3D integrable lattice models and the Bazhanov-Stroganov model

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joint work with S. Pakuliak, S. Sergeev

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Many 2D integrable lattice spin models known: Ising, XXZ, RSOS, etc. Systematic construction of solutions of Y-B equations: quantum groups.

Not many integrable 3D models: almost all generalizations of $\mathbb{Z}_N$-Zamolodchikov-Baxter-Bazhanov (ZBB) model A.B.Z. 1981, B-B 1992

Tetrahedron eqs. (3D analog of Y-B-eqs.) very restrictive, no analog to quantum group construction of solutions known.

Baxter-Bazhanov (1992) recognized intimate relation of ZBB model to 2D integrable chiral Potts model.

Elegant formulation of generalized BZZ-model:

Vertex formulation in terms of Sergeev mapping, satisfying modified tetrahedron equation (MTE).

Mangazeev, Stroganov 1993, Sergeev, Mangazeev, Stroganov 1995, Sergeev 1999

Aim of this talk: Use 3D Sergeev mapping to find new results for 2D BS-model.

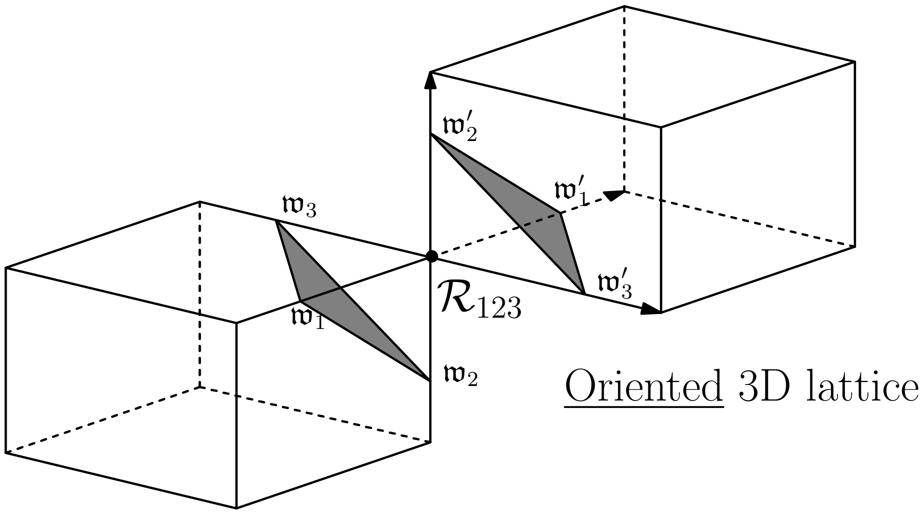
1. ZBB-model, Sergeev mapping
2. BS-model
3. BS-L-operator from Sergeev Linear Problem
4. BS-S-operator from 3D mapping $\mathbf{R}$
5. Intertwining of classical BS-L-operator from 3D $\mathcal{R}^{(f)}$
6. Isospectral transformation of BS-L-operator

1 Sergeev formulation of the 3D vertex ZBB-model

Quantum variables: elements of ultralocal Weyl algebra at root of unity:

$$\mathbf{u}_j \cdot \mathbf{w}_j = \omega\, \mathbf{w}_j \cdot \mathbf{u}_j; \qquad \omega = e^{2\pi i/N}; \qquad \mathbf{u}_i \cdot \mathbf{w}_j = \mathbf{w}_j \cdot \mathbf{u}_i \ \text{ for } \ i \neq j.$$

Attach also scalar κ_j to each link, together: $\mathfrak{w}_j = (\mathbf{u}_j, \mathbf{w}_j, \kappa_j)$.



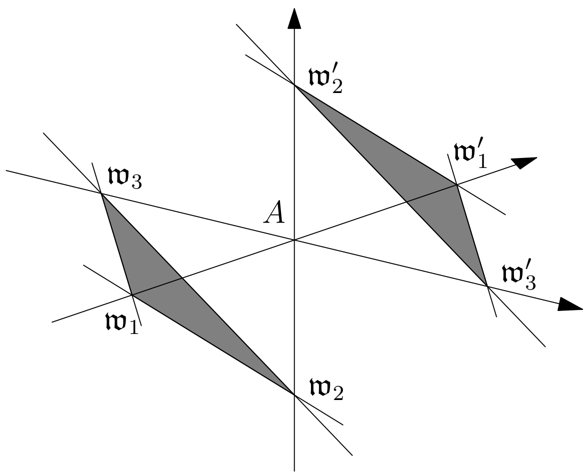
Key object: Canonical invertible rational mapping operator $\mathcal{R}_{123}$

