Conformal Triality and AdS/CFT^3 Correspondence

Han-Ying Guo
Bin Zhou

Collaborators: Zhan Xu, Chao-Guang Huang and Yu Tian

On the 23rd International Conference of DGMTP

August 25, 2005

Outline

♢ The Beltrami model of dS and AdS; proposal of special relativity on them

♢ Conformal Mink-, dS- and AdS-spaces from a projective subspace

♢ Conformal triality and AdS/CFT^3 correspondence

CFT^3: three kinds of CFTs.

♢ Null geodesics in conformal Mink-, dS- and AdS-spaces

1 The Beltrami Model of dS/AdS and Special Relativity on Them

◇ References.

▶ H.-Y. Guo, C.-G. Huang, Z. Xu and B. Zhou, "On Special Relativity with Cosmological Constant", *Phys. Lett.* **A331** (2004) 1–7, hep-th/0403171.

▶ H.-Y. Guo, C.-G. Huang, Z. Xu and B. Zhou, "On Beltrami Model of de Sitter Spacetime", *Mod. Phys. Lett.* **A19** (2004) 1701, hep-th/0311156;

▶ Basic ideas traced back to:
K. H. Look (Qi-Keng Lu), "Why the Minkowski metric must be used?", (1970), unpublished.

▶ The Beltrami coordinates:
H. S. Snyder, *Phys. Rev.* **71** (1947) 38.
W. de Sitter, "Einstein's theory of gravitation III", *Roy. Astr. Soc. Month. Not.* **78** (1917) 3.
W. Pauli ('20), *Theory of Relativity*, Pergamon, 1958.

1.1 Why could there be a special relativity on dS/AdS?

1. Analogy in geometry.

Euclidean geometry	SR on Minkowski
Lobachevskian geometry	SR on AdS?
Riemann's spherical geometry	SR on dS?

▶ Three kinds of SR from inverse Wick rotation — hep-th/0508094.

2. Experiments and observations.

▶ SR: supported by local experiments;

▶ the cosmos: asymptotically dS.

Isn't it a good thing to have a theory supported by all these experiments and observations?

3. Different features of SR and GR.

(1) In SR:

► existence of inertial reference frames;
► with the principle of SR related to them;
► metric as a background only;
► spacetime symmetry: the Poincaré group;
► symmetry group of dynamics: the Poincaré group;
► the Minkowski coordinates $\Leftrightarrow$ inertial reference frames;

. . .

(2) In GR:

► no inertial reference frames, even no global frames in general;
► the principle of GR;
► metric — background and dynamical;
► spacetime symmetry: generally, no;
► dynamics: diffeomorphic-invariant;
► coordinate independent;

. . .

(3) A question: how the features of GR turn out to be those of SR when the curvature is zero?

(4) The correct relation of SR and GR:

SR is a theory which describes a special solution of GR.

$$\lim_{\mathbf{R}\to 0} \mathrm{GR} \neq \mathrm{SR}. \tag{1}$$

Then why there could not be theories of "SR" to describe other spacetimes?

1.2 dS and Its Geometry

♢ Usually viewed as a hypersurface in $\mathcal{M}^{1,4}$:

$$\eta_{AB}\,\xi^A\xi^B = -R^2. \tag{2}$$

$$(\eta_{AB})_{A,B=0,1,\ldots,4} = \mathrm{diag}(1,-1,-1,-1,-1).$$

♢ Equivalently, $dS \cong \mathcal{S}/\sim$

▶ $\mathcal{S} \subset \mathcal{M}^{1,4}$: the set of spacelike vectors;

▶ $\sim$: the equivalence relation similar to that in projective geometry:

$$\xi' \sim \xi \quad \Leftrightarrow \quad \exists c > 0 \text{ s.t. } \xi'^A = c\,\xi^A. \tag{3}$$

▶ dS: the set of spacelike rays from the origin of $\mathcal{M}^{2,4}$.

▶ Why antipodal points not identified?

Orientable and time orientable.

♢ The Beltrami coordinates — inhomogeneous coordinates.

$$x^{\mu} := R\,\frac{\xi^{\mu}}{\xi^{4}}, \quad (\mu = 0, 1, 2, 3) \tag{4}$$

on the regions $U_{\pm 4}$ where $\xi^4 > 0$ or $\xi^4 < 0$.

▶ Other coordinate neighborhoods.

▶ $\sigma(x) > 0$ where

$$\sigma(x) := 1 - \frac{1}{R^2}\,\eta_{\mu\nu}\,x^{\mu}x^{\nu}. \tag{5}$$

♢ The $O(1,4)$ transformations:

