A Complete Catalog of Geometrically Non-Isomorphic 18-run Orthogonal Arrays

Ko-Jen Tsai
Department of Applied Math and Statistics
SUNY at Stony Brook

Kenny Q. Ye
Department of Epidemiology and Population Health
Albert Einstein College of Medicine

William Li
Operations and Management Science Department,
University of Minnesota

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Abstract

1 Introduction

Factorial designs have wide applications in industrial and scientific studies. Traditionally, they are classified based their combinatoric properties. Two designs are considered to be isomorphic if one can be obtained by permuting of the factors and level permutation within factors. Such isomorphism treats levels of each factor nominal without natural ordering and is often referred as *combinatoric isomorphism.* (Chang and Ye, 2004).

However, many researchers has realized that when factors are quantitative, level permutations of a factorial design often result changes in design properties such as estimation efficiencies, and the *combinatoric isomorphism* is not appropriate for classifying factorial designs with quantitative factors, except for two-level designs. More noticeably, Cheng and Wu (2001) defined "statistical isomorphism" based on the statistical properties, and Tsai, Gilmore, and Mead (2000) implicitly defined an isomorphism in their paper as well, both for three-level factorial designs. Furthermore, Cheng and Ye(2004) argued that such a change of design properties is due to the change of the geometric structure induced by level permutations. They defined *geometric isomorphism* on general factorial designs and introduced a polynomial form of indicator functions that uniquely represent factorial designs to characterize their geometric structures. For three-level factorial designs, *geometric isomorphism* is equivalent to the "design family" defined in Tsai et al (2000).

Among all factorial designs that are not two-level, the L_{18} array is arguably the most popular choice among experimenters because of its ability of entertaining up to seven 3-level factors in only 18-runs. All three papers mentioned above attempted to further classify 18-run orthogonal arrays beyond combinatorial isomorphism. However, none of them gives a complete classification. The classification criteria used by Cheng and Wu (2001) and Tsai et al (2000) depend on specification of certain statistical model. Although the latter defined an isomorphism equivalent to geometric isomorphism, they used an easy-to-compter surrogate measure so called *Q Criteria*. Cheng and Ye (2004) did not perform complete isomorphism check either, instead, they used a generalized word-length pattern, *β-WLP* for classification. Moreover, Tsai et al (2000) only considered 18-run orthogonal arrays without the two-level factor, and Cheng and Ye (2004) only classified designs with 3 or 4 three-level factors.

The main result of this paper is a complete catalog of all geometrically non-isomorphic 18-run orthogonal arrays. The biggest obstacle in complete classification of factorial designs, regardless of combinatoric or geometric isomorphism, is the computational burden to check isomorphism between two designs, which requires large number of permutations on the array. We developed an efficient algorithm for checking isomorphism based on indicator representation of a design, as proposed in Cheng and Ye (2004). As a result, our algorithm does not require row permutation on an array, and with proper choice of data structure, the column permutation and level reversal of an array can be handled very efficiently in the computer program.

The remainder of this paper is organized as follows. Section 2 briefly reviews the definition of geometric isomorphism. Section 3 summarizes the complete catalog of 18-run orthogonal arrays. Section 4 describes our algorithms for check isomorphism, which is based on the indicator functions. Section 5 describes how we construct the catalog.

2 Geometric Isomorphism

Throughout this paper, levels of orthogonal arrays are treated as quantitative. Consider an n-run factorial design with k 3-level (evenly spaced) quantitative factors, denoted by a matrix form $[X_1, \cdots, X_k]$, which can also be viewed as a collection of n points in the k dimension Euclidean space. It is reasonable to consider its geometric structure being preserved under two-types of operations, level reversal and factor exchange. Geometrically, if the levels of a factor, say X_p, are reversed, i.e. the low level becomes the high level and vise versa, the design points are reflected over the plane $X_p = 0$. If two factors, X_p and X_q are exchanged, geometrically the designs points are reflected over the plane $X_p = X_q$. Therefore, a new design obtained through a series of these two types of operations has the same geometric structure as the original design. This is true for general factorial designs if we limit the factor exchange among factors with the same number of levels. Cheng and Ye(2004) defined *geometric isomorphism* of general factorial designs as following.

