose that $[D(r+1,k+1;p), D_1(r+1,k_1;s_1), \ldots, D_m(r+1,k_m;s_m)]$ is a normal mixed difference matrix based on a normal orthogonal array $L_p(s_1 \cdots s_m) = (c_1, \ldots, c_m)$ and that $D(r,r;r)$ is a difference matrix. Then we can obtain a normal mixed difference matrix

$$[D(r(r+1), k(k+1); p), D_1(r(r+1), k_1(r+k+1); s_1), \ldots, D_m(r(r+1), k_m(r+k+1); s_m)]$$

based on the normal orthogonal array $L_p(s_1 \cdots s_m)$.

Proof. From Lemma 4, we assume that

$$[D(r+1,k+1;p), D(r+1,k_t;s_t)] = \begin{pmatrix} 0 & 0 \\ 0 & H \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & (h_{ij})_{r\times r_t} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & h_1 & \cdots & h_{r_t} \end{pmatrix},$$

where $r_t = k + k_t$ and $H = (A_{r\times k}, B_{r\times k_t})$. And set $F = (\sigma(d_{ij})H)_{1\le i\le r, 1\le j\le r_t}$, where $\sigma(d_{ij})$ is as in (2). Then we construct a matrix K as follows

$$K = \begin{pmatrix} 0 & H \oplus 0_{r_t}^T \\ H \oplus 0_r & F \end{pmatrix}.$$

And in the K, set $K_0 = \begin{pmatrix} 0 \\ H \oplus 0_r \end{pmatrix}$ and $K_j = \begin{pmatrix} h_j \oplus 0_{r_t}^T \\ \sigma(d_{1j})H \\ \cdots \\ \sigma(d_{rj})H \end{pmatrix}$ for $j = 1,\ldots,r_t$. Moreover, if h_j is a p-level column i.e. a column of $A_{r\times k}$, we take K_j as follows

$$K_j = \begin{pmatrix} h_j \oplus 0_k^T, \phi_t(h_j) \oplus 0_{k_t}^T \\ \sigma(d_{1j})(A,B) \\ \cdots \\ \sigma(d_{rj})(A,B) \end{pmatrix},$$

and if h_j is a s_t-level column i.e. a column of $B_{r\times k_t}$, we take K_j as follows

$$K_j = \begin{pmatrix} h_j \oplus 0_{r_t}^T \\ \sigma(d_{1j})(\phi_t(A), B) \\ \cdots \\ \sigma(d_{rj})(\phi_t(A), B) \end{pmatrix}.$$

Through some column permutation, K can be written as

$$K = [D(r(r+1), k(k+1); p), D_t(r(r+1), k_t(r_t+k+1); s_t)].$$

It easily follows from Lemma 3 that all the p-level columns of K constitute a difference matrix $D(r(r+1), k(k+1); p)$.

Now we prove that $[\phi_t(D(r(r+1), k(k+1); p)), D_t(r(r+1), k_t(r_t+k+1); s_t)]$ is a difference matrix based on the group G_t.

Let

$$[\phi_t(D(r+1, k+1; p)), D_t(r+1, k_t; s_t)] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \phi_t(A) & B \end{pmatrix}.$$

By using $D(r, r; r)$ and Lemma 3, we can construct an s_t-level difference matrix

$$K_{\phi_t} = [D(r(r+1), k(k+1) + k_t(r_t+k+1); s_t)] = \begin{pmatrix} 0 & (\phi_t(A), B) \oplus 0_{r_t}^T \\ (\phi_t(A), B) \oplus 0_r & F_1 \end{pmatrix},$$

where $F_1 = (\sigma(d_{ij})(\phi_t(A), B))_{1 \le i \le r, 1 \le j \le r_t}$. On the other hand, through some column permutation, we just have that K_{ϕ_t} can be written as $K_{\phi_t} = [\phi_t(D(r(r+1), k(k+1); p)), D_t(r(r+1), k_t(r_t+k+1); s_t)]$. Hence, to take $t = 1, \ldots, m$, respectively, we can get a normal mixed difference matrix

$$[D(r(r+1), k(k+1); p), D_1(r(r+1), k_1(r_1+k+1); s_1), \ldots, D_m(r(r+1), k_m(r_m+k+1); s_m)]$$

based on the normal orthogonal array $L_p(s_1 \cdots s_m)$.

Example 3. Construction of normal mixed difference matrices $[D_0(132, 30; 6), D_1(132, 42; 3), D_2(132, 102; 2)]$ and $[D_0(132, 20; 6), D_1(132, 112; 3), D_2(132, 112; 2)]$.