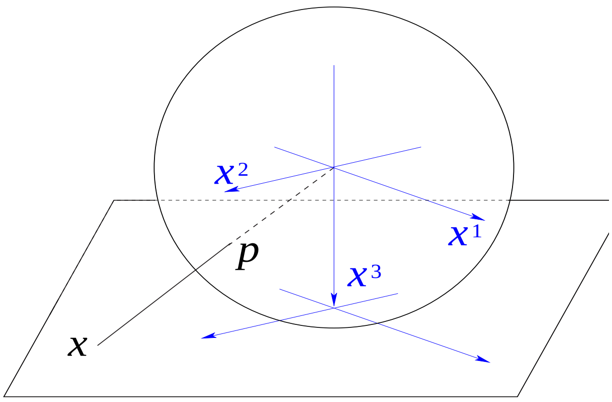
$$x'^{\mu} = \pm\frac{\sqrt{\sigma(a)}\, D^{\mu}{}_{\nu}(x^{\nu} - a^{\nu})}{\sigma(a,x)}, \tag{6}$$

$$D^{\mu}{}_{\nu} = L^{\mu}{}_{\nu} + \frac{R^{-2} L^{\mu}{}_{\rho}\, a^{\rho} a_{\nu}}{\sigma(a) + \sqrt{\sigma(a)}}, \tag{7}$$

$$\sigma(a,x) := 1 - \frac{1}{R^2}\,\eta_{\mu\nu}\, a^{\mu} x^{\nu}, \tag{8}$$

$$(L^{\mu}{}_{\nu}) \in O(1,3), \qquad \sigma(a) > 0$$

► Fractional linear.
► The same form as the $O(3)$ transformations on S^2 or $\mathbb{R}P^2$.
► Turn out to be Poincaré transformations up to the order of $1/R^2$.



♢ The $O(1,4)$-invariant metric:

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu\, dx^\nu \tag{9}$$

$$g_{\mu\nu}(x) := \frac{\eta_{\mu\nu}}{\sigma(x)} + \frac{\eta_{\mu\alpha}\,\eta_{\nu\beta}\, x^\alpha x^\beta}{R^2\sigma(x)}. \tag{10}$$

- Induced from the hypersurface, or
- derived from the invariant cross ratio.
- The same form as that on S^2 or $\mathbb{R}P^2$.
- $O(1,4)$-invariant:

$$\frac{\partial x^\alpha}{\partial x'^\mu}\,\frac{\partial x^\beta}{\partial x'^\nu}\, g_{\alpha\beta}(x) = g_{\mu\nu}(x'). \tag{11}$$

- $g_{\mu\nu}(x) = \eta_{\mu\nu} + O(1/R^2)$

♢ Geodesics

▶ Geodesics are "projective" staight lines, and vice versa.

$$x^\mu = x_0^\mu + w(s)\, u^\mu, \tag{12}$$

$$\frac{dw}{ds} = \sigma(x) = 1 - \frac{1}{R^2}\, \eta_{\mu\nu}\, x^\mu x^\nu. \tag{13}$$

▶ $w \sim s$ if all $x^\mu \ll R$.

▶ Preserved quantities:

$$P^\mu = \frac{m}{\sigma(x)}\, \frac{dx^\mu}{ds}, \quad L^{\mu\nu} = x^\mu P^\nu - x^\nu P^\mu. \tag{14}$$

▶

$$\eta_{\mu\nu}\, P^\mu P^\nu - \frac{1}{2R^2}\, \eta_{\mu\rho}\, \eta_{\nu\sigma}\, L^{\mu\nu} L^{\rho\sigma} = m^2. \tag{15}$$

1.3 Proposal of SR on dS/AdS

► Method of projective geometry works, nearly without use of DG.

► Inertial motions and inertial reference frames (IRF) can be defined intrinsically.

► An IRF can be transformed to be another IRF by $O(1,4)$ transformations.

► The Beltrami coordinates and the Minkowski coordinates cannot be distinguished in small scales.

► All the formulae can be degenerated to those in SR in small scales.

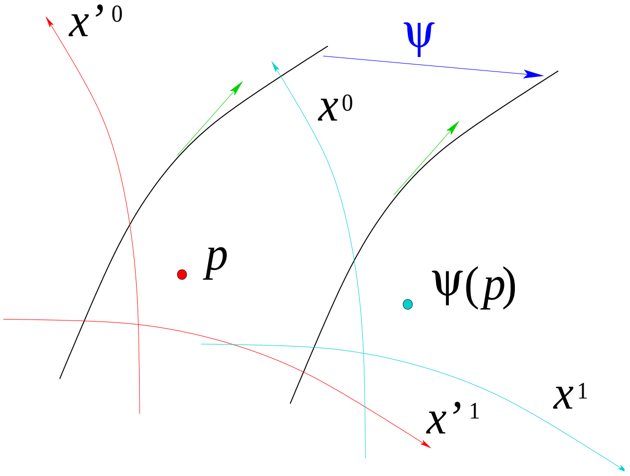
♢ If SR on dS/AdS is accepted, then nonzero curvature does not necessarily mean gravitation.

2 Conformal Transformations

♢ For a $(M, \mathbf{g})$, $\psi : M \to M$ is a conformal transformation if

(1) it is a diffeomorphism and

(2) $\psi^* \mathbf{g} = \rho^2 \, \mathbf{g}$.



$x^\mu \longrightarrow x'^\mu := \psi^* x^\mu = x'^\mu(x)$, then

$$\frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} g_{\rho\sigma}(x') = \rho^2(x) \, g_{\mu\nu}(x), \tag{16}$$

$$ds'^2 = \rho^2(x) \, ds^2. \tag{17}$$

♢ The conformal Minkowski space: Special conformal transformations are not diffeomorphisms on the Minkowski space. Additional points (points at infinity, ideal points) must be added.

♢ Similar for conformal dS- and AdS-spaces.

