Construction of fractional factorial split-plot designs with weak minimum aberration

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Abstract

Fractional factorial split-plot (FFSP) designs with minimum aberration have received much attention in industrial experiments. But they are not so easy to be constructed for the cases when there are many whole plot (or sub-plot) factors and only few sub-plot (or whole plot) factors. Weak minimum aberration is a weak version of minimum aberration. Based on the theory of complementary designs, this paper provides some theoretical results which are useful for constructing FFSP designs with weak minimum aberration for these cases. From these results, many such FFSP designs are constructed and tabulated, and it is further shown that quite a few of them are also minimum aberration designs.

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Keywords: Factorial design; Split-plot; Weak minimum aberration.

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1 Introduction

A 2^{n-k} design usually denotes a regular two-level fractional factorial (FF) design with n 2-level factors and 2^{n-k} runs. Such designs are commonly used for factorial experiments. But if the levels of some of the factors are difficult to be changed or controlled, it may be

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impractical or even impossible to perform the experimental runs of FF designs in a completely random order (Cornell, 1988; Letsinger et al., 1996 and Bisgaard and Steinberg, 1997). In situations of this kind, fractional factorial split-plot (FFSP) designs, which involve a two-phase randomization, can be conveniently used to reduce costs and hence represent a practical design option. Suppose we wish to run an experiment with n two-level factors in 2^{n-k} runs, and there are $n_1 (1 \leq n_1 < n)$ hard-to-change factors with $2^{n_1-k_1}$ distinct level-combinations. To perform such a design, we often first randomly choose one of the level-combinations of these n_1 hard-to-change factors and then run all of the level-combinations of the remaining $n_2 (= n - n_1)$ factors in a random order with the n_1 factors fixed. This is repeated for each level-combination of the n_1 factors. Then the design is said to be a $2^{(n_1+n_2)-(k_1+k_2)}$ FFSP design, where the n_1 and n_2 factors are called whole plot (WP) factors and sub-plot (SP) factors, respectively, and there are k_1 and $k_2 (= k - k_1)$ WP and SP factorial defining words, respectively. In such a design, a WP defining word can not involve any SP factor, but a SP defining word has to involve at least two SP factors. Here we refer to Box and Jones (1992) for an illuminating discussion of FFSP designs in industrial experiments.

FFSP designs have received much attention in recent years, see for example, Huang et al. (1998), Bingham and Sitter (1999a,b, 2001), Bisgaard (2000), Mukerjee and Fang (2002), Bingham and Mukerjee (2006) and Yang et al. (2006) for details. As to what is a good FFSP design, different criteria have been proposed, such as minimum aberration (Huang et al., 1998; Bingham and Sitter, 1999a), clear effect criterion (Yang et al., 2006) and D-optimal criterion (Goos, 2002). Huang et al. (1998) and Bingham and Sitter (1999) gave several algorithms to search FFSP designs with minimum aberration. But when there are many WP (or SP) factors and only few SP (or WP) factors, the construction of FFSP designs with minimum aberration is not so easy to implement. Based on the theory of complementary designs, this paper provides some theoretical results which are useful for constructing FFSP designs with weak minimum aberration, and hence many such FFSP designs are constructed. Section 2 introduces some notion and existing results. Section 3 contains the theoretical results. The constructed FFSP designs with weak minimum aberration are are tabulated in Section 4, most of which are also shown to have minimum aberration.