Definition 1 *Let $\mathcal{A}$ and $\mathcal{B}$ be two factorial designs from the same design space $\mathcal{D}$, where the design space $\mathcal{D}$ is a full factorial design $OA(N, s_1 s_2 \ldots s_k)$. Designs A and B are said to be geometrically isomorphic if one can be obtained from the other by factor exchange among factors with the same number of factors and/or reversing the level order of one or more factors.*

3 Summary of the complete catalog of 18-run Orthogonal Designs

The main results of this paper, the complete catalog of geometrically non-isomorphic 18-run orthogonal designs, is summarized in Table 1. There are a total of 13 geometrically isomorphic OA$(18, 3^3)$s, 133 OA$(18, 3^4)$s, 332 OA$(18, 3^5)$, 478 OA$(18, 3^6)$ and 284 OA$(18, 3^7)$s, and a total of 119 geometrically isomorphic OA$(18, 2^1 3^3)$s, 1836 OA$(18, 2^1 3^4)$s, 1332 OA$(18, 2^1 3^5)$s, 1617 OA$(18, 2^1 3^6)$s and 726 OA$(18, 2^1 3^7)$s.

In Table 1, we compared our complete catalog to the non-isomorphic $OA(18, 3^p)$ obtained by Tsai et al (2000) using so called *Q Criterion*, as listed in http://userweb.nhri.org.tw/ n930839/design.html. There are 4, 12, 38, 31 new cases for $p = 4, 5, 6, 7$ respectively. We also examined the β-Word Length Patterns, proposed by Cheng and Ye (2004) to rank and classify mixed-level and multi-level factorial designs with quantitative factors, of all designs in the catalog, and list the number of distinct β-word length pattern (WLP) of all 18-run orthogonal arrays in Table 1. Overall, the number of distinct β-WLP of $OA(18, 3^m)$ designs is 90% of the number of geometrical non-isomorphic $OA(18, 3^m)$, the number of distinct β-WLP of $OA(18, 2^1 3^m)$ designs is 82% of the number of geometrical non-isomorphic $OA(18, 2^1 3^m)$. And for each m, the number is no less than 70% of the number of geometrical isomorphic designs. Therefore, both Q Criterion and β-WLP are fairly good surrogate criteria to classify geometrical non-isomorphic designs.

We also found 44 $OA(18, 3^4)$ and 852 $OA(18, 2^1 3^4)$ maximal (Resolution III) designs, i.e., no more columns can be augmented to create a $OA(18, 3^5)$. Therefore, these designs cannot be found by taking projections from the well-known L_{18} array. Note that all maximal (Resolution III) designs are with four three-level factors. Hence, all designs in the catalog with $m > 4$, can be created by level permutation, factor exchanging, and projection from the L_{18} array. Cheng and Ye (2004) searched for the minimum aberration (based on β-WLP) designs of $OA(18, 3^m)$ and $OA(18, 2^1 3^m)$ by evaluating all projections (with level permutations) of L_{18}. We compared their list with the β-WLPs of MA designs found from the complete catalog and find better $OA(18, 2^1 3^4)$s according to β-WLP. The best β-WLP of $OA(18, 2^1 3^4)$ we found is $(\beta_3, \beta_4, \beta_5) = (0, 2, 4)$, compare to $(\beta_3, \beta_4, \beta_5) = (0, 3.75, 0)$ found in Cheng and Ye (2004). Not surprisingly, this new MA design is a maximal design and is shown in Table 2. We also confirmed that all other MA designs found by Cheng and Ye (2004) are indeed global MA designs.