3 Conformal dS/AdS-Spaces

♢ As hypersurfaces in $M^{1,4}$ and $M^{2,3}$, respectively,

$$H_\theta^{1,3}: \quad \eta_{\theta\, AB}\, \xi^A \xi^B = -\theta\, R^2. \tag{18}$$

$$\theta = \pm, \quad (\eta_{\theta\, AB})_{A,B=0,\dots,4} = \mathrm{diag}(J, -\theta), \tag{19}$$

$$J = (\eta_{\mu\nu}) = \mathrm{diag}(1, -1, -1, -1). \tag{20}$$

♢ In $\mathcal{M}^{2,4}$, if set

$$\zeta^A = \kappa\, \xi^A, \ (A = 0, \dots, 4) \quad \zeta^5 = \kappa\, R, \tag{21}$$

then eq. (18) $\Rightarrow$

$$\mathcal{N}_\theta: \quad \eta_{\theta\, \hat{A}\hat{B}}\, \zeta^{\hat{A}} \zeta^{\hat{B}} = 0. \tag{22}$$

$$(\zeta^0, \dots, \zeta^5) \sim (\xi^0, \dots, \xi^4, R), \tag{23}$$

$$dS \cong H_+^{1,3} = \mathcal{N}_+ \cap \mathcal{P}, \tag{24}$$

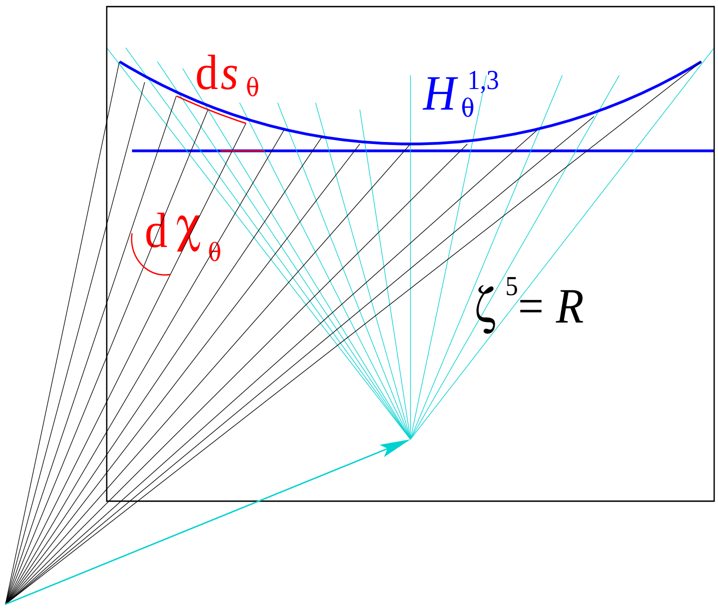
$$AdS \cong H_-^{1,3} = \mathcal{N}_- \cap \mathcal{P}, \tag{25}$$

$$\mathcal{P}: \ \zeta^5 = R. \tag{26}$$

♢ Induced metric:

$$d\chi_\theta^2 = \kappa^2\, ds_\theta^2 = \left(\frac{\zeta^5}{R}\right)^2\, ds_\theta^2, \qquad (27)$$

where
$$d\chi_\theta^2 := \eta_{\theta\ \hat{A}\hat{B}}\, d\zeta^{\hat{A}} d\zeta^{\hat{B}}, \qquad (28)$$
$$ds_\theta^2 := \eta_{AB}\, d\xi^A d\xi^B. \qquad (29)$$



Each $O(2,4)$ transformation induces a conformal transformation on $H_\theta^{1,3}$:

$$ds_\theta'^2 = \rho^2\, ds_\theta^2, \qquad \rho = \frac{\zeta^5}{\zeta'^5}. \qquad (30)$$

◇ Conformal transformations on BdS and BAdS:

$$\zeta'^{\hat{A}} = C^{\hat{A}}_{\ \hat{B}}\, \zeta^{\hat{B}}, \tag{31}$$
$$\zeta'^{\mu} = C^{\mu}_{\ \nu}\, \zeta^{\nu} + C^{\mu}_{\ 4}\, \zeta^{4} + C^{\mu}_{\ 5}\, \zeta^{5}, \tag{32}$$
$$\zeta'^{4} = C^{4}_{\ \alpha}\, \zeta^{\alpha} + C^{4}_{\ 4}\, \zeta^{4} + C^{4}_{\ 5}\, \zeta^{5}; \tag{33}$$
$$\zeta'^{5} = C^{5}_{\ \alpha}\, \zeta^{\alpha} + C^{5}_{\ 4}\, \zeta^{4} + C^{5}_{\ 5}\, \zeta^{5}. \tag{34}$$
$$x'^{\mu} = R\,\frac{\xi'^{\mu}}{\xi'^{4}} = R\,\frac{\zeta'^{\mu}}{\zeta'^{4}} \tag{35}$$

$$x'^{\mu} = \frac{C^{\mu}_{\ \nu}\, x^{\nu} + C^{\mu}_{\ 4}\, R \pm C^{\mu}_{\ 5}\, R\sqrt{\sigma(x)}}{C^{4}_{\ 4} + \frac{1}{R}\, C^{4}_{\ \alpha}\, x^{\alpha} \pm C^{4}_{\ 5}\, \sqrt{\sigma(x)}}, \tag{36}$$
$$\rho = \frac{\pm\sqrt{\sigma(x)}}{C^{5}_{\ 4} + \frac{1}{R}\, C^{5}_{\ \alpha}\, x^{\alpha} \pm C^{5}_{\ 5}\, \sqrt{\sigma(x)}}. \tag{37}$$

♢ Examples.