2 Existing results

In $2^{(n_1+n_2)-(k_1+k_2)}$ FFSP design, the numbers $1, 2, \ldots, n_1 + n_2$ attached to the factors are called letters and a product (juxtaposition) of any subset of these letters is called a word. Considering the structure of FFSP designs, only words involing none or at least two SP factors are permitted. The number of letters in a word is called the length of the word. Associated with every $2^{(n_1+n_2)-(k_1+k_2)}$ FFSP design is a set of $k_1 + k_2$ words called generators. The set of distinct words formed by all possible products involving the $k_1 + k_2$ generators gives the defining contrast subgroup of the design. Let A_i denote the number of words of length i in the defining contrast subgroup, then the vector $W = (A_3, A_4, \ldots, A_{n_1+n_2})$ is called the word length pattern of the FFSP design. The resolution of a $2^{(n_1+n_2)-(k_1+k_2)}$ FFSP design is defined to be the smallest r such that $A_r > 0$. Such a design is said to have maximum resolution if no other $2^{(n_1+n_2)-(k_1+k_2)}$ FFSP design has larger resolution than it. Also a $2^{(n_1+n_2)-(k_1+k_2)}$ FFSP design with maximum resolution $R_{\max}$ is said to have weak minimum aberration if it has the smallest $A_{R_{\max}}$ among all candidate $2^{(n_1+n_2)-(k_1+k_2)}$ FFSP designs and is said to have minimum aberration if it minimizes A_i sequentially for $i = 3, 4, \cdots, n_1 + n_2$. The definitions of resolution, weak minimum aberration and minimum aberration for FFSP designs are extensions of the corresponding ones for FF designs (see Box and Hunter, 1961; Chen and Hedayat, 1996 and Fries and Hunter, 1980, respectively).

Let P be a regular two-level saturated design with resolution III and $n - k$ independent columns, e.g. for the case of $n - k = 3$, we have $P = \{1, 2, 12, 3, 13, 23, 123\}$, where 12, 13, 23 and 123 denote the columns generated by the element-by-element product of the corresponding independent columns 1, 2 and 3, respectively. Suppose D is an FF design chosen from P. Then the design formed by all the columns not involved in D, denoted by $\overline{D}$, is called the complementary design of D (Tang and Wu, 1996). Hence, for a saturated design P, we have $P = D \cup \overline{D}$. The following result is due to Tang and Wu (1996).

Lemma 1. *For a 2^{n-k} FF design D, we have*

$$A_3(D) \quad = \quad C - A_3(\overline{D}), \tag{1}$$

where $\overline{D}$ is the complementary design of D and C is a constant not depending on the choice of D.

This lemma established the relation of A_3 values between design D and its complementary design and will be the main tool in the following for constructing FFSP designs with weak minimum aberration.

3 Main results

Now let $P = P_1 \cup P_2$ be a saturated design with $(n_1 - k_1) + (n_2 - k_2)$ independent columns, where P_1 involves $n_1 - k_1$ independent columns and all the columns generated from these independent ones. Furthermore, P_1 and P_2 can be divided into two parts, respectively, i.e. $P_1 = D_1 \cup \overline{D}_1$, $P_2 = D_2 \cup \overline{D}_2$. We, therefore, have $P = P_1 \cup P_2 = D_1 \cup \overline{D}_1 \cup D_2 \cup \overline{D}_2$. Following this way, a $2^{(n_1+n_2)-(k_1+k_2)}$ FFSP design can be represented by $D_1 \cup D_2$ with $n_1 = |D_1|$ and $n_2 = |D_2|$, where $|S|$ means the number of columns in S and D_i contains $n_i - k_i$ independent columns for $i = 1, 2$ (Yang et al., 2006).

3.1 Connection of A_3 between $D_1 \cup D_2$ and $\overline{D}_1 \cup D_2$

According to the definition of A_3 and the structure of $2^{(n_1+n_2)-(k_1+k_2)}$ FFSP designs, we have

$$A_3(D_1 \cup D_2) = A_3(D_1) + A_3(D_2) + M_2(D_1, D_2). \tag{2}$$

Similarly,

$$A_3(\overline{D}_1 \cup D_2) = A_3(\overline{D}_1) + A_3(D_2) + M_2(\overline{D}_1, D_2). \tag{3}$$

Here, $M_2(Q_1, Q_2)$ means the number of words of length three with one column from Q_1 and the other two from Q_2. Then from (1), we have

$$A_3(D_1 \cup D_2) + A_3(\overline{D}_1 \cup D_2) = C + 2A_3(D_2) + M_2(P_1, D_2),$$

i.e.

$$A_3(D_1 \cup D_2) = C + 2A_3(D_2) + M_2(P_1, D_2) - A_3(\overline{D}_1 \cup D_2). \tag{4}$$

So we have the following theorem.

Theorem 1. *For a $2^{(n_1+n_2)-(k_1+k_2)}$ FFSP design $D_1 \cup D_2$, minimizing $A_3(D_1 \cup D_2)$ is equivalent to minimizing $2A_3(D_2) + M_2(P_1, D_2) - A_3(\overline{D}_1 \cup D_2)$.*

This theorem is very useful in the construction of FFSP designs especially when D_1 has much more columns than $\overline{D}_1$ and D_2 involves only few ones.