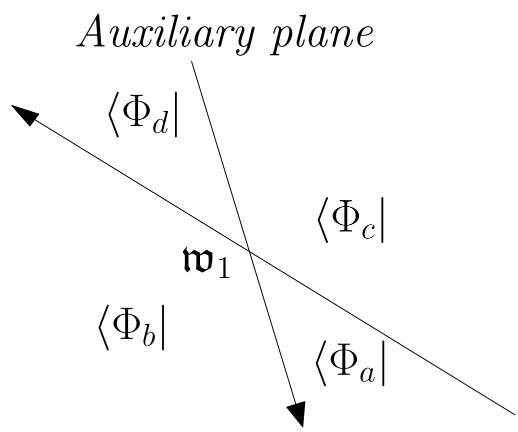
$$(\mathcal{R}_{123} \circ \Psi)\,(\mathbf{u}_1, \mathbf{w}_1, \mathbf{u}_2, \ldots, \mathbf{w}_3) \;=\; \Psi(\mathbf{u}_1', \mathbf{w}_1', \mathbf{u}_2', \ldots, \mathbf{w}_3').$$

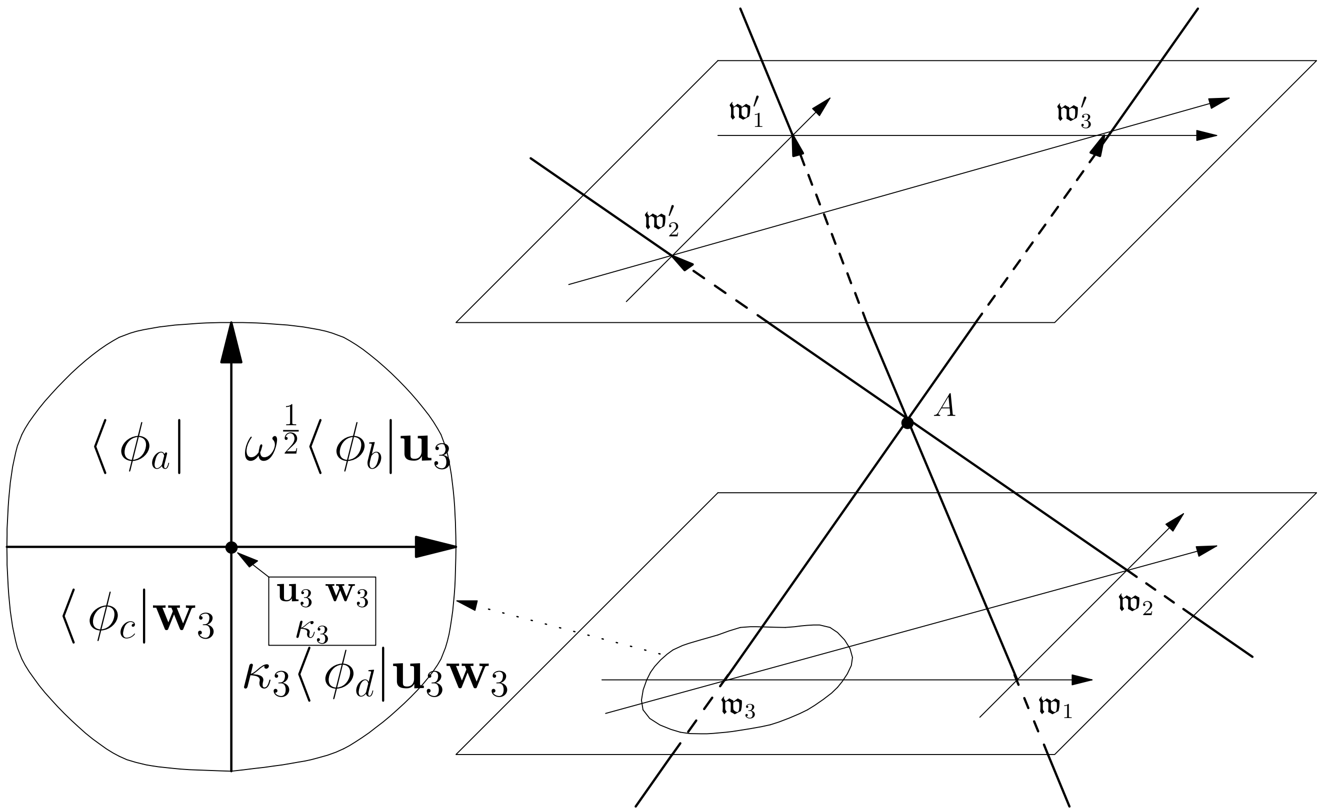
$\mathcal{R}_{123}$ is uniquely determined from postulates:

1: Baxter Z-invariance (lines may be shifted respect to each other)

2: Linear Problem: $0 \;=\; \langle \Phi_a| + \omega^{1/2} \langle \Phi_b|\, \mathbf{u}_1 + \langle \Phi_c|\, \mathbf{w}_1 + \kappa_1 \langle \Phi_d|\, \mathbf{u}_1\, \mathbf{w}_1.$





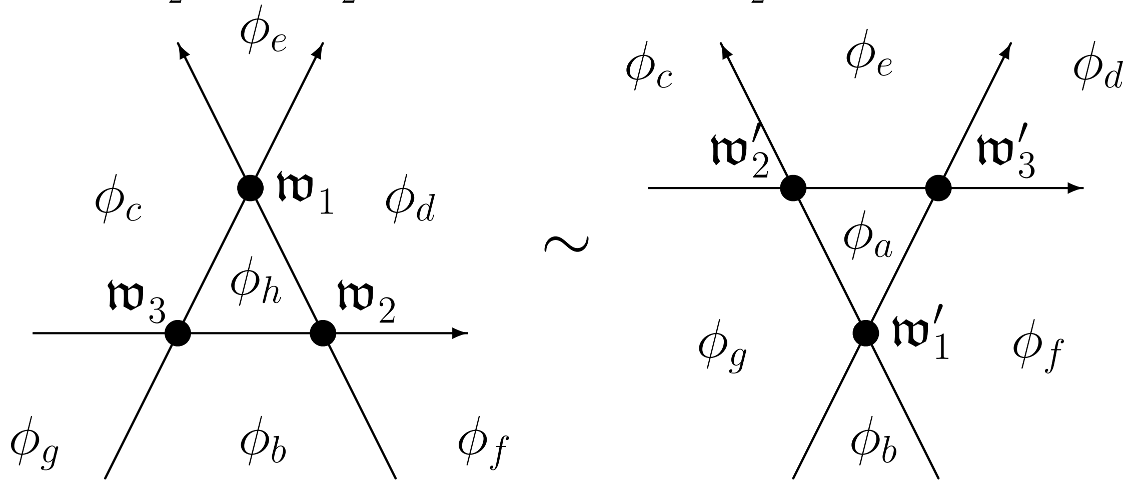


Linear problem: 2D-analog to QIS L-operator relation

$$0 \;=\; \langle\Phi^{(k)}|L^{(k)}(x) \;-\; \langle\Phi^{(k+1)}|\ell^{(k)}.$$

$\nabla = \Delta$ gives six equations: eliminate co-currents obtain unique solution:

$$\mathbf{u}_1'^{\,-1} = \frac{\kappa_1}{\kappa_2}\mathbf{u}_2^{-1} + \frac{\kappa_3}{\kappa_2}\mathbf{u}_1^{-1}\mathbf{w}_2^{-1}\mathbf{w}_3 - \omega^{1/2}\frac{\kappa_1\kappa_3}{\kappa_2}\mathbf{u}_2^{-1}\mathbf{w}_2^{-1}\mathbf{w}_3,\ etc.$$



Rational mapping $\mathcal{R}_{123}$ looks complicated in terms of Weyl operators:

$$\kappa_2 \mathbf{u}_1'^{\,-1} = \kappa_1 \mathbf{u}_2^{-1} + \kappa_3 \mathbf{u}_1^{-1}\mathbf{w}_2^{-1}\mathbf{w}_3 - \omega^{1/2}\kappa_1\kappa_3 \mathbf{u}_2^{-1}\mathbf{w}_2^{-1}\mathbf{w}_3,$$

$$\mathbf{u}_2'^{\,-1} = \mathbf{u}_1^{-1} - \omega^{1/2}\mathbf{u}_1^{-1}\mathbf{w}_1\mathbf{u}_3^{-1} + \kappa_1 \mathbf{w}_1 \mathbf{u}_2^{-1}\mathbf{u}_3^{-1};$$

$$\mathbf{w}_1' = \mathbf{w}_1\mathbf{w}_2\mathbf{w}_3^{-1} - \omega^{1/2}\mathbf{w}_2\mathbf{u}_3\mathbf{w}_3^{-1} + \kappa_3 \mathbf{u}_3$$

$$\mathbf{w}_1'\mathbf{w}_2' = \mathbf{w}_2\mathbf{w}_1; \qquad \mathbf{u}_3'\mathbf{u}_2' = \mathbf{u}_2\mathbf{u}_3; \qquad \mathbf{w}_3'^{-1}\mathbf{u}_1' = \mathbf{u}_1\mathbf{w}_3^{-1}.$$