$$C = \begin{pmatrix} I & 0 & 0 \\ 0 & \gamma & \gamma\beta \\ 0 & \gamma\beta & \gamma \end{pmatrix}, \tag{38}$$

$$x'^{\mu} = \frac{x^{\mu}\sqrt{1-\beta^2}}{1 \pm \beta\sqrt{\sigma(x)}}; \tag{39}$$

$$C = \begin{pmatrix} \delta^{\mu}_{\nu} - \frac{\theta}{R^2}\frac{b^{\mu}b_{\nu}}{1+\sqrt{\sigma(b)}} & 0 & \frac{b^{\mu}}{R} \\ 0 & 1 & 0 \\ -\frac{b_{\nu}}{R} & 0 & \sqrt{\sigma(b)} \end{pmatrix}, \tag{40}$$

$$x'^{\mu} = x^{\mu} - \frac{\theta(b \cdot x)}{1+\sqrt{\sigma(b)}}\,\frac{b^{\mu}}{R^2} \pm b^{\mu}\sqrt{\sigma(x)}. \tag{41}$$

4 Triality

4.1 Outline and review

(1) A null-cone $\mathcal{N}$ in $\mathcal{M}^{2,4}$: $\qquad \eta_{\hat{A}\hat{B}}\,\zeta^{\hat{A}}\zeta^{\hat{B}} = 0.$
(2) A 4-dim $[\mathcal{N}]$ is resulted in:

$$[\mathcal{N}] = \mathcal{N} - \{0\}/\sim, \qquad [\mathcal{N}] \cong S^1 \times S^3.$$

(3) An action of $O(2,4)$ is induced on $[\mathcal{N}]$.
(4) The Minkowski space is $\mathcal{N} \cap \mathcal{P}$, with $\mathcal{P}$ a null hyperplane

$$\zeta^- := \frac{1}{\sqrt{2}}\,(-\zeta^4 + \zeta^5) = R.$$

(5) dS-space is $\mathcal{N} \cap \mathcal{P}$, with $\mathcal{P}$ a hyperplane $\zeta^5 = R$. The normal vector of $\mathcal{P}$ is timelike.
(6) For AdS-space, the normal vector of $\mathcal{P}$ is spacelike.
(7) Relation of metrics: For Minkowski and dS/AdS, respectively,

$$d\chi_M^2 = \left(\frac{\zeta^-}{R}\right)^2 ds_M^2, \qquad d\chi_M^2 = \kappa^2\, ds_\theta^2.$$

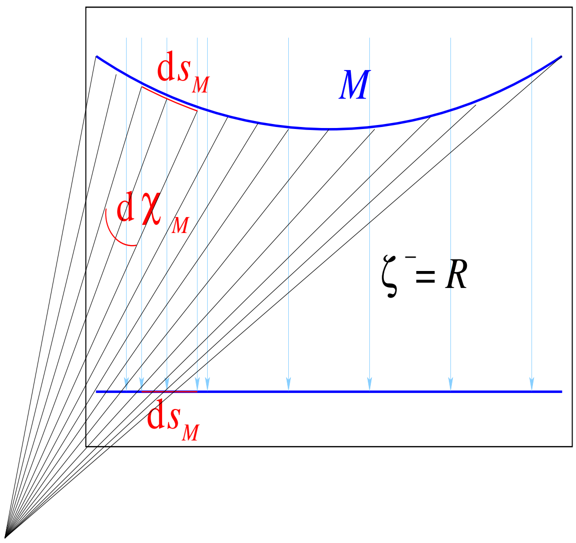
The induced $O(2,4)$ transformations on $\mathcal{N} \cap \mathcal{P}$ are conformal:

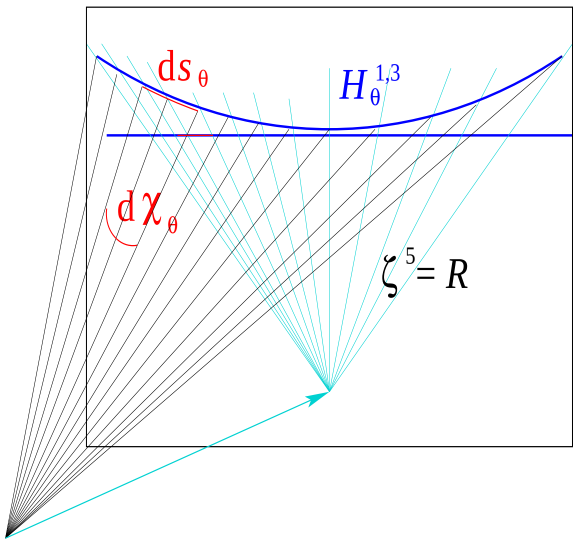
$$\left(\frac{\zeta'^-}{R}\right)^2 ds'^2_M = d\chi'^2_M \equiv d\chi^2_M = \left(\frac{\zeta^-}{R}\right)^2 ds^2_M \quad \Rightarrow$$

$$ds'^2_M = \rho^2\, ds^2_M, \qquad \rho := \frac{\zeta^-}{\zeta'^-};$$

$$\kappa'^2\, ds'^2_\theta = d\chi'^2 \equiv d\chi^2 = \kappa^2\, ds^2_\theta, \quad \Rightarrow$$

$$ds'^2_\theta = \rho^2\, ds^2_\theta, \quad \rho := \frac{\kappa}{\kappa'} = \frac{\zeta^5}{\zeta'^5}.$$





(8) Transformation law in terms of the Minkowksi or Beltrami coordinates.