Corollary 1. *In the case of $n_2 - k_2 = 1$, for given n_1 and n_2, minimizing $A_3(D_1 \cup D_2)$ is equivalent to maximizing $A_3(\overline{D}_1 \cup D_2)$.*

Proof. For this special case, we have $A_3(D_2) = 0$ and $M_2(P_1, D_2) = 0$ from the structure of a $2^{(n_1+n_2)-(k_1+k_2)}$ FFSP design. Then the assertion follows directly from Theorem 1. □

3.2 Connection of A_3 between $D_1 \cup D_2$ and $D_1 \cup \overline{D}_2$

With all the results in mind, the connection of A_3 between $D_1 \cup D_2$ and $D_1 \cup \overline{D}_2$ can be easily obtained, which is the following theorem.

Theorem 2. *For a $2^{(n_1+n_2)-(k_1+k_2)}$ FFSP design $D_1 \cup D_2$, minimizing $A_3(D_1 \cup D_2)$ is equivalent to maximizing $2A_3(\overline{D}_2) + M_2(P_1, \overline{D}_2) - A_3(D_1 \cup \overline{D}_2)$.*

Proof. According to (1), we have

$$A_3(D_1 \cup D_2) = C_1 - A_3(\overline{D}_1 \cup \overline{D}_2).$$

Then from Theorem 1,

$$A_3(\overline{D}_1 \cup \overline{D}_2) = C_2 + 2A_3(\overline{D}_2) + M_2(P_1, \overline{D}_2) - A_3(D_1 \cup \overline{D}_2).$$

Combining the above two equations, we have

$$A_3(D_1 \cup D_2) = C - (2A_3(\overline{D}_2) + M_2(P_1, \overline{D}_2) - A_3(D_1 \cup \overline{D}_2)),$$

where C_1, C_2, C are constants that do not depend on the choice of D_1 and D_2. Thus, the conclusion follows directly. $\square$

This theorem can be used to construct FFSP designs with weak minimum aberration, when the SP section in an FFSP design has almost as many factors as P_2 has but the WP section contains few ones.

4 Applications

In this section, designs with weak minimum aberration will be constructed based on the above theoretical results. For a $2^{(n_1+n+2)-(k_1+k_2)}$ FFSP design, Yang et al. (2006) pointed out that when $n_1 \geq 2^{n_1-k_1-1}+1$ or $n_1+n_2 \geq 2^{(n_1+n_2)-(k_1+k_2)-1}+1$, the maximum resolution is III. Therefore, for the parameters satisfying these conditions, by minimizing A_3, we can obtain the corresponding FFSP designs with weak minimum aberration. Based on Theorems 1, 2 and Corollary 1, we have constructed many such FFSP designs with weak minimum aberration, which are tabulated in Tables 1 to 5. Note that we only list one weak minimum aberration FFSP design and its corresponding word length pattern for each case. For simplicity, only the first five elements of each word length pattern are given in these tables except for the case of $n_1 = 5, n_2 = 1$ in Table 1. Comparing with the FF designs listed in Chen, Sun and Wu (1993), we can find that many designs in our tables are optimal in terms of minimum aberration.

Example 1. Let us take the $2^{(6+4)-(3+3)}$ FFSP design as an example to illustrate how to obtain a weak minimum aberration design. In this case, we have $|\overline{D}_1| = 1, |D_2| = 4$. To minimize $A_3(D_1 \cup D_2)$, from Corollary 1, we can maximize $A_3(\overline{D}_1 \cup D_2)$. So our purpose is to choose D_2 to constitute as many length-three words with one factor from $\overline{D}_1$ and two from D_2 as possible. One possible choice is to let $\overline{D}_1 = \{12\}$ and $D_2 = \{4, 124, 134, 234\}$. The resulting design $D_1 \cup D_2$ has the same word length pattern as that of the minimum aberration 2^{10-6} FF design listed in Chen, Sun and Wu (1993). And hence, the constructed FFSP design with weak minimum aberration is also an FFSP design with minimum aberration.

