Mathematical Results Inspired by String Duality

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During the summer of 1985, after the opening ceremony of Nankai Institute of Mathematics, a beautiful lecture of S.-S. Chern on Chern classes and the Atiyah-Singer index formula introduced me into the beautiful fields of geometry and topology.

Today, exactly 20 years later, it is a great honor for me to speak at this opening ceremony of the magnificent new building of Nankai Institute.

I will discuss some results we have obtained recently. All of these results are related to Chern in one way or another.

String Theory, as the unified theory of all fundamental forces, should be unique. But now there are five different looking string theories.

$$I, \ IIA, \ IIB, \ E_8 \times E_8, \ SO(32).$$

Physicists: these theories should be equivalent, in a way dual to each other: their "partition functions" should be "equivalent".

The identifications of computations in different theories have produced many surprisingly beautiful mathematical formulas like the famous **mirror formula**, following from the mirror principle inspired by duality between IIA and IIB string theory.

Let us first briefly look at the mirror formula, then we will discuss several recent results.

Example: Mirror Formula.

Let X be a Calabi-Yau manifold, compact or toric open. We want to compute certain characteristic numbers K_d^g of certain naturally induced vector bundles on the moduli spaces of stable maps of degree d from curves of genus g into X. Actually all of these characteristic numbers are Chern numbers.

Some of these Chern numbers give the numbers of rational curves in X, a problem of very long history in mathematics.

Motivated by the famous Candelas formula from string theory, mathematicians have developed new theories and methods for such computations: Gromov-Witten theory. The theory of moduli spaces of stable maps and integrations on such spaces have been fully developed in mathematics.

Mirror Principle: Compute

$$F(T,\lambda) = \sum_{d,\,g} K_d^g \, \lambda^g \; e^{d\cdot T}$$

in terms of hypergeometric series. Here λ, T formal variables.

Mirror principle (Lian-Liu-Yau 1997-1999) with **explicit formulas** was proved for the most interesting cases:

(1). For toric manifolds and $g = 0 \Longrightarrow$ all mirror conjectural formulas from physics.

(2). Grassmannian manifolds: Hori-Vafa formula: reduced to projective spaces.

(3). Direct sum of positive line bundles on $\mathbf{P}^n$ including the Candelas formula: (by Givental, Lian-Liu-Yau.)

Example: Calabi-Yau quintic X given by the zero locus of a generic section of the line bundle $\mathcal{O}(5)$ on $\mathbf{P}^4$. $\mathcal{O}(5)$ induces a sequence of vector bundles on the moduli spaces. The corresponding hypergeometric series is:

$$HG[B](t) = e^{H\,t} \sum_{d=0}^{\infty} \frac{\prod_{m=0}^{5d}(5H+m)}{\prod_{m=1}^{d}(H+m)^5}\, e^{d\,t},$$

where H: hyperplane class on $\mathbf{P}^4$; t: parameter. Introduce

$$\mathcal{F}(T) = \frac{5}{6}T^3 + \sum_{d>0} K_d^0\, e^{d\,T}.$$

Take expansion in H:

$$HG[B](t) = H\{f_0(t)+f_1(t)H+f_2(t)H^2+f_3(t)H^3\}.$$

Candelas Formula: *With parameter change* $T = f_1(t)/f_0(t)$, *we have*

$$\mathcal{F}(T) = \frac{5}{2}(\frac{f_1}{f_0}\frac{f_2}{f_0} - \frac{f_3}{f_0}).$$

The numbers K_d^0 encode the numbers of rational curves of all degrees in quintic X:

$n_d = 2875, 609250, 317206375, 242467530000...$

The first two numbers already took mathematicians more than 100 years to get. But now we can get all of them with one stroke, through joint efforts with string theorists.

One key point of our proof of mirror principle is to apply functorial localization formula (to be explained later) to connect the computations of mathematicians to that of the physicists. This method has been very successful in proving conjectures from string theory.

Hypergeometric series naturally appear from localizations on certain linear moduli spaces, a sequence of projective spaces in the quintic case.

Today I will give an overview of some recent results we obtained in proving conjectures originated from the duality between large N Chern-Simons gauge theory and Calabi-Yau in string theory.