4.2 A generic description

(1) $\mathcal{P}$: a hyperplane, not passing through the origin, orientation induced.
n: a normal vector of $\mathcal{P}$.

Items	Minkowski	dS	AdS
n	null	timelike	spacelike
$\mathcal{P}$	null	spacelike	timelike
signature	$0, +, -, -, -$	$1 + 4$	$2 + 3$
$\mathcal{N} \cap \mathcal{P}$	Minkowski	dS	AdS
status of **n**	up to a scalar	determined	determined

(2) For null $\mathcal{P}$,

$\mathbf{l}$: a null vector pointing to $\mathcal{P}$, and $\mathbf{l} \cdot \mathbf{n} = 1$. ($\mathbf{n}$ determined by $\mathbf{l}$.)

$\mathbf{e}_\mu$ ($\mu = 0, 1, 2, 3$): orthonormal, tangent to $\mathcal{P}$,

$$\mathbf{l} \cdot \mathbf{e}_\mu = \mathbf{n} \cdot \mathbf{e}_\mu = 0. \tag{42}$$

$\{\mathbf{e}_\mu, \mathbf{n}, \mathbf{l}\}$ — orientation. $\mathbf{e}_0$ future pointed.

♣ $\boldsymbol{\zeta} \in \mathcal{P}$ can be expressed as

$$\boldsymbol{\zeta} = x^\mu \mathbf{e}_\mu + x^+ \mathbf{n} + R\,\mathbf{l}, \quad \boldsymbol{\zeta} \in \mathcal{N} \cap \Leftrightarrow x^+ = -\frac{1}{2R}\,\eta_{\mu\nu}\,x^\mu x^\nu. \tag{43}$$

♣ $C \in O(2,4)$ transforms $\boldsymbol{\zeta} \in \mathcal{N} \cap \mathcal{P}$ to

$$\boldsymbol{\zeta}' = x'^\mu \mathbf{e}_\mu + x'^+ \mathbf{n} + R\,\mathbf{l} \quad = \quad \rho\, C \boldsymbol{\zeta}. \tag{44}$$

$$ds'^2_M = \rho^2\, ds^2_M. \tag{45}$$

♣ $\{\mathcal{P}, -\mathcal{P}\}$ C-invariant $\Leftrightarrow$ C induces a Poincaré transformation on $\mathcal{N} \cap \mathcal{P}$.

♣ Change of $\mathbf{l}$ induces a Poincaré coordinate transformation:

$$R\,\mathbf{l}' = a^\mu \mathbf{e}_\mu + a^+ \mathbf{n} + R\,\mathbf{l} \qquad \Rightarrow \tag{46}$$

$$\mathbf{e}'_\mu = \mathbf{e}_\mu - \frac{\eta_{\mu\nu}}{R}\, a^\nu\, \mathbf{n}, \qquad x'^\mu = x^\mu - a^\mu. \tag{47}$$

(3) For a spacelike $\mathcal{P}$: $(\mathcal{N} \cap \mathcal{P} \cong dS)$
$\{\mathbf{e}_A, \mathbf{n} \,|\, A = 0, 1, \ldots, 4\}$ — oriented orthonormal basis,
$\mathbf{n}$ — unit timelike normal vector pointing to $\mathcal{P}$,
$\mathbf{e}_0$ — future pointed.
♣ $\boldsymbol{\zeta} \in \mathcal{P}$:

$$\boldsymbol{\zeta} = \xi^A \, \mathbf{e}_A + R\,\mathbf{n}, \qquad \boldsymbol{\zeta} \in \mathcal{N} \cap \mathcal{P} \quad \Leftrightarrow \quad \eta_{AB}\, \xi^A \xi^B = -R^2. \tag{48}$$

♣ Beltrami coordinates can be defined. E.g.,

$$x^\mu := R\,\frac{\xi^\mu}{\xi^4}. \quad \Rightarrow \quad \xi^4 = \pm \frac{R}{\sqrt{\sigma(x)}}. \tag{49}$$

$$\sigma(x) > 0, \qquad \sigma(x) := 1 - \frac{1}{R^2}\,\eta_{\mu\nu}\, x^\mu x^\nu. \tag{50}$$

♣ $C \in O(2,4)$ transforms $\boldsymbol{\zeta} \in \mathcal{N} \cap \mathcal{P}$ to

$$\boldsymbol{\zeta}' = \xi'^A \, \mathbf{e}_A + R\,\mathbf{n} \quad = \quad \rho\, C \boldsymbol{\zeta}, \tag{51}$$

$$ds'^2_+ = \rho^2 \, ds^2_+. \tag{52}$$

♣ $\{\mathcal{P}, -\mathcal{P}\}$ C-invariant $\Leftrightarrow$ C induces a de Sitter transformation on $\mathcal{N} \cap \mathcal{P}$.

