Energy in Gravitational Waves

Xiao Zhang
Institute of Mathematics
Academy of Mathematics and Systems Science
Chinese Academy of Sciences

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1 Gravitation waves

Gravitational waves:

(i) Wave-like spacetime metrics - time dependent solutions of the Einstein field equations;

(ii) Radiation or transport of energy.

Gravitational waves have not yet been detected.

An indirect proof of the existence of gravitational waves comes from observations of the pulsar PSR 1913+16. This binary system rotate rapidly, therefore should emit appreciable amounts of gravitational quadrupole radiation, hence lose energy and rotate faster. The observed relative change in period of $-2.422(\pm 0.006) \cdot 10^{-12}$ is in agreement with the theoretical value remarkably.

We try to understand the following two questions:

1. Can gravitational waves carry away more energy than they have initially in isolated gravitational systems?

2. How does the energy transport when an isolated gravitational system moves from the low speed towards the light speed?

2 Einstein field equations

Spacetime $(L^{3,1}, \mathbf{g})$**:** $L^{3,1}$ is a 4-dimensional manifold, $\mathbf{g}$ is a Lorentzian metric which satisfies **the Einstein field equations**

$$Ric(\mathbf{g}) - \frac{1}{2}\, R(\mathbf{g})\, \mathbf{g} = T.$$

where T is the energy-momentum tensor.

Vacuum: $T = 0$.

Exact vacuum solutions:

(i) Minkowski spacetime:

$$\mathbf{g}_{Mink} = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\psi^2).$$

(ii) Schwarzschild spacetime:

$$\begin{aligned} \mathbf{g}_{Sch} &= -(1-\frac{2m}{r})dt^2 + (1-\frac{2m}{r})^{-1}dr^2 \\ &\quad + r^2(d\theta^2 + \sin^2\theta d\psi^2) \end{aligned}$$

where constant m is the total mass.

(iii) Kerr spacetime:

$$\begin{aligned} \mathbf{g}_{Kerr} &= -\Big(1-\frac{2mr}{\Sigma}\Big)dt^2 - \frac{4mar\sin^2\theta}{\Sigma}dtd\varphi \\ &\quad + \frac{\Sigma}{\triangle}dr^2 + \Sigma d\theta^2 \\ &\quad + \Big(r^2 + a^2 + \frac{2mra^2\sin^2\theta}{\Sigma^2}\Big)\sin^2\theta d\varphi^2 \end{aligned}$$

where

$$\Sigma \equiv r^2 + a^2\cos^2\theta, \quad \triangle \equiv r^2 - 2mr + a^2$$

and constant m is the total mass, constant a is the total angular momentum per unit mass.

3 The positive mass theorem at spatial infinity

Asymptotically flat spacetimes: Asymptotic to the Minkowski spacetime in certain sense. For example, the Schwarzschild spacetime.

Spatial infinity: Infinity of a complete noncompact spacelike hypersurface.

Basic quantities (Arnowitt-Deser-Misner, etc.): Total energy, total linear momentum, total angular momentum at spatial infinity.

Asymptotically flat initial data set: Let (M^3, g, h) be a spacelike hypersurface in $L^{3,1}$ where g is the induced Riemannian metric and h is the second fundamental form. Outside a compact subset, M is diffeomorphic to $R^3 \backslash B_R$ and g, h satisfy

$$g_{ij} = \delta_{ij} + O(\frac{1}{r}), \partial_k g_{ij} = O(\frac{1}{r^2}), \partial_l \partial_k g_{ij} = O(\frac{1}{r^3}), h_{ij} = O(\frac{1}{r^2}), \partial_k h_{ij} = O(\frac{1}{r^3}).$$

ADM total energy-momentum: The total energy E and the total linear momentum P_k of an asymptotically flat initial data set are defined as

$$\begin{aligned} E &= \frac{1}{16\pi} \lim_{r\to\infty} \int_{S_r} (\partial_j g_{ij} - \partial_i g_{jj}) * dx^i, \\ P_k &= \frac{1}{8\pi} \lim_{r\to\infty} \int_{S_r} (h_{ki} - g_{ki} h_{jj}) * dx^i, \end{aligned}$$

where S_r is the sphere of radius r in R^3.