By Lemma 4, we can make the normal mixed difference matrix $[D_0(12, 6; 6), D_1(12, 3; 3), D_2(12, 6; 2)]$ in Wang (1996) the following form

$$\begin{pmatrix} 0 & 0 \\ 0 & H \end{pmatrix}.$$

Then using the difference matrix $D(11, 11; 11)$ and Theorem 2, we can construct a normal mixed difference matrix $[D_0(132, 30; 6), D_1(132, 42; 3), D_2(132, 102; 2)]$.

Similarly by using $[D_0(12, 5; 6), D_1(12, 7; 3), D_2(12, 7; 2)]$ in Wang (1996), we can obtain a normal mixed difference matrix $[D_0(132, 20; 6), D_1(132, 112; 3), D_2(132, 112; 2)]$.

Remark In theorem 3, if $r(r+1)-1$ is a prime or prime power, then we can use the method again to construct a new normal mixed difference matrix. For example, by using difference matrix $D(131, 131; 131)$ and Theorem 3 again, we can construct a normal mixed difference matrix $[D_0(131 \times 132, 19 \times 20; 6), D_1(131 \times 132, 112 \times (131+20); 3), D_2(131 \times 132, 112 \times (131+20); 2)] = [D_0(17292, 380; 6), D_1(17292, 16912; 3), D_2(17292, 16912; 2)]$.

Example 4. Construction of normal mixed difference matrix $[D_0(23\times 24, 380; 4), D_1(23\times 24, 172; 2), D_2(23\times 24, 172; 2),\ D_3(23\times 24, 172; 2)]$

By Lemma 4, we can make the $[D_0(24, 20; 4), D_1(24, 4; 2), D_2(24,\ 4; 2),\ D_3(24, 4; 2)]$ in Example 2 the following form

$$\begin{pmatrix} 0 & 0 \\ 0 & H \end{pmatrix}.$$

Then using the difference matrix $D(23, 23; 23)$ and Theorem 2, we can construct a normal mixed difference matrix $[D_0(23\times 24, 380; 4), D_1(23\times 24, 172; 2), D_2(23\times 24, 172; 2),\ D_3(23\times 24, 172; 2)]$.

When difference matrix $D(r, r, , r)$ does not exist, we state the following theorem.

Theorem 4. Under the conditions of Lemma 3, suppose that $[D(r + 1, k + 1; p), D_1(r + 1, k_1; s_1), \ldots, D_m(r + 1, k_m; s_m)]$ is a normal mixed difference matrix based on a normal orthogonal array $L_p(s_1 \cdots s_m) = (c_1, \ldots, c_m)$ and that there exists an orthogonal array $L_{r^2}(r^{x+1}) = [(r) \otimes 1_r, Q_1(1_r \otimes (r)), \cdots, Q_x(1_r \otimes (r))]$ where $Q_j = diag(\sigma_{1j}, \cdots, \sigma_{rj})$ is a permutation matrix satisfying $Q_j((r) \otimes 1_r) = (r) \otimes 1_r$. Then we can obtain a normal mixed difference matrix

$$[D(r(r + 1), k(k + 1); p), D_1(r(r + 1), y_1; s_1), \ldots, D_m(r(r + 1), y_m; s_m)]$$

based on the normal orthogonal array $L_p(s_1 \cdots s_m)$, where

$$y_t = \begin{cases} k_t(2k + k_t + 1) & \text{if } x \geq k + k_t \\ k_t(k + 1) + (x - k)(k + k_t) & \text{if } k \leq x < k + k_t \end{cases}$$

If $x < k$, the above normal mixed difference matrix becomes

$$[D(r(r + 1), k(x + 1); p), D_1(r(r + 1), k_1(x + 1); s_1), \ldots, D_m(r(r + 1), k_m(x + 1); s_m)].$$

Proof. We only prove the case $k \leq x < k + k_t$, the other cases are similar to Theorem 3 and the above case.

From Lemma 4, we assume that

$$[D(r + 1, k + 1; p), D(r + 1, k_t; s_t)] = \begin{pmatrix} 0 & 0 \\ 0 & H \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & (h_{ij})_{r\times r_t} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & h_1 & \cdots & h_{r_t} \end{pmatrix},$$

where $r_t = k + k_t$ and $H = (A_{r\times k}, B_{r\times k_t})$. And set $F = (Q_1(1_r \otimes H), \ldots, Q_x(1_r \otimes H))$. Then we construct a matrix K as follows

$$K = \begin{pmatrix} 0 & (h_1, \cdots, h_x) \oplus 0^T_{r_t} \\ H \oplus 0_r & F \end{pmatrix}.$$