At root of unity, affine Weyl operators have $N \times N$-matrix representation:

$$\mathbf{u} \equiv u\,\mathbf{X}; \qquad \mathbf{w} \equiv w\,\mathbf{Z}; \qquad u,\, w \in \mathbb{C}; \qquad \mathbf{X}\,\mathbf{Z} = \omega\,\mathbf{Z}\,\mathbf{X}$$

$$\mathbf{X}\,|\beta\rangle = \omega^{\beta}\,|\beta\rangle; \qquad \mathbf{Z}\,|\beta\rangle = |\beta+1\rangle; \qquad \langle\alpha|\beta\rangle = \delta_{\alpha,\beta}; \qquad \mathbf{X}^N = \mathbf{Z}^N = 1.$$

N-th powers of Weyl are centers: $\mathbf{u}_j^N = u_j^N \equiv U_j$; $\mathbf{w}_j^N = w_j^N \equiv W_j$.

Now if we take N-th powers of the mapping relations:

$$(\mathbf{u} + \mathbf{w})^N = U + W,$$

$\mathcal{R}_{123}$ induces *functional mapping* $\mathcal{R}_{123}^{(f)}$ of the centers $(K_j \equiv \kappa_j^N)$:

$$\frac{U_1'}{U_1} = \frac{W_3'}{W_3} = \frac{K_2U_2W_2}{K_1U_1W_2 + K_3U_2W_3 + K_1K_3U_1W_3};$$

$$\frac{W_1}{W_1'} = \frac{W_2'}{W_2} = \frac{W_1W_3}{W_1W_2 + U_3W_2 + K_3U_3W_3};$$

$$\frac{U_2'}{U_2} = \frac{U_3}{U_3'} = \frac{U_1U_3}{U_2U_3 + U_2W_1 + K_1U_1W_1}.$$

Remarkable feature: $\mathcal{R}_{123}$ decomposes into a matrix conjugation $\mathbf{R}_{123}$ ($\mathbf{R}_{123}$ is $N^3 \times N^3$-matrix) and $\mathcal{R}_{123}^{(f)}$: (Baxter, Bazhanov, Reshetikhin, Bobenko, Sergeev, Mangazeev, Stroganov 1995)

$$\mathcal{R}_{123} \circ \Psi = \mathbf{R}_{123} \left(\mathcal{R}_{123}^{(f)} \circ \Psi\right) \mathbf{R}_{123}^{-1}.$$

$\mathbf{R}_{123}$ has compact expression in of root-of-unity q-Gamma-functions $w_p(n)$:

$$\mathbf{R}_{i_1i_2i_3}^{j_1j_2j_3} = \delta_{i_2+i_3,j_2+j_3}\,\omega^{(j_1-i_1)j_3}\,\frac{w_{p_1}(i_2-i_1)\,w_{p_2}(j_2-j_1)}{w_{p_3}(j_2-i_1)\,w_{p_4}(i_2-j_1)}.$$

Each $w_p(n)$ $(n \in \mathbb{Z}_N)$ depends on point $p = (x, y)$ on Fermat curve $x^N + y^N = 1$:

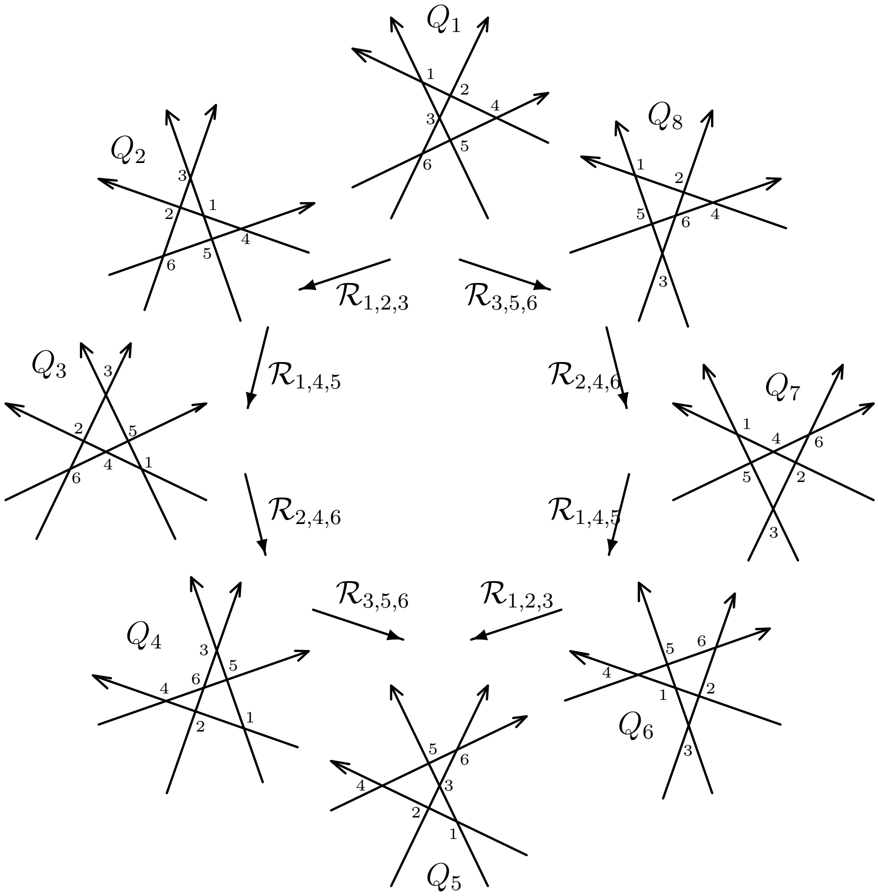
$$w_p(n) \;=\; \prod_{j=1}^{n} \frac{y}{1-\omega^n\, x}, \qquad w_p(0) \;=\; 1\,; \qquad w_p(n+N) \;=\; w_p(n)\,.$$

$\mathbf{R}_{123}$ is a weighted cross-ratio of *four* w_{p_i} , only *three* Fermat points are independent: $x_1\, x_2 \;=\; \omega\, x_3\, x_4$.

The Fermat points are related to initial and final Weyl variables $\mathfrak{w}_j$, $\mathfrak{w}'_j$ by

$$x_1 = \frac{1}{\omega^{1/2}} \frac{u_2}{\kappa_1 u_1}\;; \quad x_2 = \frac{1}{\omega^{1/2}} \frac{\kappa_2 u'_2}{u'_1}\;; \quad x_3 = \frac{1}{\omega} \frac{u'_2}{u_1}\;; \qquad \frac{y_3}{y_1} = \frac{\kappa_1 w_1}{u'_3}\,.$$

The <u>Tetrahedron equation</u> follows *without calculation* considering four succesive transformations $\mathcal{R}_{ijk}$ of quadrangles Q_m in different order:



$$\mathcal{R}_{123} \cdot \mathcal{R}_{145} \cdot \mathcal{R}_{246} \cdot \mathcal{R}_{356} \;=\; \mathcal{R}_{356} \cdot \mathcal{R}_{246} \cdot \mathcal{R}_{145} \cdot \mathcal{R}_{123}\,.$$

6 spaces, each $\mathcal{R}_{ijk}$ depends on 3 param., there are 5 different param. in the TE

2 The Bazhanov-Stroganov model

Bazhanov-Stroganov 1990 (Korepanov 1985): standard 6-v R-matrix

$$R(\lambda,\nu) \;=\; \begin{pmatrix} \lambda-\omega\nu & 0 & 0 & 0 \\ 0 & \omega(\lambda-\nu) & \lambda(1-\omega) & 0 \\ 0 & \nu(1-\omega) & \lambda-\nu & 0 \\ 0 & 0 & 0 & \lambda-\omega\nu \end{pmatrix}.$$

intertwines not only the 6-vertex L-operators, *but at* $\omega=e^{2\pi i/N}$ *also*

$$L(\lambda,\mathbf{a}) \;=\; \begin{pmatrix} 1+\lambda\, b_1\mathbf{Z}; & \lambda\,\mathbf{X}^{-1}(a_1-b_2\mathbf{Z}) \\ \\ \mathbf{X}(a_2-b_3\mathbf{Z}); & \lambda\, a_1a_2+b_2b_3b_1^{-1}\mathbf{Z} \end{pmatrix}; \;\; \mathbf{X}\,\mathbf{Z}=\omega\,\mathbf{Z}\,\mathbf{X}\,,$$

with entries $\mathbf{X},\ \mathbf{Z}$: *Weyl elements*, $\mathbf{a} \;=\; (a_1,\ldots,b_3)\in\mathbb{C}^5$.