The dominant energy condition: For any timelike vector w,

(i) $T_{uv} w^u w^v \geq 0$;

(ii) $T^{uv} w_u$ is a non-spacelike vector.

Restricted on (M^3, g, h), the dominant energy condition implies that

$$\frac{1}{2}\Big(R + (h^i{}_i)^2 - h_{ij}h^{ij}\Big) \geq \sqrt{\sum_i (\nabla^j h_{ij} - \nabla_i h^j{}_j)^2}.$$

The Schoen-Yau's positive mass theorem (Schoen-Yau, 1979; Witten, 1981): If the spacetime satisfies the dominant energy condition, then, for asymptotically flat initial data set (M^3, g, h),

$$E \geq \sqrt{\sum_k P_k^2}.$$

Equality implies that $L^{3,1}$ is flat over M.

In 1999, **Xiao Zhang** defined the total angular momentum in the global sense and generalized the positive mass theorem to the spacetimes including the total angular momentum.

The idea is to prove a positive mass theorem for a nonsymmetric initial data set (M^3, g, p) where p is not necessarily symmetric.

This basically relates to the Einstein-Cartan theory in general relativity − When the spacetime connection has torsion, the second fundamental forms of a spacelike hypersurface is not necessary symmetric.

4 The positive mass conjecture at null infinity – Question 1

Most physical systems cannot radiate away more energy than they have initially. This is usually a trivial consequence of a conserved stress-energy tensor with a positive timelike component.

However, the gravitational field does not have a well-defined stress-energy tensor. It is possible that an isolated gravitational system might be able to radiate arbitrarily large amounts of energy.

The positive mass conjecture at null infinity: An isolated gravitational system cannot radiate away more energy than they have initially.

There is no mathematical setting available of this conjecture in general spacetimes except in the vacuum Bondi's radiating spacetime.

Bondi's radiating spacetime: A spacetime equipped with the following *Bondi's radiating metric*

$$\begin{aligned}
\mathbf{g}_{Bondi} \;=\; & -\Big(-\frac{V}{r}e^{2\beta} + r^2 e^{2\gamma} U^2 \cosh 2\delta \\
& + r^2 e^{-2\gamma} W^2 \cosh 2\delta + 2r^2 UW \sinh 2\delta\Big) du^2 \\
& -2e^{2\beta} du dr \\
& -2r^2 \Big(e^{2\gamma} U \cosh 2\delta + W \sinh 2\delta\Big) du d\theta \\
& -2r^2 \Big(e^{-2\gamma} W \cosh 2\delta + U \sinh 2\delta\Big) \sin\theta du d\psi \\
& + r^2 \Big(e^{2\gamma} \cosh 2\delta d\theta^2 + e^{-2\gamma} \cosh 2\delta \sin^2\theta d\psi^2 \\
& + 2 \sinh 2\delta \sin\theta d\theta d\psi\Big)
\end{aligned}$$

where $\beta, \gamma, \delta, U, V, W$ are functions of

$$x^0 = u, \;\; x^1 = r, \;\; x^2 = \theta, \;\; x^3 = \psi$$

which are smooth for $r \geq r_0 > 0$, $0 \leq \theta \leq \pi$, $0 \leq \psi \leq 2\pi$. u is a retarded coordinate, r is Euclidean distance, θ and ψ are spherical coordinates.

Physically, $u = constant$ **requires to be null hypersurfaces.**

In the Minkowski spacetime, $u = t - r$,

$$\mathbf{g}_{Mink} = -du^2 - 2dudr + r^2(d\theta^2 + \sin^2\theta d\psi^2).$$

In the Schwarzschild spacetime, $u = t - r - 2m \ln |r - 2m|$,

$$\mathbf{g}_{Sch} = -(1 - \frac{2m}{r})du^2 - 2dudr + r^2(d\theta^2 + \sin^2\theta d\psi^2).$$

It is still open what u is in the Kerr spacetime.