The mathematical proofs of these results also depend on **Localization Techniques** on various finite dimensional moduli spaces. More precisely integrals of Chern classes on moduli spaces.

These are related to various mathematics: Chern-Simons knot invariants, combinatorics of symmetric groups, Kac-Moody algebras' representations, Calabi-Yau, geometry and topology of moduli space of stable maps....

We use again the simple *Functorial Localization* to transfer computations on complicated spaces to simple spaces.

I will first talk about the Mariño-Vafa formula, and then the other results.

Papers Containing the Results:

(1). *A Proof of a conjecture of Mariño-Vafa on Hodge Integrals*, JDG 2003.

(2). *A Formula of Two Partition Hodge Integrals*, math.AG/0310273.

C.-C. Liu, K. Liu and J. Zhou.

(3). *A Mathematical Theory of Topological Vertex*, math.AG/0408426.

(4). *Topological String Partition Functions as Equivariant Indices*, math.AG/0412089.

J. Li, C.-C. Liu, K. Liu and J. Zhou.

(5). *A Simple Proof of the Witten Conjecture through Localization*, math.AG/0508384.

Y.-S. Kim and K. Liu

Brief Description of the Results:

(1). *Duality:* Gauge theory, Chern-Simons $\Longleftrightarrow$ Calabi-Yau in String theory.

(2). Differential equation: *cut-and-join* equation encoded in the moduli spaces and in the combinatorics of symmetric groups.

(3). Mathematical theory of *Topological Vertex*: Vafa group's works on duality and geometric engineering for the past several years.

(4). *Integrality* in GW invariants $\Leftrightarrow$ *Indices* of elliptic operators in Gauge theory. (Gopakumar-Vafa conjecture).

(5). Simple proof of the Witten conjecture *by localization and asymptotics.*

The Mariño-Vafa Conjecture.

It is an important problem to compute Hodge integrals (i.e. intersection numbers of λ classes and ψ classes) on the Deligne-Mumford moduli space of stable curves $\overline{\mathcal{M}}_{g,h}$.

A point in $\overline{\mathcal{M}}_{g,h}$ consists of $(C, x_1, \ldots, x_h)$, a (nodal) curve and h smooth points on C.

The Hodge bundle $\mathbb{E}$ is a rank g vector bundle over $\overline{\mathcal{M}}_{g,h}$ whose fiber over $[(C, x_1, \ldots, x_h)]$ is holomorphic one forms $H^0(C, \omega_C)$. The λ classes are Chern Classes:

$$\lambda_i = c_i(\mathbb{E}) \in H^{2i}(\overline{\mathcal{M}}_{g,h}; \mathbb{Q}).$$

The cotangent line $T^*_{x_i}C$ of C at the i-th marked point x_i gives a line bundle $\mathbb{L}_i$ over $\overline{\mathcal{M}}_{g,h}$. The ψ classes are also Chern classes:

$$\psi_i = c_1(\mathbb{L}_i) \in H^2(\overline{\mathcal{M}}_{g,h}; \mathbb{Q}).$$

Introduce total Chern class

$$\Lambda_g^\vee(u) = u^g - \lambda_1 u^{g-1} + \cdots + (-1)^g \lambda_g.$$

Mariño-Vafa formula: Generating series of triple Hodge integrals for all genera:

$$\int_{\overline{\mathcal{M}}_{g,h}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(\tau)\Lambda_g^\vee(-\tau-1)}{\prod_{i=1}^h (1-\mu_i\psi_i)},$$

can be expressed by close formulas of **finite** expression in terms of representations of symmetric groups, or Chern-Simons knot invariants. Here τ is a parameter.

Conjectured from large N duality between Chern-Simons and string theory.

Remark: Mumford first computed some low genus intersection numbers in early 80s. Witten conjecture in early 90s is about the integrals of the ψ classes.