Although we do not intend to extend the scope of this paper beyond presenting the catalog and the constructing algorithms, we would like to show that the geometric non-isomorphic designs indeed carry different statistical properties so that our effort to carry out the complete classification add values for practical purpose. We evaluated two Information Capacity (IC) criteria for $OA(18, 3^6)$. Criterion IC_3 averages $|X'X|$ over all $\binom{6}{3} = 20$ three factors second order polynomial models, where X is normalized model matrix of the model with 10 degrees of freedoms (constant term, 6 main effects and 3 bilinear terms). Similarly, criterion IC_4 averages over all $\binom{6}{4} = 15$ four factors

Table 1: Summary of All Geometrically Non-Isomorphic 18-run Orthogonal Arrays

OA(18,3^m)					
Number Of 3-Level Factors	3	4	5	6	7
Number Of Non-Isomorphic Designs	13	133	332	478	284
Number Of Different $\beta-$WLP	13	128	332	420	223
Number Of Maximal Designs	0	44	0	0	0
Number of Designs (Tsai et al, 2000)	13	129	320	440	253
OA(18, $2^1 3^m$)					
Number Of 3-Level Factors	3	4	5	6	7
Number Of Non-Isomorphic Designs	119	1836	1332	1617	726
Number Of Distinct $\beta-$WLP	118	1293	1274	1406	556
Number of Maximal Designs	0	852	0	0	0

second order polynomial models with 15 degrees of freedoms (constant term, 8 main effects and 6 bilinear terms). All but two of 476 non-isomorphic designs $OA(18, 3^6)$ have equal values in both IC_3 and IC_4. Interestingly, these two designs do not have the same β-WLP. Therefore, 58 new designs not distinguishable by β-WLP have different statistical properties, and this is very much true for the 38 new designs not found by Tsai et al (2000).

Because the total number of non-isomorphic 18-run orthogonal designs is close to 7000, we are unable to publish our entire catalog in print. The complete catalog of design matrices are available from the correspondent author upon request. More statistical properties of those designs can be found in Tsai (2005) and forthcoming papers.

Table 2: Minimum Aberration (β-WLP) 18-run OA$(18, 2^1 3^4)$

1	0	0	0	1
-1	0	0	1	0
-1	0	1	0	2
1	0	1	2	2
1	0	2	1	0
-1	0	2	2	1
1	1	0	2	0
-1	1	0	2	2
1	1	1	1	1
-1	1	1	1	1
-1	1	2	0	0
1	1	2	0	2
-1	2	0	0	1
1	2	0	1	2
1	2	1	0	0
-1	2	1	2	0
-1	2	2	1	2
1	2	2	2	1

4 Algorithm for checking geometric isomorphism

4.1 Indicator Function

The indicator function was originally proposed by Fontana, Pistone, and Rogantin(2000) to represent a factorial design. The efficiency of our algorithm for checking geometric isomorphism largely depends on this representation. To make the present paper more self-contained, we will briefly introduce indicator functions before presenting our algorithm. For more details on this topic , please see Cheng and Ye(2004).

Definition 2 *Let $\mathcal{A}$ be a design in the design space $\mathcal{D}$. The indicator function $F_{\mathcal{A}}(x)$ of $\mathcal{A}$ is a function defined on $\mathcal{D}$, such that for $x \in \mathcal{D}$, the value of $F_{\mathcal{A}}(x)$ is the number of appearances of point x in design $\mathcal{A}$.*

Furthermore, the indicator function can be written as a linear combination of an orthonormal basis. Let $\mathcal{A}$ be a factorial design in design space $\mathcal{D} = OA(N, s_1 s_2 \cdots s_k)$, where s_i is the number of levels of ith factor.

Let ${C_0}^i(x), {C_1}^i(x), \cdots, {C_{s_i-1}}^i(x)$ be the orthogonal contrasts of the ith factor, such that

$$\sum_{x \in \{0,1,\ldots,s_i-1\}} {C_u}^i(x){C_v}^i(x) = \begin{cases} 0, & \text{if } u \neq v, \\ s_i, & \text{if } u = v, \end{cases}$$

where u and $v \in T_i = \{0, 1, ..., s_i - 1\}$. Then $\{C_{\mathbf{t}}(\mathbf{x}) = \prod_{i=1}^{k} {C_{t_i}}^i(x_i) | \mathbf{t} \in \mathcal{T}\}$, where $\mathbf{t} = (t_1, t_2, ..., t_k) \in \mathcal{T} = T_1 \times T_2 \times ... \times T_k$, forms the orthonormal contrast basis of the functional space on $\mathcal{D}$ such that for $\mathbf{u}$ and $\mathbf{v} \in \mathcal{T}$

$$\sum_{\mathbf{x} \in \mathcal{D}} C_{\mathbf{u}}(\mathbf{x}) C_{\mathbf{v}}(\mathbf{x}) = \begin{cases} 0, & \text{if } \mathbf{u} \neq \mathbf{v}, \\ N, & \text{if } \mathbf{u} = \mathbf{v}. \end{cases}$$