And in the K, set $K_0 = \begin{pmatrix} 0 \\ H \oplus 0_r \end{pmatrix}$ and $K_j = \begin{pmatrix} h_j \oplus 0^T_{r_t} \\ Q_j(1_r \otimes H) \end{pmatrix}$ for $j = 1, \ldots, x$. Moreover, if h_j is a p-level column i.e. a column of $A_{r\times k}$, we take K_j as follows

$$K_j = \begin{pmatrix} h_j \oplus 0^T_k, \phi_t(h_j) \oplus 0^T_{k_t} \\ Q_j(1_r \otimes H) \end{pmatrix},$$

and if h_j is a s_t-level column i.e. a column of $B_{r\times k_t}$, we take K_j as follows

$$K_j = \begin{pmatrix} h_j \oplus 0_{r_t}^T \\ Q_j(1_r \otimes (\phi_t(A), B)) \end{pmatrix}.$$

Through some column permutation, K can be written as

$$K = [D(r(r+1), k(k+1); p), D_t(r(r+1), k_t(k+1) + (x-k)r_t; s_t)].$$

It easily follows from Lemma 3 that all the p-level columns of K constitute a difference matrix $D(r(r+1), k(k+1); p)$.

Now we prove that $[\phi_t(D(r(r+1), k(x+1); p)), D_t(r(r+1), k_t(k+1) + (x-k)r_t; s_t)]$ is a difference matrix based on the group G_t.

Let

$$[\phi_t(D(r+1, k+1; p)), D_t(r+1, k_t; s_t)] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \phi_t(A) & B \end{pmatrix}.$$

By using orthogonal array and similar to Theorem 3, we can construct an s_t-level difference matrix

$$\begin{aligned} K_{\phi_t} &= [D(r(r+1), k(k+1) + k_t(k+1) + (x-k)r_t; s_t)] \\ &= \begin{bmatrix} 0 & (\phi_t(A), B) \oplus 0_x^T \\ (\phi_t(A), B) \oplus 0_r & (Q_1(1_r \otimes (\phi_t(A), B)), \ldots, Q_x(1_r \otimes (\phi_t(A), B))) \end{bmatrix}. \end{aligned}$$

On the other hand, through some column permutation, we just have that K_{ϕ_t} can be written as $K_{\phi_t} = [\phi_t(D(r(r+1), k(k+1); p)), D_t(r(r+1), k_t(k+1) + (x-k)r_t; s_t)]$. Hence, to take $t = 1, \ldots, m$, respectively, we can get a normal mixed difference matrix $[D(r(r+1), k(k+1); p), D_1(r(r+1), k_1(k+1) + (x-k)r_1; s_1), \ldots, D_m(r(r+1), k_m(k+1) + (x-k)r_m; s_m)]$ based on the normal orthogonal array $L_p(s_1 \cdots s_m)$.

Acknowledgements

The work was partly done when the first author visited Nagoya University. The first author would like to thank Professor Masakazu Jimbo for his discussion and suggestions. This work is supported by the National Natural Science Foundation of China (Grant No. 10571045) and NSF of Henan Province, China (0411011100).

References

[1] Bose, R.C., Bush, K. A., Orthogonal arrays of strength two and three. Ann. Math. Stat. 1952, 23: 508-524.

[2] Hedayat, A.S., Sloane, N.J.A., Stufken, J., Orthogonal arrays: Theory and Applications, Springer, New York, 1999.

[3] Pang, S.Q.,Zhang, Y.S.,Liu S.Y., Further Results on the Orthogonal Arrays Obtained by Generalized Hadamard Product, Statist. Probab. Lett., 2004a, 68(1): 17-25.

[4] Pang, S.Q.,Zhang, Y.S.,Liu S.Y., Normal Mixed Difference Matrix and the Construction of Orthogonal Arrays. Statist. Probab. Lett.,2004b, 69(4): 431-437.

[5] Wang, J.C., Mixed difference matrices and the construction of orthogonal arrays. Statist. Probab. Lett., 1996, 28: 121-126.

[6] Wang, J.C., Wu, C.F.J. An approach to the construction of asymmetrical orthogonal arrays. J. Amer. Statist. Assoc., 1991 6: 450-456.

[7] Chung-yi Suen and Warren F. Kuhfeld (2005). On the construction of mixed orthogonal arrays of strength two. J. Statist. Plann. Inference, 133, 555-560.

[8] Wu,C.F.J., Zhang, R., Wang, R. Construction of asymmetrical orthogonal array of the type OA$(s^k, s^m(s_1^r)^{n_1} \cdots (s_t^r)^{n_t})$. Statist. Sinica., 1992, 1: 203-219.

[9] Zhang, Y.S., Duan, L., Lu, Y.Q. Construction of Generalized Hadamard Matrices $D(r^m(r+1), r^m(r+1); p)$. J. of Statist. Plann. Inference, 2002, 104:239-258.

[10] Zhang, Y.S., Pang, S.Q., Wang, Y.P. Orthogonal arrays obtained by generalized Hadamard product. Discrete Math., 2001, 238: 151-170.