Conifold transition: Resolve singularity in two ways:

Conifold X

$$\left\{\begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathbf{C}^4 : xw - yz = 0\right\}$$

(1). Deformed conifold T^*S^3

$$\left\{\begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathbf{C}^4 : xw - yz = \epsilon\right\}$$

(ϵ real positive number)

(2). Resolved conifold $\tilde{X} = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{P}^1$

$$\left\{([Z_0, Z_1], \begin{pmatrix} x & y \\ z & w \end{pmatrix}) \in \mathbf{P}^1 \times \mathbf{C}^4 : \begin{array}{l} (x, y) \in [Z_0, Z_1] \\ (z, w) \in [Z_0, Z_1] \end{array}\right\}$$

$$\begin{array}{ccc} \tilde{X} & \subset & \mathbf{P}^1 \times \mathbf{C}^4 \\ \downarrow & & \downarrow \\ X & \subset & \mathbf{C}^4 \end{array}$$

Witten 92: The open topological string theory on the deformed conifold T^*S^3 is equivalent to Chern-Simons gauge theory on S^3.

Gopakumar-Vafa 98, Ooguri-Vafa 00: The open topological string theory on the deformed conifold T^*S^3 is equivalent to the closed topological string theory on the resolved conifold $\tilde{X}$.

Vafa and his collaborators 98-04: For the past several years, Vafa et al developed these duality ideas into the most powerful and effective tool to get closed formulas for the Gromov-Witten invariants on all local toric Calabi-Yau manifolds: *Topological Vertex.*

We have a rather complete mathematical theory of topological vertex. Start with Mariño-Vafa formula.

Mathematical Consequence of the Duality:
Chern-Simons partition function:

$$\langle Z(U,V)\rangle = \text{exp}(-F(\lambda,t,V))$$

U: holonomy of the $U(N)$ Chern-Simons gauge field around the knot $K \subset S^3$; V: $U(M)$ matrix

$\langle Z(U,V)\rangle$: Chern-Simons knot invariants of K.

$F(\lambda,t,V)$: Generating series of the open Gromov-Witten invariants of $(\tilde{X},L_K)$, where L_K is a Lagrangian submanifold of the resolved conifold $\tilde{X}$ "canonically associated to" the knot K. (Taubes).

t'Hooft large N expansion, and canonical identifications of parameters similar to mirror formula: duality identification at level k:

$$\lambda = \frac{2\pi}{k+N}, \quad t = \frac{2\pi i N}{k+N}.$$

Special case: When K is the unknot, $\langle Z(U,V)\rangle$ was computed in the zero framing by Ooguri-Vafa and in any framing $\tau \in \mathbb{Z}$ by Mariño-Vafa.

Comparing with Katz-Liu's computations of $F(\lambda, t, V)$, Mariño-Vafa conjectured a striking formula about triple Hodge integrals in terms of Chern-Simons: representations and combinatorics of symmetric groups.

The framing in Mariño-Vafa's computations corresponds to choice of the circle action on the pair $(\tilde{X}, L_{\text{unknot}})$ in Katz-Liu's localization computations. Both choices are parametrized by an integer τ.

Remark: Iqbal conjecture and Zhou's conjecture of the two partition analogue of the Mariño-Vafa conjecture, in between the MV conjecture and topological vertex.

The Mariño-Vafa Formula:

Geometric side: For every partition $\mu = (\mu_1 \geq \cdots \geq \mu_{l(\mu)} \geq 0)$, define triple Hodge integral:

$$G_{g,\mu}(\tau) = A(\tau) \cdot \int_{\overline{\mathcal{M}}_{g,l(\mu)}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(-\tau-1)\Lambda_g^\vee(\tau)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)},$$

with

$$A(\tau) = -\frac{\sqrt{-1}^{|\mu|+l(\mu)}}{|\operatorname{Aut}(\mu)|}[\tau(\tau+1)]^{l(\mu)-1} \prod_{i=1}^{l(\mu)} \frac{\prod_{a=1}^{\mu_i-1}(\mu_i\tau+a)}{(\mu_i-1)!}.$$

Introduce generating series

$$G_\mu(\lambda;\tau) = \sum_{g\geq 0} \lambda^{2g-2+l(\mu)} G_{g,\mu}(\tau).$$

Special case when $g = 0$:

$$\int_{\overline{\mathcal{M}}_{0,l(\mu)}} \frac{\Lambda_0^\vee(1)\Lambda_0^\vee(-\tau-1)\Lambda_0^\vee(\tau)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)}$$

$$= \int_{\overline{\mathcal{M}}_{0,l(\mu)}} \frac{1}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)} = |\mu|^{l(\mu)-3}$$

for $l(\mu) \geq 3$, and we use this expression to extend the definition to the case $l(\mu) < 3$.