Their new intertwining relation is:

$$\sum_{j_1,j_2,\beta} R_{i_1j_1,i_2j_2}(\lambda,\nu)\, L^{\alpha_1\beta}_{j_1k_1}(\lambda,\mathbf{a})\, L^{\beta\alpha_2}_{j_2k_2}(\nu,\mathbf{a})$$

$$= \sum_{j_1,j_2,\beta} L^{\alpha_1\beta}_{i_2j_2}(\nu,\mathbf{a})\, L^{\beta\alpha_2}_{i_1j_1}(\lambda,\mathbf{a})\, R_{j_1k_1,j_2k_2}(\lambda,\nu)\,.$$

$\alpha_1,\alpha_2,\beta = \;0,1,\ldots,N-1$ and $i_1,i_2,j_1,\ldots \;=\; 0,1\,.$

Further they noticed: for special choice of $\mathbf{a}$ there exists $\mathbf{S}$

intertwining two L in the Weyl (quantum) space:

$$\sum_{\beta_1\beta_2,k}\mathbf{S}_{\alpha_1\alpha_2;\beta_1\beta_2}(p,p',q,q')\, L^{\beta_1\gamma_1}_{i_1k}(\lambda;p,p')\, L^{\beta_2,\gamma_2}_{k\,i_2}(\lambda;q,q')$$

$$= \sum_{\beta_1\beta_2,k} L^{\alpha_2\beta_2}_{i_1k}(\lambda;q,q')\, L^{\alpha_1\beta_1}_{k\,i_2}(\lambda;p,p')\;\; \mathbf{S}_{\beta_1,\beta_2;\gamma_1,\gamma_2}(p,p',q,q').$$

$\mathbf{S}\,L\,L\,=\,L\,L\,\mathbf{S}$ if

$$L(\lambda;\,q,q') \;=\; \begin{pmatrix} 1+\lambda\dfrac{y_q y_{q'}}{\mu_q\mu_{q'}}\mathbf{Z} & \lambda\,\mathbf{X}^{-1}\left(x_q-\dfrac{y_{q'}}{\mu_q\mu_{q'}}\mathbf{Z}\right) \\ \mathbf{X}\left(\omega x_{q'}-\dfrac{y_q}{\mu_q\mu_{q'}}\mathbf{Z}\right) & \lambda\,\omega\,x_q x_{q'}+\dfrac{1}{\mu_q\mu_{q'}}\mathbf{Z} \end{pmatrix}$$

where $x_q,\,y_q,\,\mu_q,\,x_{q'},\,y_{q'},\,\mu_{q'}$ fulfil the high-genus chiral Potts conditions:

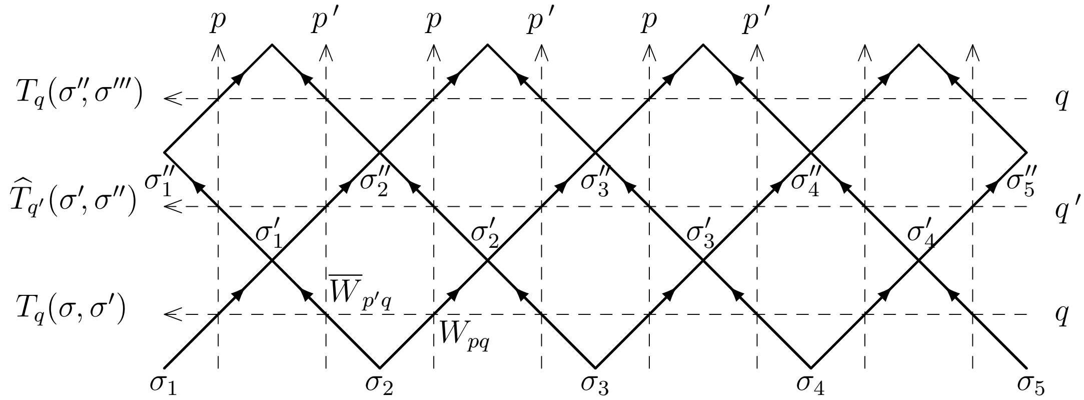
$$x_q^N+y_q^N=k(1+x_q^N y_q^N); \qquad k x_q^N = 1-k'\mu_q^{-N}; \qquad k y_q^N = 1-k'\mu_q^N; \qquad (*)$$

$$k^2+k'^2=1, \qquad k\,:\ \text{fixed temperature-like parameter}.$$

Intertwining matrix $\mathbf{S}$ is product of four *chiral Potts*-Boltzmann weights:

$$\mathbf{S}_{\alpha_1\alpha_2,\beta_1\beta_2}(p,p',q,q') \\ =\; W_{pq'}(\alpha_1-\alpha_2)W_{p'q}(\beta_2-\beta_1)\overline{W}_{pq}(\beta_2-\alpha_1)\overline{W}_{p'q'}(\beta_1-\alpha_2).$$