(4) For a timelike $\mathcal{P}$: $(\mathcal{N}\cap\mathcal{P}\cong AdS)$
$\{\mathbf{e}_\mu, \mathbf{n}, \mathbf{e}_4\}$ — oriented orthonormal basis,
$\mathbf{n}$ — unit spacelike normal vector pointint to $\mathcal{P}$,
$\mathbf{e}_0$ — future pointed.
♣ $\boldsymbol{\zeta}\in\mathcal{P}$:

$$\boldsymbol{\zeta}=\xi^A\,\mathbf{e}_A+R\,\mathbf{n},\qquad \boldsymbol{\zeta}\in\mathcal{N}\cap\mathcal{P}\ \Rightarrow\ \eta_{AB}\,\xi^A\xi^B=R^2. \tag{53}$$

♣ Beltrami-Hua-Lu coordinates can be defined. E.g.,

$$x^\mu := R\,\frac{\xi^\mu}{\xi^4}.\quad\Rightarrow\quad \xi^4=\pm\frac{R}{\sqrt{\sigma(x)}}. \tag{54}$$

$$\sigma(x)>0,\qquad \sigma(x):=1+\frac{1}{R^2}\,\eta_{\mu\nu}\,x^\mu x^\nu. \tag{55}$$

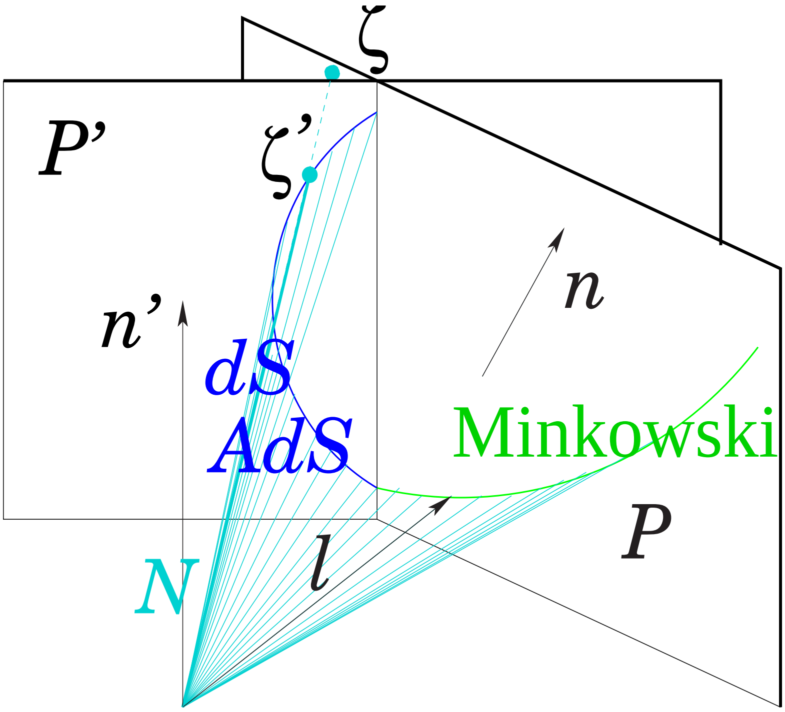
♣ $C\in O(2,4)$ transforms $\boldsymbol{\zeta}\in\mathcal{N}\cap\mathcal{P}$ to

$$\boldsymbol{\zeta}'=\xi'^A\,\mathbf{e}_A+R\,\mathbf{n}\quad=\quad\rho\,C\boldsymbol{\zeta}, \tag{56}$$

$$ds_-'^2=\rho^2\,ds_-^2. \tag{57}$$

♣ $\{\mathcal{P},-\mathcal{P}\}$ is C-invariant $\Leftrightarrow$ C induces an anti-de Sitter transformation on $\mathcal{N}\cap\mathcal{P}$.

4.3 Conformal maps between conformal Minkowski, dS and AdS



(1) Conformal map from the Minkowski to dS/AdS.
$\mathcal{P}$: null, $\mathcal{N} \cap \mathcal{P}$ Minkowski,
$\mathcal{P}'$: spacelike or timelike, $\mathcal{N} \cap \mathcal{P} \cong dS$ or AdS, respectively.
♣ $\boldsymbol{\zeta} = x^\mu \mathbf{e}_\mu + x^+ \mathbf{n} + R\, \mathbf{l} \in \mathcal{N} \cap \mathcal{P} \quad \sim$
$\boldsymbol{\zeta}' = \xi^A \mathbf{e}'_A + R'\, \mathbf{n}' \in \mathcal{N} \cap \mathcal{P}'$,
$\boldsymbol{\zeta}' = \rho\, \boldsymbol{\zeta} \qquad \Rightarrow$

$$x'^\mu = R' \, \frac{C^\mu{}_\nu\, x^\nu - \frac{1}{2^{\frac{3}{2}} R} (C^\mu{}_4 + C^\mu{}_5)\,(x \cdot x) - \frac{1}{\sqrt{2}} (C^\mu{}_4 - C^\mu{}_5)\, R}{C^4{}_\alpha\, x^\alpha - \frac{1}{2^{\frac{3}{2}} R} (C^4{}_4 + C^4{}_5)\,(x \cdot x) - \frac{1}{\sqrt{2}} (C^4{}_4 - C^4{}_5)\, R}, \qquad (58)$$

$$\rho = \frac{R'}{C^5{}_\alpha\, x^\alpha - \frac{1}{2^{\frac{3}{2}}\, R}\,(C^5{}_4 + C^5{}_5)\,(x\cdot x) - \frac{1}{\sqrt{2}}\,(C^5{}_4 - C^5{}_5)\,R}, \qquad (59)$$

$$ds'^2_\theta = \rho^2\, ds^2_M, \qquad \theta = \pm 1. \qquad (60)$$

♣ A special example.

$$x'^\mu = -\frac{x^\mu}{1 + \frac{1}{4R'^2}\,(x\cdot x)}, \qquad ds'^2_\theta = \frac{ds^2_M}{[1 - \frac{1}{4R'^2}\,(x\cdot x)]^2}. \quad (61)$$

(2) The conformal map from dS to AdS.