Note that for the constructed FFSP designs, many of the word length patterns are inferior to that of the corresponding FF designs given in Chen, Sun and Wu (1993). One reason lies in the additional constraints imposed by the structure of FFSP designs. In other words, many minimum aberration FF designs have no such a structure required by FFSP designs. However, all the FFSP designs given in our tables have weak minimum aberration, which are guaranteed by Theorems 1, 2 or Corollary 1.

In the tables, each word length pattern marked with an asteria (*) means that the corresponding FFSP design has minimum aberration by comparing with the word length pattern in Chen, Sun and Wu (1993). And the one marked with superscript ($^+$) also means that the corresponding design has minimum aberration. But this can be confirmed by the following two ways: one is that the design can be declared to have minimum aberration if its word length pattern is the best among those of the corresponding FF designs given in Chen, Sun and Wu (1993) with the same A_3 value; the other is for the case when the SP section has only one factor (i.e. $n_2 = 1$). In this case, the $2^{(n_1+n_2)-(k_1+k_2)}$ FFSP design is judged to have minimum aberration if its word length pattern is the same with that of the minimum aberration $2^{n_1-k_1}$ FF design.

Note that for some of the parameters given in our tables, minimum aberration designs were also derived by Huang et al. (1998) and Bingham and Sitter (1999a). Comparing with their methods, we see that our approaches have the following two advantages. Firstly, for some specific parameters, the methods in Huang et al. (1998) cannot produce minimum aberration FFSP designs if the corresponding minimum aberration FF designs have no split structure. Ours, however, have no this restriction. Secondly, Bingham and Sitter (1999a) obtained minimum aberration FFSP designs by search algorithms. But here we construct the FFSP designs based on the theoretical results, which may be regarded as an initial work in this direction.

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Table 1: FFSP designs with weak minimum aberration obtained through Corollary 1 ($n_1 - k_1 = 3, n_2 - k_2 = 1$)

n_1	n_2	k_1	k_2	$\lvert\overline{D}_1\rvert$	$\lvert D_2\rvert$	$\overline{D}_1$	D_2	W
7	1	4	0	0	1	ϕ	$\{4\}$	$(7\ 7\ 0\ 0\ 1)^+$
6	1	3	0	1	1	$\{123\}$	$\{4\}$	$(4\ 3\ 0\ 0\ 0)^+$
5	1	2	0	2	1	$\{123, 13\}$	$\{4\}$	$(2\ 1\ 0\ 0)^+$
7	2	4	1	0	2	ϕ	$\{4, 14\}$	$(8\ 10\ 4\ 4\ 4)^+$
6	2	3	1	1	2	$\{12\}$	$\{4, 124\}$	$(4\ 6\ 4\ 0\ 0)$
5	2	2	1	2	2	$\{12, 13\}$	$\{4, 124\}$	$(2\ 3\ 2\ 0\ 0)^*$
7	3	4	2	0	3	ϕ	$\{4, 14, 24\}$	$(10\ 16\ 12\ 12\ 10)$
6	3	3	2	1	3	$\{12\}$	$\{4, 14, 24\}$	$(6\ 10\ 8\ 4\ 2)$
5	3	2	2	2	3	$\{12, 13\}$	$\{4, 124, 134\}$	$(3\ 7\ 4\ 0\ 1)^+$
7	4	4	3	0	4	ϕ	$\{4, 14, 24, 34\}$	$(13\ 25\ 25\ 27\ 23)^+$
6	4	3	3	1	4	$\{12\}$	$\{4, 124, 134, 234\}$	$(8\ 18\ 16\ 8\ 8)^*$
5	4	2	3	2	4	$\{12, 13\}$	$\{4, 124, 134, 234\}$	$(4\ 14\ 8\ 0\ 4)^*$