Integrable CP-model: (Baxter, Perk, Au-Yang 1988)



different left $\overline{W}_{pq}(\sigma_2-\sigma_1')$ and right $W_{pq}(\sigma_2-\sigma_2')$ Boltzmann weights:

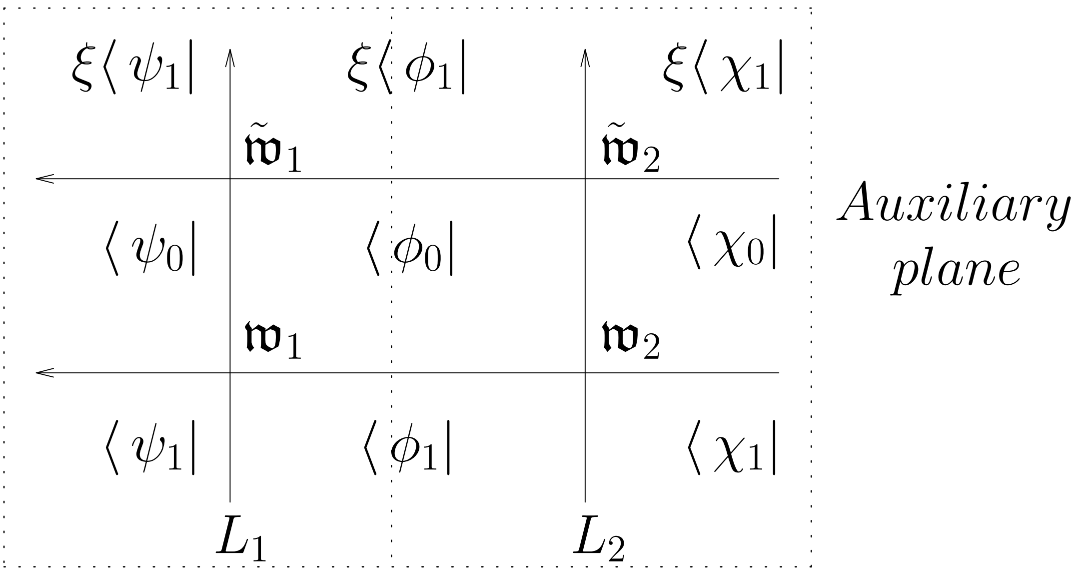
$$W_{pq}(n)=\left(\frac{\mu_p}{\mu_q}\right)^n\prod_{j=1}^{n}\frac{y_q-x_p\omega^j}{y_p-x_q\omega^j}; \quad \overline{W}_{pq}(n)=(\mu_p\mu_q)^n\prod_{j=1}^{n}\frac{\omega x_p-x_q\omega^j}{y_q-y_p\omega^j},$$

where $x_q, y_q, \mu_q,$ *etc.* satisfy (*) .

W, $\overline{W}$ are cyclic: $\quad W_{pq}(n+N)=W_{pq}(n); \qquad \overline{W}_{pq}(n+N)=\overline{W}_{pq}(n),$

Special Y-B-eq.: $\quad \sum_d \overline{W}_{qr}(d-b)W_{pr}(d-a)\overline{W}_{pq}(c-d)=R_{pqr}\,W_{pq}(b-a)\overline{W}_{pr}(c-b)W_{qr}(c-a)\,.$

3 BS-L-operator from Sergeev Linear Problem



Consider four quantum variables $\mathfrak{w}_1,\,\tilde{\mathfrak{w}}_1,\,\mathfrak{w}_2,\,\tilde{\mathfrak{w}}_2$ on links of the basic 3D-lattice, in the auxiliary plane through these variables.

Assume lattice *periodic* in vertical direction, quasi-momentum ξ.

Linear problem at $\mathfrak{w}_1$ and $\tilde{\mathfrak{w}}_1$ (direction of lines important!):

$$\begin{aligned} 0 &= \langle\,\psi_0| + \xi\omega^{1/2}\langle\,\psi_1|\tilde{\mathbf{u}}_1 + \langle\,\phi_0|\tilde{\mathbf{w}}_1 + \xi\tilde{\kappa}_1\langle\,\phi_1|\tilde{\mathbf{u}}_1\tilde{\mathbf{w}}_1\;, \\ 0 &= \langle\,\psi_1| + \;\omega^{1/2}\langle\,\psi_0|\mathbf{u}_1 + \langle\,\phi_1|\mathbf{w}_1 \;+\; \kappa_1\langle\,\phi_0|\mathbf{u}_1\mathbf{w}_1\;. \end{aligned}$$