If the vacuum Bondi's radiating spacetime satisfies the *outgoing radiation conditions*, then, as $r \to \infty$,

$$\begin{aligned}
\gamma &= \frac{c(u,\theta,\psi)}{r} + O(\frac{1}{r^3}), \\
\delta &= \frac{d(u,\theta,\psi)}{r} + O(\frac{1}{r^3}), \\
\beta &= -\frac{c^2 + d^2}{4r^2} + O(\frac{1}{r^4}), \\
U &= -\frac{l(u,\theta,\psi)}{r^2} + O(\frac{1}{r^3}), \\
W &= -\frac{\bar{l}(u,\theta,\psi)}{r^2} + O(\frac{1}{r^3}), \\
V &= -r + 2M(u,\theta,\psi) + O(\frac{1}{r}),
\end{aligned}$$

where

$$\begin{aligned}
l &= c_{,2} + 2c\cot\theta + d_{,3}\csc\theta, \\
\bar{l} &= d_{,2} + 2d\cot\theta - c_{,3}\csc\theta
\end{aligned}$$

and M, c and d are functions of u, θ and ψ.

Under these conditions, the vacuum Bondi's radiating metric has the following asymptotic form

$$\begin{aligned}
\mathbf{g}_{Bondi} \quad = \quad & -\Big(1 - \frac{2M}{r} + O(\frac{1}{r^2})\Big)du^2 \\
& -2\Big(1 - \frac{c^2+d^2}{4r^2} + O(\frac{1}{r^4})\Big)dudr \\
& +2\Big(l + \frac{2cl + 2d\bar{l}}{r} + O(\frac{1}{r^2})\Big)dud\theta \\
& +2\Big(\bar{l} - \frac{2c\bar{l} - 2dl}{r} + O(\frac{1}{r^2})\Big)\sin\theta dud\psi \\
& +r^2\Big(1 + \frac{2c}{r} + O(\frac{1}{r^2})\Big)d\theta^2 \\
& +r^2\Big(1 - \frac{2c}{r} + O(\frac{1}{r^2})\Big)\sin^2\theta d\psi^2 \\
& +r^2\Big(\frac{4d}{r} + O(\frac{1}{r^2})\Big)\sin\theta d\theta d\psi.
\end{aligned}$$

Regularity assumptions

- **Condition A :**
  Each of the six functions β, γ, δ, U, V, W together with its derivatives up to the second orders are equal at $\psi = 0$ and 2π.

- **Condition B:**
  For all u, $\theta_0 = 0$, or π,

$$\int_0^{2\pi} c(u, \theta_0, \psi)d\psi = 0.$$

At null infinity, the Bondi energy-momentum of u_0-slice is defined as

$$m_\nu(u_0) = \frac{1}{4\pi}\int_{S^2} M(u_0, \theta, \psi)n^\nu dS$$

for $\nu = 0, 1, 2, 3$, where

$$n^0 = 1, \quad n^1 = \sin\theta\cos\psi, \quad n^2 = \sin\theta\sin\psi, \quad n^3 = \cos\theta.$$

In the Minkowski spacetime, $m_\nu(u_0) = 0$. In the Schwartzschild spacetime, $m_0(u_0) = m$, $m_i(u_0) = 0$.

The Bondi energy-momentum is the total energy-momentum measured after the loss due to the gravitational radiation up to that time.

In 1962, Bondi proved that, in vacuum Bondi's radiating spacetime,

$$M_{,0} = -\Big((c_{,0})^2 + (d_{,0})\Big)^2 + \frac{1}{2}\Big(l_{,2} + l\cot\theta + \bar{l}_{,3}\csc\theta\Big)_{,0}$$

where $_{,0} = \frac{d}{du}$.