♣ Generic transformation.

$$x'^{\mu} = R' \frac{C^{\mu}{}_{\nu}\, x^{\nu} + C^{\mu}{}_{4}\, R \pm C^{\mu}{}_{5}\, R\sqrt{\sigma(x)}}{C^{4}{}_{\alpha}\, x^{\alpha} + C^{4}{}_{4}\, R \pm C^{4}{}_{5}\, R\sqrt{\sigma(x)}}, \quad (62)$$

$$\rho = \frac{\pm R' \sqrt{\sigma(x)}}{C^{5}{}_{\alpha}\, x^{\alpha} + C^{5}{}_{4}\, R \pm C^{5}{}_{5}\, R\sqrt{\sigma(x)}}, \quad (63)$$

$$ds'^2_{-} = \rho^2\, ds^2_{+}. \quad (64)$$

♣ A special example:

$$x'^{\mu} = \pm \frac{R'}{R} \frac{x^{\mu}}{\sqrt{\sigma(x)}}, \qquad \rho = \pm \frac{R'}{R} \sqrt{\sigma(x)}. \quad (65)$$

The inverse map:

$$x^{\mu} = \pm \frac{R}{R'} \frac{x'^{\mu}}{\sqrt{\sigma'(x')}}, \quad (66)$$

$$\rho' = \pm \frac{R}{R'} \sqrt{\sigma'(x')}. \quad (67)$$

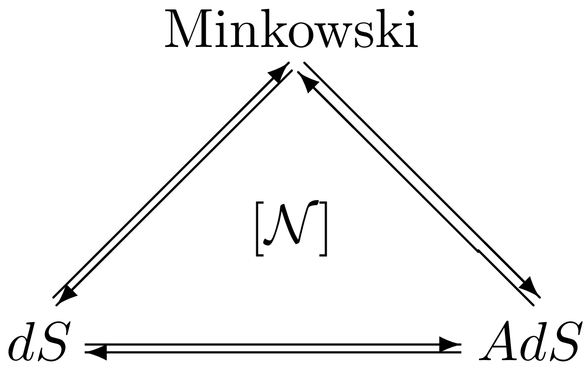
$$\sigma'(x')\, \sigma(x) = 1, \quad (68)$$

♡

$$\sigma'(x') = 1 + \frac{1}{R'^2} \eta_{\mu\nu}\, x'^{\mu} x'^{\nu}, \quad (69)$$

$$\sigma(x) = 1 - \frac{1}{R^2} \eta_{\mu\nu}\, x^{\mu} x^{\nu}. \quad (70)$$

♢ Triality.



5 Null Geodesics, Massive Particles

$\diamondsuit$ Null geodesics are conformally transformed to each other.
Timelike or spacelike geodesics cannot be.
Is there a clear picture?
$\diamondsuit$ Momentum and angular momentum as conserved quantities in inertial motions. What if under conformal transformations/maps?
$\diamondsuit$ For two linear independent $\zeta_0, \zeta_1 \in \mathcal{N}$,

$$[\zeta_0] \neq [\zeta_1]. \quad \Rightarrow$$

A 2-dim linear supspace Σ spanned by ζ_0 and ζ_1, or a projective straight line in $\mathcal{M}^{2,4} - \{0\}/\sim$.
Is it in $[\mathcal{N}]$?
$\clubsuit$ ζ_0, $\zeta_1 \in \mathcal{N} \Rightarrow \zeta_0 \cdot \zeta_0 = 0$, $\zeta_1 \cdot \zeta_1 = 0$.
$\forall \zeta \in \Sigma$, $\zeta = x\zeta_0 + y\zeta_1$. Since $\zeta \cdot \zeta = 2xy\, \zeta_0 \cdot \zeta_1$,

$$\Sigma \subset \mathcal{N} \quad \Leftrightarrow \quad \zeta_0 \cdot \zeta_1 = 0 \quad \Leftrightarrow$$

$\Sigma \cap \mathcal{P}$ is a straight line in $\mathcal{N} \cap \mathcal{P}$.

♢ The equation of $\Sigma \cap \mathcal{P}$: Assume ζ_0, $\zeta_1 \in \mathcal{P}$,

$$\zeta = (1-t)\,\zeta_0 + t\,\zeta_1. \tag{71}$$

In terms of the Minkowski/Beltrami coordinates:

$$x^\mu = a^\mu + w(s)\,u^\mu, \qquad \frac{ds}{dw} = \frac{1}{\sigma(x)}. \tag{72}$$

Conserved momentum and angular momentum along geodesics:

$$P^\mu := \frac{1}{\sigma(x)}\frac{dx^\mu}{ds}, \quad L^{\mu\nu} := x^\mu P^\nu - x^\nu P^\mu \tag{73}$$

For BdS,

$$\mathcal{L}^{\mu\nu} = L^{\mu\nu}, \qquad \mathcal{L}^{4\nu} = -\mathcal{L}^{\nu 4} = RP^\nu. \tag{74}$$

♢ For Σ, define $\mathcal{L} = \boldsymbol{\zeta}_0 \wedge \boldsymbol{\zeta}_1$:

$$\mathcal{L}^{\hat{A}\hat{B}} := \zeta_0^{\hat{A}} \zeta_1^{\hat{B}} - \zeta_0^{\hat{B}} \zeta_1^{\hat{A}}. \tag{75}$$

When $\{\boldsymbol{\zeta}_0, \boldsymbol{\zeta}_1\} \to \{\boldsymbol{\zeta}'_0, \boldsymbol{\zeta}'_1\}$, $\boldsymbol{\zeta}'_i = A^j{}_i\, \boldsymbol{\zeta}_j$,

$$\mathcal{L}' = (\det A)\, \mathcal{L}. \tag{76}$$

Σ is specified by $\mathcal{L}^{\hat{A}\hat{B}}$ up to a nonzero scaling constant.