Introduce formal variables $p = (p_1, p_2, \ldots, p_n, \ldots)$, and define

$$p_\mu = p_{\mu_1} \cdots p_{\mu_{l(\mu)}}$$

for any partition μ. ($\Leftrightarrow \mathrm{Tr}\, V^{\mu_j}$)

Generating series for all genera and all possible marked points:

$$G(\lambda; \tau; p) = \sum_{|\mu| \geq 1} G_\mu(\lambda; \tau) p_\mu.$$

Representation side: χ_μ: the character of the irreducible representation of symmetric group $S_{|\mu|}$ indexed by μ with $|\mu| = \sum_j \mu_j$,

$C(\mu)$: the conjugacy class of $S_{|\mu|}$ indexed by μ.

Introduce:

$$\mathcal{W}_\mu(\lambda) = \prod_{1\le a<b\le l(\mu)} \frac{\sin\left[(\mu_a - \mu_b + b - a)\lambda/2\right]}{\sin\left[(b-a)\lambda/2\right]}$$

$$\cdot \frac{1}{\prod_{i=1}^{l(\nu)} \prod_{v=1}^{\mu_i} 2\sin\left[(v - i + l(\mu))\lambda/2\right]}.$$

This has an interpretation in terms of *quantum dimension* in Chern-Simons knot theory.

Define:

$$R(\lambda;\tau;p) = \sum_{n\ge 1} \frac{(-1)^{n-1}}{n} \sum_\mu \Big[\sum_{\cup_{i=1}^n \mu^i = \mu}$$

$$\prod_{i=1}^n \sum_{|\nu^i|=|\mu^i|} \frac{\chi_{\nu^i}(C(\mu^i))}{z_{\mu^i}} e^{\sqrt{-1}(\tau+\frac{1}{2})\kappa_{\nu^i}\lambda/2} \mathcal{W}_{\nu^i}(\lambda)\Big] p_\mu$$

where μ^i are sub-partitions of μ, $z_\mu = \prod_j \mu_j! j^{\mu_j}$ and $\kappa_\mu = |\mu| + \sum_i (\mu_i^2 - 2i\mu_i)$ for a partition μ: standard for representations of symmetric groups.

Theorem: (Liu-Liu-Zhou 2003) *Mariño-Vafa Conjecture is true, that is:*

$$G(\lambda;\tau;p) = R(\lambda;\tau;p).$$

Remark: (0). This is a formula:

G: Geometry = R: Representations

Representations of symmetric groups are essentially combinatorics.

(1). Witten conjecture is about KdV equations. But the Mariño-Vafa formula gives *closed formula!*

(2). Equivalent expression:

$$G(\lambda;\tau;p)^{\bullet} = \exp\left[G(\lambda;\tau;p)\right] = \sum_{\mu} G(\lambda;\tau)^{\bullet} p_{\mu} =$$

$$\sum_{|\mu|\geq 0} \sum_{|\nu|=|\mu|} \frac{\chi_{\nu}(C(\mu))}{z_{\mu}} e^{\sqrt{-1}(\tau+\frac{1}{2})\kappa_{\nu}\lambda/2} \mathcal{W}_{\nu}(\lambda) p_{\mu}$$

(3). Each $G_\mu(\lambda,\tau)$ is given by a **finite and closed** expression in terms of representations of symmetric groups:

$$G_\mu(\lambda,\tau) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \sum_{\cup_{i=1}^n \mu^i = \mu} \prod_{i=1}^n$$

$$\sum_{|\nu^i|=|\mu^i|} \frac{\chi_{\nu^i}(C(\mu^i))}{z_{\mu^i}} e^{\sqrt{-1}(\tau+\frac{1}{2})\kappa_{\nu^i}\lambda/2} \mathcal{W}_{\nu^i}(\lambda).$$

$G_\mu(\lambda,\tau)$ gives triple Hodge integrals for moduli spaces of curves of all genera with $l(\mu)$ marked points.