Under **Condition A** and **Condition B**, this implies the following Bondi's mass loss formula **(Bondi(1962), Wenling Huang-S.T. Yau-Xiao Zhang (2004))**:

$$\frac{d}{du}\Big(m_0(u) - \sqrt{\sum_{1\leq i\leq 3} m_i^2(u)}\Big) = -\int_{S^2}\Big((c_{,0})^2 + (d_{,0})^2\Big) \leq 0,$$

i.e., the Bondi mass is non-increasing with respect to u, and more and more energy is radiated away.

In vacuum Bondi's radiating spacetime, **the positive mass conjecture at null infinity** is equivalent that **the Bondi mass must be nonnegative**.

Main ideas of the proof:

1982, Schoen-Yau: Solving the Jang's equation–prescribing the mean curvatures.

1982-, Israel-Nester, Horowitz-Perry, Ashtekar-Horowitz, Renla-Tod, Ludvigsen-Vickers, etc.: Witten's method–the Dirac operator.

The idea is to choose certain spacelike hypersurfaces approaching to null infinity. These spacelike hypersurfaces are asymptotically hyperbolic with the nontrivial second fundamental forms in the Bondi's radiating spacetimes. Therefore, it requires to establish the positive mass theorem for these spacelike hypersurfaces.

Two approaches: Schoen-Yau's method and Witten's method.

In 2002, by using Witten's method, **Xiao Zhang** was able to find a complete and rigorous proof of this positive mass theorem near null infinity.

In the Minkowski spacetime, the spacelike hypersurface $t = \sqrt{1+r^2}$ has the hyperbolic metric $\breve{g}$ and the nontrivial second form $\breve{h}$

$$\breve{g} = \breve{h} = \frac{dr^2}{1+r^2} + r^2(d\theta^2 + \sin^2\theta d\psi^2)$$

in polar coordinate. Denote by $\breve{e}_i$ the associated frame and by $\breve{e}^i$ the associated coframe.

Based on the above model, we can define an asymptotically null initial data set: **An initial data set** (M^3, g, p) **(**p **is not necessarily symmetric) is asymptotically null of order** τ if, outside a compact subset, M is diffeomorphic to $R^3 \backslash B_R$ and the metric g and the 2-tensor p are

$$g(\breve{e}_i, \breve{e}_j) = \breve{g}(\breve{e}_i, \breve{e}_j) + a_{ij}, \quad p(\breve{e}_i, \breve{e}_j) = \breve{p}(\breve{e}_i, \breve{e}_j) + b_{ij}$$

where a_{ij} and b_{ij} satisfy

$$a_{ij} = O(\frac{1}{r^\tau}), \breve{\nabla}_k a_{ij} = O(\frac{1}{r^\tau}), \breve{\nabla}_k \breve{\nabla}_l a_{ij} = O(\frac{1}{r^\tau}), b_{ij} = O(\frac{1}{r^\tau}), \breve{\nabla}_k b_{ij} = O(\frac{1}{r^\tau})$$

where $\breve{\nabla}$ is the Levi-Civita connection of $\breve{g}$ and $\breve{\nabla}_{\breve{e}_i}$ is denoted by $\breve{\nabla}_i$. Denote

$$\begin{aligned} \mathcal{E} &= \breve{\nabla}^j a_{1j} - \breve{\nabla}_1 tr_{\breve{g}}(a) - (a_{11} - g_{11} tr_{\breve{g}}(a)), \\ \mathcal{P}_k &= b_{k1} - g_{k1} tr_{\breve{g}}(b). \end{aligned}$$

The total energy E_ν and the total linear momentum $P_{\nu,k}$ are

$$\begin{aligned} E_\nu &= \frac{1}{16\pi} \lim_{r\to\infty} \int_{S_r} \mathcal{E} n^\nu r \breve{\omega}_2 \wedge \breve{\omega}_3, \\ P_{\nu,k} &= \frac{1}{8\pi} \lim_{r\to\infty} \int_{S_r} \mathcal{P}_k n^\nu r \breve{\omega}_2 \wedge \breve{\omega}_3, \end{aligned}$$

where S_r is the sphere of radius r in $\mathbb{R}^3$, $\nu = 0, 1, 2, 3$, $k = 1, 2, 3$.