Cheng and Ye (2004) showed that the indicator function of a factorial design $\mathcal{A}$ can be written as

$$F_{\mathcal{A}}(\mathbf{x}) = \sum_{\mathbf{t} \in \mathcal{T}} b_{\mathbf{t}} C_{\mathbf{t}}(\mathbf{x}), \tag{1}$$

where $b_{\mathbf{t}} = \frac{1}{N}\sum_{\mathbf{x}\in\mathcal{A}} C_{\mathbf{t}}(\mathbf{x})$.

Furthermore, they gave a Theorem that connected geometric isomorphism to indicator functions. We presented it below since it is essential to our algorithm for isomorphism check between two designs.

Theorem 3 *Let $\mathcal{A}$ and $\mathcal{B}$ be two factorial designs of the design space $\mathcal{D}$, and $C_{\mathbf{t}}(\mathbf{x})$ be an orthogonal polynomial basis (OPB) defined on $\mathcal{D}$. Let $F_{\mathcal{A}}(\mathbf{x}) = \sum a_{\mathbf{t}} C_{\mathbf{t}}(\mathbf{x})$ and $F_{\mathcal{B}}(\mathbf{x}) = \sum b_{\mathbf{t}} C_{\mathbf{t}}(\mathbf{x})$ be the indicator functions of $\mathcal{A}$ and $\mathcal{B}$, respectively. Designs $\mathcal{A}$ and $\mathcal{B}$ are geometrically isomorphic if and only if there exists a permutation $(i_1 i_2 ... i_k)$ and a binary vector $(j_1 j_2 ... j_k)$, where j_l's are either 0 or 1, such that*

$$a_{t_1 t_2 \dots t_k} = (\prod_{i=1}^{k} (-1)^{j_l t_{i_l}}) b_{t_{i_l} t_{i_2} \dots t_{i_k}} \qquad (2)$$

for all $\mathbf{t} = (t_1 t_2 ... t_k) \in \mathcal{T}$.

4.2 An Algorithm for Checking Geometric Isomorphism

To identify whether two design matrices are isomorphic, traditionally one has to consider all possible row permutations, column permutations, and level permutations (level reversal in the cases of geometric isomorphism) within each column. Our approach using indicator functions to represent the designs only requires the consideration of column permutations and level reversals. We also implemented several measures to further improve the efficiency so that the non-isomorphism of two designs are determined without going through all possible permutations and level reversals. First, no level reversal will be needed if no permutation can be found to match the absolute values of the indicator function coefficients of two designs. Second, we can quickly determine if such permutation exists using the following proposition.

The algorithm presented here applies to general factorial designs. Without loss of generality, we limited our discussion to designs with only three-level factors.

Table 3: Grouping of coefficients of the indicator function of OA$(n, 3^3)$

Coefficient index $\mathbf{t}$	Group
111	1
222	2
112	3
121	3
211	3
122	4
212	4
221	4

Proposition 4 *Let $\mathcal{A}$ and $\mathcal{B}$ be two k-factor three-level factorial designs of the design space $\mathcal{D}$, and $C_{\mathbf{t}}(\mathbf{x})$ be an orthogonal polynomial basis(OPB) defined on $\mathcal{D}$. $F_{\mathcal{A}}(\mathbf{x}) = \sum a_{\mathbf{t}} C_{\mathbf{t}}(\mathbf{x})$ and $F_{\mathcal{B}}(\mathbf{x}) = \sum b_{\mathbf{t}} C_{\mathbf{t}}(\mathbf{x})$ be the indicator functions of $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $T(\mathbf{t}) = (T_0(\mathbf{t}), T_1(\mathbf{t}), T_2(\mathbf{t}))$, where $T_i(\mathbf{t}) = \#\{t_j = i \mid 0 \leq j \leq k\}$ for $i = 0, 1,$ or 2. If designs $\mathcal{A}$ and $\mathcal{B}$ are geometrically isomorphic, then $\{|a_{\mathbf{t}}| \mid$ for all $\mathbf{t}$ where $T(\mathbf{t}) = (l_0, l_1, l_2)\} = \{|b_{\mathbf{t}}| \mid$ for all $\mathbf{t}$ where $T(\mathbf{t}) = (l_0, l_1, l_2)\}$ for all (l_1, l_2, l_3) such that $l_1 + l_2 + l_3 = k$.*