(4). Mariño-Vafa formula gives explicit values of many interesting Hodge integrals up to three Hodge classes:

• Taking limit $\tau \longrightarrow 0$ we get the λ_g conjecture (Faber-Pandhripande: Annals of Math 2005),

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1}\cdots\psi_n^{k_n}\lambda_g = \binom{2g+n-3}{k_1,\ldots,k_n} \frac{2^{2g-1}-1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!},$$

for $k_1+\cdots+k_n=2g-3+n$, and the following identity for Hodge integrals:

$$\int_{\overline{\mathcal{M}}_g}\lambda_{g-1}^3=\int_{\overline{\mathcal{M}}_g}\lambda_{g-2}\lambda_{g-1}\lambda_g$$

$$=\frac{1}{2(2g-2)!}\frac{|B_{2g-2}|}{2g-2}\frac{|B_{2g}|}{2g},$$

B_{2g} are Bernoulli numbers. And other identities.

• Taking limit $\tau\longrightarrow\infty$, we get the famous ELSV formula which relates the generating series of Hurwitz numbers to Hodge integrals.

• Taking limit $\mu_i\longrightarrow\infty$ again in ELSV, Okounkov-Pandhripande gave a different approach to the Witten conjecture proved by Kontsevich.

I will present a simple and direct proof of the Witten conjecture from localization and asymptotics.

The idea to prove the Mariño-Vafa formula is to prove that both G and R satisfy the *Cut-and-Join* equation:

Theorem : *Both R and G satisfy the following differential equation:*

$$\frac{\partial F}{\partial \tau} = \frac{1}{2}\sqrt{-1}\lambda \sum_{i,j=1}^{\infty}\left((i+j)p_i p_j \frac{\partial F}{\partial p_{i+j}} + ijp_{i+j}\left(\frac{\partial F}{\partial p_i}\frac{\partial F}{\partial p_j} + \frac{\partial^2 F}{\partial p_i \partial p_j}\right)\right)$$

Initial Value: $\tau = 0$, Ooguri-Vafa formula:

$$G(\lambda, 0, p) = \sum_{d=1}^{\infty} \frac{p_d}{2d \sin\left(\frac{\lambda d}{2}\right)} = R(\lambda, 0, p).$$

This is a linear systems of ODE.

The solution is unique!

$$G(\lambda; \tau; p) = R(\lambda; \tau; p).$$

Cut-and-Join operator denoted by (CJ) in variables p_j on the right hand side: a beautiful match of Combinatorics and Geometry from collecting the following operations:

Combinatorics: Cut and join of cycles:

Cut: a k-cycle is cut into an i-cycle and a j-cycle, denote the set by $C(\mu)$:

Join: an i-cycle and a j-cycle are joined to an $(i+j)$-cycle, denote the set by $J(\mu)$:

Geometry: How curves stably vary,

Cut: One curve split into two lower degree or lower genus curves.

Join: Two curves joined together to give a higher genus or higher degree curve.

Remark: Cut-and-join equation is encoded in the geometry of the moduli spaces of stable maps: convolution formula of the form: (disconnected version: $G^\bullet = \exp G$)

$$G^\bullet_\mu(\lambda, \tau) = \sum_{|\nu|=|\mu|} \Phi^\bullet_{\mu,\nu}(-\sqrt{-1}\tau\lambda) z_\nu K^\bullet_\nu(\lambda)$$

where $\Phi^\bullet_{\mu,\nu}$ is series of double Hurwitz numbers, z_ν the combinatorial constants. Equivalently this gives the explicit solution of the cut-and-join equation, with initial value $K^\bullet(\lambda)$, the integrals of Euler classes on moduli of relative stable maps.

The proof of the combinatoric cut-and-join equation is a direct computation.