The total angular momentum can be also defined when p is symmetric.

The Positive Mass Theorem (Xiao Zhang, 2002): Let (M, g_{ij}, p_{ij}) be a 3-dimensional asymptotically null initial data set of order $\tau = 3$. Denote

$$\mu = \frac{1}{2}(R + (p_i^{\ i})^2 - p_{ij}p^{ij}), \varpi_j = \nabla^i p_{ji} - \nabla_j p_i^{\ i}, \sigma_j = 2\nabla^i (p_{ij} - p_{ji}).$$

If the initial data set satisfies the dominant energy condition

$$\mu \geq \max\Big\{\sqrt{\sum_j \varpi_j^2}, \sqrt{\sum_j (\varpi_j + \sigma_j)^2}\Big\},$$

then,

$$E_0 - P_{0,1} \geq \sqrt{\sum_{i=1,2,3} (E_i - P_{i,1})^2}.$$

If equality holds, then

$$R_{ijkl} = -p_{ik}p_{jl} + p_{il}p_{jk}, \ \nabla_i p_{jk} = \nabla_j p_{ik}, \ \nabla^j p_{ij} = \nabla^j p_{ji}.$$

Important Remark: The proof of the theorem can still go through if the order $\tau > \frac{3}{2}$ and the $E_\nu - P_{\nu,1}$ are finite for $\nu = 0, 1, 2, 3$.

Recently, **Wenling Huang, S. T. Yau and Xiao Zhang** were able to find suitable asymptotically null, spacelike hypersurface (M^3, g, h) in vacuum Bondi's radiating spacetimes where g is the induced asymptotically hyperbolic metric and h the second fundamental form. Then the above theorem indicates that

Theorem (Huang-Yau-Zhang, 2004): If there exists u_0 in vacuum Bondi's radiating spacetime such that

$$c|_{u=u_0} = d|_{u=u_0} = 0$$

for r sufficiently large, then

$$m_0(u) \geq \sqrt{\sum_{i=1,2,3} m_i^2(u)}$$

for all $u \leq u_0$. Equality implies that the spacetime is flat in the region $\{u \leq u_0\}$.

Huang-Yau-Zhang also work on Schoen-Yau's method to solve the **Jang's equation** over (M^3, g, h).

$$\Big(g^{ij} - \frac{f^i f^j}{1+|\nabla f|^2}\Big)\Big(\frac{f_{,ij}}{\sqrt{1+|\nabla f|^2}} - h_{ij}\Big) = 0$$

under the boundary condition

$$f \to r + p(\theta, \psi) \ln r$$

as $r \to \infty$, where

$$p(\theta, \psi) = 2M(u, \theta, \psi) - \Big(l_{,2} + l \cot\theta + \bar{l}_{,3} \csc\theta\Big)$$

for fixed u.

Theorem (Huang-Yau-Zhang, 2005): If there exists u_0 in vacuum Bondi's radiating spacetime such that

$$M(u_0, \theta, \psi) - \frac{1}{2}\Big(l_{,2} + l \cot\theta + \bar{l}_{,3} \csc\theta\Big)_{u=u_0}$$

is constant for r sufficiently large, then

$$m_0(u) \geq \sqrt{\sum_{i=1,2,3} m_i^2(u)}$$

for all $u \leq u_0$. Equality implies that the spacetime is flat in the region $\{u \leq u_0\}$.

5 The ADM mass and the Bondi mass – Question 2

In 1979, assumed that the spacetime can be conformally compactified, and asymptotically empty and flat at null and spatial infinity in certain sense, Ashtekar and Magnon-Ashtekar demonstrated the energy-momentum at spatial infinity is the past limit of the Bondi energy-momentum. Here, the "past limit" means

$$\lim_{u \to -\infty} m_\nu(u).$$

(In 2003, Hayward proved this theorem in a new framework for spacetime asymptotics, replacing the Penrose conformal factor by a product of advanced and retarded conformal factors.)