♣ When $\boldsymbol{\zeta}_0$, $\boldsymbol{\zeta}_1 \in \mathcal{P}$, $\Sigma \cap \mathcal{P}$ is

$$\boldsymbol{\zeta}(t) = (1-t)\, \boldsymbol{\zeta}_0 + t\, \boldsymbol{\zeta}_1, \qquad \frac{d\boldsymbol{\zeta}}{dt} = \boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_0.$$

$\mathcal{L}$ is the conserved angular momentum:

$$\mathcal{L} = \boldsymbol{\zeta}(t) \wedge \frac{d\boldsymbol{\zeta}}{dt}. \tag{77}$$

♣ When $\boldsymbol{\zeta}_0 \cdot \boldsymbol{\zeta}_1 = 0$, $\Sigma \cap \mathcal{P}$ is a null geodesic in $\mathcal{N} \cap \mathcal{P}$.

♣ If $\mathcal{N} \cap \mathcal{P}$ is Minkowski, with $\mathcal{P}$ being $\zeta^- = R$,

$$x^\mu(t) = x_0^\mu + tv^\mu, \qquad v^\mu = x_1^\mu - x_0^\mu. \tag{78}$$

Define

$$L^{\mu\nu} := x^\mu(t)v^\nu - x^\nu(t)v^\mu, \qquad P^\mu = v^\mu. \tag{79}$$

Then

$$\mathcal{L}^{\mu\nu} = L^{\mu\nu}, \tag{80}$$

$$\mathcal{L}^{-\mu} = RP^\mu, \tag{81}$$

$$\mathcal{L}^{+\mu} = -\frac{\eta_{\rho\sigma}}{2R}\,(x_0^\rho x_0^\sigma v^\mu - 2x_0^\rho v^\sigma x_0^\mu), \tag{82}$$

$$\mathcal{L}^{+-} = \eta_{\mu\nu} x_0^\mu v^\nu. \tag{83}$$

♣ When $\mathcal{N} \cap \mathcal{P}$ is dS/AdS,

$$x^\mu(t) = x_0^\mu + w(t) v^\mu, \quad v^\mu := x_1^\mu - x_0^\mu, \tag{84}$$

$$w(t) := \frac{t\xi_1^4}{\xi_0^4 + t(\xi_1^4 - \xi_0^4)}. \tag{85}$$

$$P^\mu := \frac{1}{\sigma(x)} \frac{dx^\mu}{dt} = v^\mu, \qquad L^{\mu\nu} := x^\mu P^\nu - x^\nu P^\mu. \tag{86}$$

$$\mathcal{L}^{\mu\nu} = \frac{L^{\mu\nu}}{\sqrt{\sigma(x_0)\,\sigma(x_1)}}, \quad \mathcal{L}^{5\nu} = \pm\left(\frac{x_1^\nu}{\sqrt{\sigma(x_1)}} - \frac{x_0^\nu}{\sqrt{\sigma(x_0)}}\right), \tag{87}$$

$$\mathcal{L}^{4\nu} = \frac{RP^\nu}{\sqrt{\sigma(x_0)\,\sigma(x_1)}}, \quad \mathcal{L}^{45} = \mp\left(\frac{R^2}{\sqrt{\sigma(x_1)}} - \frac{R^2}{\sqrt{\sigma(x_0)}}\right). \tag{88}$$

$\diamondsuit$ Massive case.
(1) Δ: $(1+2)$-dim or $(2+1)$-dim.
(2) $(2+1)$-dim: timelike straight line ($\boldsymbol{n} \in \Delta$) or hyperbola.
(3) $(1+2)$-dim: spacelike straight line ($\boldsymbol{n} \in \Delta$) or conic curve.

6 AdS_5/CFT^3 Correspondence

♢ AdS_5: a "projective" subspace
♢ $[\mathcal{N}] = \partial AdS_5$
♢ $[\mathcal{N}]$: the conformal Minkowski, dS- or AdS-space.
♢ Conformal maps, conformal triality.
♢ CFT on Minkowski = CFT on dS= CFT on AdS.
♢ Three kinds of AdS_5/CFT correspondence.
♢ Can be generalized to AdS_{d+1}/CFT^3_d correspondence.

7 Conclusion

♢ Conformal Minkowski, dS- and AdS-spaces are the same thing.

♢ They are three landscapes from different point of view.

♢ They can be linked mutually to each other and to themselves by conformal maps.

♢ Minkowski coordinates and Beltrami coordinates are at the equal footing.

♢ Physical contents can be moved around from these spaces, provided they are conformally "covariant".

♢ Projective might be "dead" in mathematics, but it is finding a new life in physics.

Thanks!