The proof of the above Proposition is straightforward. Note that the column permutations only permute coefficients within each group. Suppose not, then we can not find any permutation to make $|a_{t_1 t_2 \ldots t_k}| = |b_{t_{i_l} t_{i_2} \ldots t_{i_k}}|$. Hence there is a contradiction.

Based on the above proposition, we divide the coefficients into distinct groups according to the number of linear and quadratic terms they involve. For example, when $k = 3$, the coefficients in the indicator functions are grouped as in Table 4.2. Note that $b_{\mathbf{0}}$ is constant for all designs with the same number of runs, and for an orthogonal array $b_{\mathbf{t}} = 0$ where $T_1(\mathbf{t}) + T_1(\mathbf{t}) \leq 2$. Therefore, we omit those coefficients in the table. In the remainder of the paper, we will refer these groups as $\{T_i\}$ groups.

Main Algorithm

In general, to check geometric isomorphism of two designs, we followed

these steps:
Step 1: Compare the number of the non-zero coefficients in each $\{T_i\}$ group. If they do not match in any one $\{T_i\}$ group, return false as the two designs are not geometrically isomorphic. Otherwise, go to step 2.
Step 2: Compare the distribution of the absolute values of coefficients within each $\{T_i\}$ group. If they are not the same, return false. Otherwise, go to step 3.
Step 3: Apply the first permutation to the coefficients of $\mathcal{A}$. Denote the new coefficients be $a'_{\mathbf{t}}$. Compare $|a'_{\mathbf{t}}|$ with $|b_{\mathbf{t}}|$. If the equation holds for all coefficients in the first $\{T_i\}$ group, check the next group. Otherwise, repeat step 3 with next permutation. If a permutation matches all coefficients, then go to Step 4. If none of the permutation matches all coefficients, return false.
Step 4: Start from the first binary vector in Z_2^k, say $(j_1, j_2, ..., j_k)$. If $j_1t_1 + j_2t_2 + ... + j_kt_k = 0 (\text{mod } 2)$ for all $a_{\mathbf{t}}{}' = b_{\mathbf{t}}$, and $j_1t_1 + j_2t_2 + ... + j_kt_k = 1 (\text{mod } 2)$ for all $a'_{\mathbf{t}} = -b_{\mathbf{t}}$, the two designs are isomorphic. If not, apply the next binary vector. If no match are found with all binary vectors, go back to Step 3 and start with next permutation. If the last permutations is reached, return false.

Note that in Step 4, we also check the equality by $\{T_i\}$ groups, as described in more details later. It is possible to expedite this step much more by eliminating the need to evaluate all possible level reversals. We did not implement such methods since our algorithm is already efficient enough to handle its current task.

Implementation

As last part of this section, we now discuss several measures we implemented that improve the efficiency of our computer program.

1. Computing indicator function coefficients. The orthogonal contrasts for three-level factors are

$$\begin{aligned} C_0(x) &= 1, \\ C_1(x) &= \sqrt{\tfrac{3}{2}}(x-1), \\ C_2(x) &= \sqrt{\tfrac{1}{2}}(3(x-1)^2 - 2). \end{aligned}$$

where x takes values $0, 1$ and 2. In our program, we ignored normalizing constant, $\sqrt{\frac{3}{2}}$ in $C_1(x)$ and $\sqrt{\frac{1}{2}}$ in $C_2(x)$ for computational conveniences. We did not calculate the coefficients b_t such that $T_1(t) + T_2(t) \leq 2$ since they are zero.