Table 2: FFSP designs with weak minimum aberration obtained through Theorem 2 ($n_1 - k_1 = 3, n_2 - k_2 = 1$)

n_1	n_2	k_1	k_2	$\lvert D_1\rvert$	$\lvert\overline{D}_2\rvert$	D_1	$\overline{D}_2$	W
3	8	0	7	3	0	$\{1, 2, 3\}$	ϕ	$(12\ 26\ 28\ 24\ 20)^*$
3	7	0	6	3	1	$\{1, 2, 3\}$	$\{1234\}$	$(9\ 16\ 15\ 12\ 7)^+$
3	6	0	5	3	2	$\{1, 2, 3\}$	$\{124, 34\}$	$(6\ 9\ 9\ 6\ 0)^+$
4	8	1	7	4	0	$\{1, 2, 3, 123\}$	ϕ	$(16\ 39\ 48\ 48\ 48)^*$
4	7	1	6	4	1	$\{1, 2, 3, 123\}$	$\{124\}$	$(12\ 26\ 28\ 24\ 20)^*$
4	6	1	5	4	2	$\{1, 2, 3, 123\}$	$\{14, 24\}$	$(8\ 18\ 16\ 8\ 8)^*$
4	5	1	4	4	3	$\{1, 2, 3, 123\}$	$\{14, 24, 34\}$	$(4\ 14\ 8\ 0\ 4)^*$

Table 3: FFSP designs with weak minimum aberration obtained through Theorem 2 ($n_1 - k_1 = 2, n_2 - k_2 = 2$)

n_1	n_2	k_1	k_2	$\lvert D_1\rvert$	$\lvert\overline{D}_2\rvert$	D_1	$\overline{D}_2$	W
2	12	0	10	2	0	$\{1, 2\}$	ϕ	$(28\ 77\ 112\ 168\ 232)^+$
2	11	0	9	2	1	$\{1, 2\}$	$\{1234\}$	$(22\ 55\ 72\ 96\ 116)^+$
2	10	0	8	2	2	$\{1, 2\}$	$\{13, 23\}$	$(16\ 39\ 48\ 48\ 48)^*$
2	9	0	7	2	3	$\{1, 2\}$	$\{13, 23, 124\}$	$(12\ 26\ 28\ 24\ 20)^*$
2	8	0	6	2	4	$\{1, 2\}$	$\{13, 24, 1234, 34\}$ or	$(8\ 18\ 16\ 8\ 8)^*$
3	12	1	10	3	0	$\{1, 2, 12\}$	ϕ	$(35\ 105\ 168\ 280\ 435)^+$
3	11	1	9	3	1	$\{1, 2, 12\}$	$\{1234\}$	$(28\ 77\ 112\ 168\ 232)^+$
3	10	1	8	3	2	$\{1, 2, 12\}$	$\{13, 23\}$	$(22\ 55\ 72\ 96\ 116)^+$
3	9	1	7	3	3	$\{1, 2, 12\}$	$\{13, 24, 1234\}$	$(16\ 39\ 48\ 48\ 48)^*$
3	8	1	6	3	4	$\{1, 2, 12\}$	$\{13, 24, 1234, 14\}$	$(12\ 26\ 28\ 24\ 20)^*$

Table 4: FFSP designs with weak minimum aberration obtained through Corollary 1 ($n_1 - k_1 = 4, n_2 - k_2 = 1$)