In 1993, Christodoulou and Klainerman proved the global existence of globally hyperbolic, strongly asymptotically flat, maximal foliated vacuum solutions of the Einstein field equations. They also proved rigorously the ADM mass at spatial infinity is the past limit of the Bondi mass in these spacetimes.

In 2004, **Xiao Zhang** studied this problem in the vacuum Bondi's radiating spacetime. He defines the spatial infinity as the t-slices where the "real" time t is defined as $t = u + r$. Under the asymptotic flatness assumptions at spatial infinity which ensure the Schoen-Yau's positive mass theorem, he proved that

$$P_\nu(t_0) = \lim_{u \to -\infty} m_\nu(u)$$

for $\nu = 0, 1, 2, 3$, where $E(t_0)$ is denoted as $P_0(t_0)$. This means the standard ADM mass at spatial infinity is the past limit of the Bondi mass.

In this case, it was proved also that the ADM total energy, the ADM total linear momentum of (spatial) t_0-slice and the Bondi energy-momentum of (null) u_0-slice satisfy

$$P_\nu(t_0) = m_\nu(u_0) + \frac{1}{4\pi} \int_{-\infty}^{u_0} \int_{S^2} \Big((c_{,0})^2 + (d_{,0})^2 \Big) n^\nu dS du$$

where $E(t_0)$ is denoted as $P_0(t_0)$. In particular, if there is news $c_{,0}$, $d_{,0}$, then the ADM total energy is always greater than the Bondi mass.

Unfortunately, the asymptotic flatness conditions at spatial infinity in all above works **preclude** gravitational radiation. The formulae are essentially meaningless.

Therefore, certain weaker asymptotic flatness conditions are assumed at spatial infinity in order to *include* gravitational radiation: Roughly speaking, it is assumed that

$$\{M, c, d, M_{,0}, c_{,0}, d_{,0}, M_{,A}, c_{,A}, d_{,A}\} = O(1)$$

as $u \to -\infty$, where $A = 2, 3$.

Under these conditions, **Xiao Zhang (2004)** proved a formula related the ADM total energy of any t_0-slice to the past limit of the Bondi mass

$$E(t_0) = m_0(-\infty) + \frac{1}{2\pi} \lim_{u \to -\infty} \int_0^{\pi} \int_0^{2\pi} (cc_{,0} + dd_{,0}) \sin\theta d\psi d\theta.$$

This may be thought as the relation between them in radiative fields:

This formula indicates that, in radiative fields,

$$E(t_0) - m_0(u_0) = \infty.$$

Very recently, **Wenling Huang, Xiao Zhang (2005)** were able to prove formulae related the ADM total linear momentum of any t_0-slice to the past limit of the Bondi momentum in radiative fields:

$$P_k(t_0) = m_k(-\infty) + \frac{1}{8\pi} \lim_{u \to -\infty} \int_0^{\pi} \int_0^{2\pi} \mathbb{P}_k d\psi d\theta$$

for $k = 1, 2, 3$, where $\mathbb{P}_k$ are functions depending on θ, ψ, c and d together with their derivatives up to the third order. Their precise expressions will be addressed elsewhere.

Remark: In general, one cannot expect the "real" time $t = u + r$ (e.g. the Schwarzschild spacetime). That the case t approaches $u + r$ asymptotically is being studied.

6 Open problems

1. What do they mean in physics that $c|_{u=u_0} = d|_{u=u_0} = 0$ and that $p(\theta, \psi)$ is constant in solving the Jang's equation? Do they preclude gravitational waves?

2. If not, gravitational waves may carry away more energy than gravitational radiating have initially. What does it indicate in physics?

3. If the nature rules out 2, we might use a wrong mathematical formulation. What is the right mathematical formulation then?

4. Positive cosmological constant + gravitational waves.

5. ......?