2. Data structure of indicator functions. An indicator function of a design

is fully determined by its coefficients. In our java program, none-zero coefficients of an indicator function are kept as a List. Each element of the list has $k + 2 + 1$ attributes, k indices of the coefficient, its absolute value and sign, and which $\{T_i\}$ group it belongs to. Note that when ith column is exchanged with jth column in a design, we only need to exchange the ith and jth indices across the list to obtain the indicator function of the new design, which is a very efficient operation.
3. Column permutations. When we evaluate all column permutations, we use an algorithm described in Nigenhuis and Wilf (1978) that generates all $k!$ permutations in the way such that the next one differs from the current one only by a pair switch.
4. Level reversal. Note that coefficients in those $\{T_i\}$ groups such that $T_1(\mathbf{t}) = 0$, are invariant to level reversal. Therefore, if the signs of coefficients in those group does not match, we will move to the next permutation immediately in Step 4.

5 Construction of the Complete Catalog of 18-run Orthogonal Designs

The construction of our complete catalog of 18-run orthogonal designs was completed in two stages. We first obtained all non-isomorphic cases of $\mathrm{OA}(18, 3^m)$. Then, we added the two-level column to the $\mathrm{OA}(18, 3^m)$s constructed in the first stage to obtain $\mathrm{OA}(18, 2^1 3^m)$s.

5.1 Construction of Non-isomorphic $OA(18, 3^m)$

Our construction of the complete catalog of 18-run orthogonal array follows the same basic idea used by Sun, Li and Ye (2002) to construct all non-isomorphic $\mathrm{OA}(12, 2^m)$s, $\mathrm{OA}(16, 2^m)$s, and $\mathrm{OA}(20, 2^m)$s. Tsai et al (2000) used the same methods to obtain a catalog of $\mathrm{OA}(18, 3^m)$s classified by “Q Criterion”. We start with the only non-isomorphic case of $OA(18, 3^2)$, then augment a column to it to construct all cases of $OA(18, 3^3)$, and subsequently add another column to find all cases of $OA(18, 3^4)$, and so on. Using this method, we are able to obtain all geometric non-isomorphic 18-run orthogonal arrays, as shown in the following proposition.

Proposition 5 *Denote* $[\mathbf{A}\ v]$ *an array formed by an array* $\mathbf{A}$ *augmented* *with a vector v. Let $\mathcal{C}$ be the collection of all geometrically non-isomorphic OA(n, p^m). We can find all geometrically non-isomorphic OA(n, p^{m+1}) among* $\{[\mathbf{A}\ v]\}$*, where* $\mathcal{A} \in \mathcal{C}$ *and v is a p-level vector.*

PROOF: Let $Y = [v_1\ v_2\ ...\ v_{m+1}]$ be an OA(n, p^{m+1}). Since $[v_1\ v_2\ ...\ v_m]$ is an OA(n, p^m), it must be geometrically isomorphic to one of the design in $\mathcal{C}$, say $\mathcal{A}$. Therefore, Y can be constructed by $[\mathbf{A}\ v]$. We have completed the proof.□

There is only one geometrically non-isomorphic OA($18, 3^2$), that is, $OA(9, 3^2)$ replicated twice. Note that by the definition of orthogonal arrays, all possible level combinations of an OA($18, 3^2$) have to appear the same number of times. Hence, each of the 9 level-combinations has to appear exactly twice. Without loss of generality, we start our construction of 18-run orthogonal arrays from the OA($18, 3^2$) with two columns

$$\begin{array}{rl} {v_1}^t = & [000000111111222222] \\ {v_2}^t = & [001122001122001122] \end{array}.$$

Proposition 5 shows that all geometrically non-isomorphic OA($18, 3^3$) can be found from OAs obtained by adding an additional column to v_1 and v_2. To achieve this, we first identified all 23436 vectors that are orthogonal to v_1 and v_2 from all $\frac{18!}{6!6!6!} = 17,153,136$ possible balanced three-level vectors. Then we found a set of geometrically non-isomorphic OA($18, 3^3$)s, denoted by $\mathcal{C}$, from the set of 23436 OAs, denoted by $\mathcal{F}$, in the following steps:
Step 1: Select an arbitrary OA from $\mathcal{F}$ into a set $\mathcal{C}$. Remove it from $\mathcal{F}$.
Step 2: Select an OA from $\mathcal{F}$ and remove it from $\mathcal{F}$. If it is geometrically isomorphic to any existing OAs in $\mathcal{C}$, discard it. Otherwise, add it to $\mathcal{C}$.
Step 3: If $\mathcal{F}$ is not empty, repeat Step 2. Otherwise, C has all geometrically non-isomorphic OA($18, 3^3$).