The proof of the geometric cut-and-join equation used *Functorial Localization Formula*:

$f: X \to Y$ equivariant map. $i_F: F \hookrightarrow Y$ a fixed component, $i_E: E \hookrightarrow f^{-1}(F)$ fixed components in $f^{-1}(F)$. Let $f_0 = f|_E$, then

Lemma: *For $\omega \in H_T^*(X)$ an equivariant cohomology class, we have identity on F:*

$$f_{0*}\{\frac{i_E^*\omega}{e_T(E/X)}\} = \frac{i_F^*(f_*\omega)}{e_T(F/Y)}.$$

Here e_T denotes the equivariant Euler class of the normal bundle. f_* and i^* denote push-forward and pull-back respectively.

This formula is similar to Riemann-Roch. It has been applied to various settings to prove various conjectures from physics. It is used to push computations on complicated moduli space to simpler moduli space: the proof of the mirror formula; the proof of the Hori-Vafa formula; the proof of the ELSV formula....

In each case we have natural equivariant maps from the complicated moduli spaces to much simpler spaces like projective spaces.

In our first proof of the Mariño-Vafa formula we used the moduli spaces of relative stable maps to $\mathbf{P}^1$ as introduced by J. Li and natural branch maps to projective spaces.

Remarks: (1). The cut-and-join equation is closely related to the Virasoro algebra.

(2). Other later approaches:

(a). Direct derivation of convolution formula (Liu-Liu-Zhou).

(b). Okounkov-Pandhripande: use ELSV formula, λ_g conjecture, and nontrivial combinatorics.

The Mariño-Vafa formula can be viewed as a duality: **Chern-Simons** $\Longleftrightarrow$ **Calabi-Yau.**

Can we go further with the ideas and methods?

Duality $\Leftrightarrow$ convolution and cut-and-join.

Yes much more!

One, two, three partitions.

Mariño-Vafa: one partition case....

Topological vertex: three partition case.

Generating series of Hodge integrals with more partitions $\Leftrightarrow$ closed formulas in terms of Chern-Simons invariants of Hopf link and more....

Topological Vertex Theory.

Mirror symmetry used periods and holomorphic anomaly to compute Gromov-Witten series, difficult for higher genera. Topological vertex gives complete answers for all genera and all degrees in the local Calabi-Yau cases in terms of Chern-Simons knot invariants.

Topological Vertex is related to a three partition analogue of the Mariño-Vafa formula. This formula gives closed formula for the generating series of the Hodge integrals involving three partitions. The corresponding cut-and-join equation has the form:

$$\frac{\partial}{\partial\tau}F^{\bullet} = (CJ)^1F^{\bullet} + \frac{1}{\tau^2}(CJ)^2F^{\bullet} + \frac{1}{(\tau+1)^2}(CJ)^3F^{\bullet}$$

where (CJ) denotes the cut-and-join operator with respect to the three groups of infinite numbers of variables associated to the three partitions.

We first derived the convolution formulas both in combinatorics and in geometry. Then we proved the identity of initial values at $\tau = 1$. Much more complicated in both geometry and combinatorics. We then derived all of its basic properties for computations, like the gluing formula.

By using gluing formula of the topological vertex, we can derive closed formulas for generating series of GW invariants, all genera and all degrees, open or closed, for all local toric Calabi-Yau, in terms Chern-Simons invariants, by simply looking at the moment map graphs of the Calabi-Yau.

Let us look at an example to see the computational power of topological vertex.

Let $N_{g,d}$ denote the GW invariants of a local toric CY, total space of canonical bundle on a toric surface S: the Euler number of the obstruction bundle on the moduli space $\overline{\mathcal{M}}_g(S,d)$ of stable maps of degree $d \in H_2(S,\mathbb{Z})$ from genus g curve into the surface S:

$$N_{g,d} = \int_{[\overline{\mathcal{M}}_g(S,d)]^v} e(V_{g,d})$$

with $V_{g,d}$ a vector bundle induced by the canonical bundle K_S: at point $(\Sigma; f) \in \overline{\mathcal{M}}_g(S,d)$, its fiber is $H^1(\Sigma, f^*K_S)$. Write