n_1	n_2	k_1	k_2	$\lvert\overline{D}_1\rvert$	$\lvert D_2\rvert$	$\overline{D}_1$	D_2	W
15	1	11	0	0	1	ϕ	$\{5\}$	$(35\ 105\ 168\ 280\ 435)^+$
15	2	11	1	0	2	ϕ	$\{5, 15\}$	(36 112 196 364 624)
15	3	11	2	0	3	ϕ	$\{5, 15, 25\}$	(38 126 252 532 1002)
15	4	11	3	0	4	ϕ	$\{5, 15, 25, 35\}$	(41 147 337 791 1597)
14	1	10	0	1	1	$\{1234\}$	$\{5\}$	$(28\ 77\ 112\ 168\ 232)^+$
14	2	10	1	1	2	$\{1234\}$	$\{5, 12345\}$	(28 84 140 224 344)
14	3	10	2	1	3	$\{123\}$	$\{5, 1235, 2345\}$	(30 96 184 348 598)
14	4	10	3	1	4	$\{12\}$	$\{5, 125, 35, 1235\}$	(32 116 256 528 992)
13	1	9	0	2	1	$\{12, 13\}$	$\{5\}$	$(17\ 39\ 48\ 65\ 86)^+$
13	2	9	1	2	2	$\{12, 13\}$	$\{5, 125\}$	(22 61 94 136 188)
13	3	9	2	2	3	$\{12, 13\}$	$\{5, 125, 135\}$	(23 73 132 216 347)
13	4	9	3	2	4	$\{12, 13\}$	$\{5, 125, 135, 145\}$	(26 88 184 356 610)
12	1	8	0	3	1	$\{12, 13, 23\}$	$\{5\}$	$(16\ 39\ 48\ 48\ 48)^+$
12	2	8	1	3	2	$\{12, 13, 23\}$	$\{5, 125\}$	(16 45 64 72 96)
12	3	8	2	3	3	$\{12, 13, 23\}$	$\{5, 125, 135\}$	(16 57 96 120 192)
12	4	8	3	3	4	$\{12, 13, 23\}$	$\{5, 125, 135, 235\}$	(16 76 144 192 352)
11	1	7	0	4	1	$\{12, 13, 23, 14\}$	$\{5\}$	$(12\ 26\ 28\ 24\ 20)^+$
11	2	7	1	4	2	$\{12, 13, 23, 14\}$	$\{5, 125\}$	(12 31 40 40 48)
11	3	7	2	4	3	$\{12, 13, 23, 14\}$	$\{5, 125, 135\}$	(12 41 64 72 104)
11	4	7	3	4	4	$\{12, 13, 23, 14\}$	$\{5, 125, 135, 235\}$	(12 57 100 120 200)
10	1	6	0	5	1	$\{12, 13, 23, 14, 24\}$	$\{5\}$	$(8\ 18\ 16\ 8\ 8\ )^+$
10	2	6	1	5	2	$\{12, 13, 23, 14, 24\}$	$\{5, 125\}$	(8 23 24 16 24)
10	3	6	2	5	3	$\{12, 13, 23, 14, 24\}$	$\{5, 125, 135\}$	(8 31 40 40 56)
10	4	6	3	5	4	$\{12, 13, 23, 14, 24\}$	$\{5, 125, 135, 235\}$	(8 45 64 72 112)
9	1	5	0	6	1	$\{12, 13, 23, 124, 134, 234\}$	$\{5\}$	$(4\ 14\ 8\ 0\ 4)^+$
9	2	5	1	6	2	$\{12, 13, 23, 124, 134, 234\}$	$\{5, 125\}$	(4 18 12 8 12)
9	3	5	2	6	3	$\{12, 13, 23, 124, 134, 234\}$	$\{5, 125, 135\}$	(4 26 20 24 28)
9	4	5	3	6	4	$\{12, 13, 23, 124, 134, 234\}$	$\{5, 125, 135, 235\}$	(4 39 32 48 56)

Table 5: FFSP designs with weak minimum aberration obtained through Theorem 2 ($n_1 - k_1 = 3, n_2 - k_2 = 2$)

n_1	n_2	k_1	k_2	$\lvert\overline{D}_1\rvert$	$\lvert D_2\rvert$	$\overline{D}_1$	D_2	W
7	2	4	0	0	2	ϕ	$\{4,5\}$	(7 7 0 0 1)
6	2	3	0	1	2	$\{123\}$	$\{4,5\}$	(4 3 0 0 0)
5	2	2	0	2	2	$\{12,13\}$	$\{4,5\}$	(2 1 0 0 0)
7	3	4	1	0	3	ϕ	$\{4,5,145\}$	(7 8 3 4 5)
6	3	3	1	1	3	$\{123\}$	$\{4,5,1234\}$	(4 6 4 0 0)
5	3	2	1	2	3	$\{12,13\}$	$\{4,5,124\}$	(2 3 2 0 0)
7	4	4	2	0	4	ϕ	$\{4,5,145,245\}$	(8 12 10 12 12)
6	4	3	2	1	4	$\{123\}$	$\{4,5,1234,1235\}$	(4 10 8 0 4)
5	4	2	2	2	4	$\{12,13\}$	$\{4,5,145,234\}$	(3 5 2 2 3)
7	5	4	3	0	5	ϕ	$\{4,5,14,245,345\}$	(9 17 21 27 27)
7	6	4	4	0	6	ϕ	$\{4,5,14,15,245,345\}$	(10 25 38 50 58)