The above procedure is repeated to find all non-isomorphic cases of OA($18, 3^p$) for $p = 4, 5, 6, 7$ subsequently. For each p, $\mathcal{F}$ initially contains OAs constructed by adding an additional column to each of the non-isomorphic OA($18, 3^{p-1}$)s. Note that the additional column is first selected from the same 23436 vectors orthogonal to v_1 and v_2, its orthogonality with other columns is subsequently checked.

5.2 Construction of Non-Isomorphic $OA(18, 2^1 3^m)$

Following the same arguments of Proposition 5, to find all geometrically non-isomorphic designs of OA($18, 2^1 3^m$), we only need to consider the orthogonal arrays obtained by augmenting a two-level column to all non-isomorphic $\mathrm{OA}(18, 3^m)$s. Moreover, we will show that two $\mathrm{OA}(18, 2^1 3^m)$s constructed from two non-isomorphic $\mathrm{OA}(18, 3^m)$s are also non-isomorphic.

Lemma 6 *Let $\mathcal{A}$ be an $OA(N, d^1 e^k)$ and $F_{\mathcal{A}}(x) = \sum a_{\mathbf{t}} C_{\mathbf{t}}(x)$ be its indicator function. Without loss of generality, the d-level factor is the first column. Let $\mathcal{A}'$ be the projection of $\mathcal{A}$ to all of its e level factors, and $F_{\mathcal{A}'}(\mathbf{x}') = \sum a_{\mathbf{t}'} C_{\mathbf{t}'}(\mathbf{x}')$ where $\mathbf{x}' = (x_2, x_3, ..., x_{k+1})$ and $\mathbf{t}' = (t_2, t_3, ..., t_{k+1})$. Then $a_{\mathbf{t}'} = d a_{0 t_2 t_3 \ldots t_{k+1}}$.*

The lemma is a special case of Corollary 2.1 in Cheng and Ye (2004).

Proposition 7 *Let $\mathcal{A}$ and $\mathcal{B}$ be two $OA(N, d^1 e^k)$ factorial designs of the design space $\mathcal{D}$. Let $\mathcal{A}'$ be the projection of $\mathcal{A}$ that contains all the e-level factors and $\mathcal{B}'$ be the projection of $\mathcal{B}$ design that contains all the e-level factors. If $\mathcal{A}'$ and $\mathcal{B}'$ are geometrically non-isomorphic, then $\mathcal{A}$ and $\mathcal{B}$ are geometrically non-isomorphic.*

PROOF: It goes without loss of generality, let the d-level factor be the first column of $\mathcal{A}$ and $\mathcal{B}$, respectively.

Let $C_{\mathbf{t}}(\mathbf{x})$ be an orthogonal polynomial basis(OPB) defined on $\mathcal{D}$. Let $F_{\mathcal{A}}(\mathbf{x}) = \sum a_{\mathbf{t}} C_{\mathbf{t}}(x)$ and $F_{\mathcal{B}}(\mathbf{x}) = \sum b_{\mathbf{t}} C_{\mathbf{t}}(\mathbf{x})$ be the indicator functions of $\mathcal{A}$ and $\mathcal{B}$, respectively. Following the similar notation used in the lemma, $F_{\mathcal{A}'}(\mathbf{x}') = \sum a_{\mathbf{t}'} C_{\mathbf{t}'}(\mathbf{x}')$ and $F_{\mathcal{B}'}(\mathbf{x}') = \sum b_{\mathbf{t}'} C_{\mathbf{t}'}(\mathbf{x}')$ be the indicator function of $\mathcal{A}'$ and B'.