$$F_g(t) = \sum_d N_{g,d}\, e^{-d\cdot t}.$$

Example: Topological vertex formula of GW generating series in terms of Chern-Simons invariants. For the total space of canonical bundle $\mathcal{O}(-3)$ on $\mathbf{P}^2$:

$$\exp\Big(\sum_{g=0}^{\infty} \lambda^{2g-2} F_g(t)\Big) = \sum_{\nu_1,\nu_2,\nu_3} \mathcal{W}_{\nu_1,\nu_2}\mathcal{W}_{\nu_2,\nu_3}\mathcal{W}_{\nu_3,\nu_1}\cdot$$

$$(-1)^{|\nu_1|+|\nu_2|+|\nu_3|} q^{\frac{1}{2}\sum_{i=1}^{3}\kappa_{\nu_i}}\, e^{t(|\nu_1|+|\nu_2|+|\nu_3|)}.$$

Here $q = e^{\sqrt{-1}\lambda}$, and $\mathcal{W}_{\mu,\nu}$ are from the Chern-Simons knot invariants of Hopf link. Sum over three partitions $\nu_1,\ \nu_2,\ \nu_3$.

Three vertices of moment map graph of $\mathbf{P}^2 \leftrightarrow$ three $\mathcal{W}_{\mu,\nu}$'s.

For general local toric Calabi-Yau, the expressions are just similar: **closed formulas**.

Another interesting conjecture from string theory:

Gopakumar-Vafa conjecture: *There exists expression:*

$$\sum_{g=0}^{\infty} \lambda^{2g-2} F_g(t) = \sum_{k=1}^{\infty} \sum_{g,d} n_d^g \frac{1}{d} (2\, \mathit{sin} \frac{d\lambda}{2})^{2g-2} e^{-kd\cdot t},$$

such that n_d^g *are integers, called instanton numbers.*

For some interesting cases we can interpret the n_d^g as equivariant indices of twisted Dirac operators on moduli spaces of anti-self-dual connections on $\mathbf{C}^2$, related to the Nekrasov conjecture. (Li-Liu-Zhou)

By using the Chern-Simons knot invariant expressions from topological vertex, the following theorem was first proved by Peng Pan:

Theorem: *The Gopakumar-Vafa conjecture is true for all (formal) local toric Calabi-Yau for all degree and all genera.*

We have seen close connection between knot invariants and Gromov-Witten invariants. There should be a more interesting and grand duality picture between Chern-Simons invariants for real three dimensional manifolds and Gromov-Witten invariants for complex three dimensional toric Calabi-Yau.

General correspondence between real dimension 3 and complex dimension 3?!.

The Witten Conjecture.

The Witten conjecture for moduli spaces states that the generating series F of the integrals of the ψ classes for all genera and any number of marked points satisfies the KdV equations and the Virasoro constraint. For example the Virasoro constraint states that F satisfies

$$L_n \cdot F = 0, \ n \geq -1$$

where L_n denote certain Virasoro operators.

Witten conjecture was first proved by Kontsevich using combinatorial model of the moduli space and matrix model, with later approaches by Okounkov-Pandhripande using ELSV formula and combinatorics, by Mirzakhani using Weil-Petersson volumes on moduli spaces of bordered Riemann surfaces.

I will present a much simpler proof by using functorial localization and asymptotics, jointly with Y.-S. Kim. This is also motivated by methods in proving conjectures from string duality. It should have more applications.

Motivated by our proof of the Mariño-Vafa formula, Y.-S. Kim has derived a beautiful recursion formula for one Hodge integrals of all genera and any numbers of marked points, which should have more interesting corollaries about Hodge integrals.

The basic idea of our proof is to directly prove the following recursion formula which, as derived in physics by Dijkgraaf using quantum field theory, implies the Virasoro and the KdV equation for the generating series F of the integrals of the ψ classes:

Theorem: *We have identity*

$$\langle\tilde{\sigma}_n \prod_{k\in S}\tilde{\sigma}_k\rangle_g = \sum_{k\in S}(2k+1)\langle\tilde{\sigma}_{n+k-1}\prod_{l\neq k}\tilde{\sigma}_l\rangle_g +$$

$$\frac{1}{2}\sum_{a+b=n-2}\langle\tilde{\sigma}_a\tilde{\sigma}_b\prod_{l\neq a,b}\tilde{\sigma}_l\rangle_{g-1} +$$

$$\frac{1}{2}\sum_{\substack{S=X\cup Y,\\ a+b=n-2,\\ g_1+g_2=g}}\langle\tilde{\sigma}_a\prod_{k\in X}\tilde{\sigma}_k\rangle_{g_1}\langle\tilde{\sigma}_b\prod_{l\in Y}\tilde{\sigma}_l\rangle_{g_2}.$$

Here $\tilde{\sigma}_n = (2n+1)!!\psi^n$ and

$$\langle\prod_{j=1}^{n}\tilde{\sigma}_{k_j}\rangle_g = \int_{\overline{\mathcal{M}}_{g,n}}\prod_{j=1}^{n}\tilde{\sigma}_{k_j}.$$

The notation $S = \{k_1, \cdots, k_n\} = X \cup Y$.

To prove the above recursion relation, similar to the proof of the Mariño-Vafa formula, we first apply the functorial localization to the natural branch map from moduli space of relative stable maps $\overline{\mathcal{M}}_g(\mathbb{P}^1, \mu)$ to projective space $\mathbf{P}^r$ where $r = 2g-2+|\mu|+l(\mu)$ is the dimension of the moduli.

We easily get the cut-and-join equation for one Hodge integral

$$I_{g,\mu} = \frac{1}{|\text{Aut } \mu|} \prod_{i=1}^{n} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\Lambda_g^\vee(1)}{\prod(1-\mu_i\psi_i)}.$$

The equation we get has the form:

$$I_{g,\mu} = \sum_{\nu \in J(\mu)} I_1(\nu) I_{g,\nu} + \sum_{\nu \in C(\mu)} I_2(\nu) I_{g-1,\nu} +$$

$$\sum_{g_1+g_2=g} \sum_{\nu^1 \cup \nu^2 \in C(\mu)} I_3(\nu^1, \nu^2) I_{g_1,\nu^1} I_{g_2,\nu^2}$$

where $I(\nu)$, $I(\nu^1, \nu^2)$ are some coefficients. This formula was first derived by Liu-Liu-Zhou. More general formulas of such type was first found and proved by Kim.

Write $\mu_i = N x_i$. Let N goes to infinity and expand in x_i, we get:

$$\sum_{i=1}^{n} \Big[\frac{(2k_i+1)!!}{2^{k_i+1} k_i!} x_i^{k_i} \prod_{j \neq i} \frac{x_j^{k_j - \frac{1}{2}}}{\sqrt{2\pi}} \int_{\overline{\mathcal{M}}_{g,n}} \prod \psi_j^{k_j} -$$

$$\sum_{j \neq i} \frac{(x_i + x_j)^{k_i + k_j - \frac{1}{2}}}{\sqrt{2\pi}} \prod_{l \neq i,j} \frac{x_l^{k_l - \frac{1}{2}}}{\sqrt{2\pi}} \int_{\overline{\mathcal{M}}_{g,n-1}} \psi^{k_i + k_j - 1} \prod \psi_l^{k_l}$$

$$-\frac{1}{2} \sum_{k+l = k_i - 2} \frac{(2k+1)!!(2l+1)!!}{2^{k_i} k_i!} x_i^{k_i} \prod_{j \neq i} \frac{x_j^{k_j - \frac{1}{2}}}{\sqrt{2\pi}}$$

$$\Big[\int_{\overline{\mathcal{M}}_{g-1,n+1}} \psi_1^k \psi_2^l \prod \psi_j^{k_j} +$$

$$\sum_{\substack{g_1 + g_2 = g, \\ \nu_1 \cup \nu_2 = \nu}} \int_{\overline{\mathcal{M}}_{g_1,n_1}} \psi_1^k \prod \psi_j^{k_j} \int_{\overline{\mathcal{M}}_{g_2,n_2}} \psi_1^l \prod \psi_j^{k_j} \Big] \Big] = 0$$

Performing Laplace transforms on the x_i's, we get the recursion formula which implies both the KdV equations and the Virasoro constraints.

THANK YOU ALL!