Suppose not, then $\mathcal{A}$ and $\mathcal{B}$ are geometrically isomorphic. By Theorem 3, there exist one permutation $P = (i_1 i_2 \cdots i_k)$ and a binary vector $R = (j_1 j_2 ... j_k)$, such that,

$$b_{t_1 t_2 \ldots t_k t_{k+1}} = (\prod_{i=1}^{k+1} (-1)^{j_l t_{i_l}}) b_{t_{i_l} t_{i_2} \ldots t_{i_k} t_{k+1}}{}' \tag{3}$$

Since the only d-level factor is the first column, we have $i_1 = 1$ in P. By the lemma, we also know that $a_{\mathbf{t}'} = d a_{0 t_2 t_3 \ldots t_{k+1}}$ and $b_{\mathbf{t}'} = d b_{0 t_2 t_3 \ldots t_{k+1}}$. Hence,

$$\begin{aligned} a_{\mathbf{t}'} = d a_{0 t_2 \ldots t_{k+1}} &= d(\prod_{l=1}^{k+1} (-1)^{j_l t_{i_l}}) b_{0 t_{i_2} \ldots t_{i_k}}{}' && (4) \\ &= (\prod_{l=2}^{k+1} (-1)^{j_l t_{i_l}}) d b_{0 t_{i_2} \ldots t_{i_k}}{}' && (5) \\ &= (\prod_{i=2}^{k+1} (-1)^{j_l t_{i_l}}) b_{\mathbf{t}'} && (6) \end{aligned}$$

Therefore, $\mathcal{A}'$ is geometrically isomorphic to $\mathcal{B}'$, after the permutation $P' = (i_2 \cdots i_k)$ and level reversal $R' = (j_2 \cdots j_k)$. It contradicts the assumption. Hence we have completed the proof.$\square$

The above proposition can be easily extended to more general cases of mixed-level designs. By the Proposition, we only need to check isomorphism between the $\mathrm{OA}(18, 2^1 3^m)$ that are constructed from the same $OA(18, 3^m)$. We constructed all non-isomorphic $\mathrm{OA}(18, 2^1 3^m)$ as following:
Step 1: pick an $\mathrm{OA}(18, 3^m)$, say A, from the set of all non-isomorphic $\mathrm{OA}(18, 3^m)$, denoted as $\mathcal{C}$.
Step 2: find all binary vectors v that is orthogonal to A to form a set of OAs $[vA]$, call this set $\mathcal{F}$,
Step 3: check the isomorphism of OAs in $\mathcal{F}$, eliminate all redundant ones, and keep all non-isomorphic cases,
Step 4: pick a new $\mathrm{OA}(18, 3^m)$ from $\mathcal{C}$, and go back to Step 2. Stop if there is no new $\mathrm{OA}(18, 3^m)$ left.

6 Concluding Remarks

In this paper, we present the *complete* catalog of geometrically non-isomorphic 18-run orthogonal arrays. What is instrumental in our construction is an efficient algorithm for check isomorphism that is based on indicator function representation of factorial designs. Our computer program is written in JAVA. When run on a Pentium 4 2.0GHz Desktop PC with 256Mb RAM, it took about 120 minutes to find all $\mathrm{OA}(18, 2^1 3^7)$ non-isomorphic designs from $\mathrm{OA}(18, 3^7)$ designs. The time taken to construct 18-run orthogonal designs of any other size is less. The total CPU time to construct the entire catalog is less than 12 hours. However, this is achieved through a great effort was spent to implement the algorithm in the most efficient way.

The completion of this catalog makes it very easy to find optimal 18-run orthogonal designs. Given any criterion, by evaluating its values on the designs in the catalog one can easily find the global optimal design, without repeating over isomorphic designs. This can be done even for the cases only combinatoric isomorphism is relevant, since all combinatoric non-isomorphic cases is a subset of our catalog. Nonetheless, it is still of theoretical interest to obtain a complete catalog of combinatoric non-isomorphic designs, and we are working with our collaborators to further classify the designs in our catalog for construct a complete catalog of combinatoric non-isomorphic 18-

run orthogonal arrays.

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