In matrix form: $\quad \langle\,\psi|\,(\omega\,\xi\,\mathbf{u}_1\,\tilde{\mathbf{u}}_1 \,-\, 1)\tilde{\mathbf{w}}_1^{-1} = \langle\,\phi|\cdot L_1(\xi)\,,$

where $\quad \langle\phi| = (\langle\phi_0|,\,\langle\phi_1|)\,; \qquad \langle\psi| = (\langle\psi_0|,\,\langle\psi_1|)\,,$

$L_1(\xi)$ has operator-valued elements:

$$L_1(\xi) \;=\; \begin{pmatrix} 1-\omega^{1/2}\xi\mathbf{u}_1\tilde{\mathbf{u}}_1\kappa_1\mathbf{w}_1\tilde{\mathbf{w}}_1^{-1} & -\mathbf{u}_1\left(\omega^{1/2}-\kappa_1\mathbf{w}_1\tilde{\mathbf{w}}_1^{-1}\right) \\ \xi\tilde{\mathbf{u}}_1\left(\tilde{\kappa}_1-\omega^{1/2}\mathbf{w}_1\tilde{\mathbf{w}}_1^{-1}\right) & -\omega^{1/2}\xi\mathbf{u}_1\tilde{\mathbf{u}}_1\tilde{\kappa}_1+\mathbf{w}_1\tilde{\mathbf{w}}_1^{-1} \end{pmatrix}.$$

Only $\mathbf{w}_1\tilde{\mathbf{w}}_1,\,\mathbf{u}_1,\,\tilde{\mathbf{u}}_1$ appear: we can use $N\times N$-rep for the operators:

$$\mathbf{w}_1\tilde{\mathbf{w}}_1^{-1} = \frac{w_1}{\tilde{w}_1}\mathbf{Z}_1\,; \qquad \mathbf{u}_1 = u_1\mathbf{X}_1\,; \qquad \tilde{\mathbf{u}}_1 = \tilde{u}_1\mathbf{X}_1^{-1}\,.$$

Inserting representation and diagonal gauge transformation gives

$$\widetilde{L}_1(\tilde{\xi}) = \left(\begin{array}{cc} 1 - \omega^{1/2}\tilde{\xi}\kappa_1 w_1 \tilde{w}_1^{-1}\mathbf{Z}_1 & -\mathbf{X}_1 \left(\omega^{1/2} - \kappa_1 w_1 \tilde{w}_1^{-1}\mathbf{Z}_1\right) \\ \tilde{\xi}\,\mathbf{X}_1^{-1}\left(\,\tilde{\kappa}_1 - \omega^{1/2} w_1 \tilde{w}_1^{-1}\mathbf{Z}_1\right) & -\omega^{1/2}\tilde{\xi}\,\tilde{\kappa}_1 + w_1\tilde{w}_1^{-1}\mathbf{Z}_1 \end{array} \right).$$

$\widetilde{L}_1(\tilde{\xi})$: depends on spectral param. $\tilde{\xi}$ and 3 variables $\kappa_1,\ \tilde{\kappa}_1,\ w_1/\tilde{w}_1$.

$\Longrightarrow$ Bazhanov-Stroganov L-operator if we identify

$$\kappa_1 = \omega^{1/2}\frac{x_q}{y_{q'}}; \qquad \tilde{\kappa}_1 = \frac{1}{\omega^{1/2}}\frac{y_q}{x_{q'}}; \qquad \frac{w_1}{\tilde{w}_1} = \frac{1}{\omega}\,\frac{y_q y_{q'}}{x_q x_{q'}\mu_q\mu_{q'}}.$$

Recall:

$$L^{BS}(\lambda;\, q, q') \;=\; \left(\begin{array}{cc} 1 \,+\, \lambda\frac{y_q y_{q'}}{\mu_q\mu_{q'}}\,\mathbf{Z} & \lambda\,\mathbf{X}^{-1}\left(x_q - \frac{y_{q'}}{\mu_q\mu_{q'}}\,\mathbf{Z}\right) \\ \mathbf{X}\left(\omega x_{q'} - \frac{y_q}{\mu_q\mu_{q'}}\,\mathbf{Z}\right) & \lambda\,\omega\, x_q x_{q'} + \frac{1}{\mu_q\mu_{q'}}\,\mathbf{Z} \end{array} \right)$$

We shall soon get these relations from $\mathbf{S}$ -operator too.

Remark: If we demand periodicity after *third* vertical step:

get L-operator of cyclic rep of $U_q(\widehat{s\ell_3})$ etc. Isaev, Sergeev 2003, Bazhanov et al, Date et al, Tarasov 1990

Successive action of two L-operators simply

$$\langle\,\psi|\cdot\tilde{\mathbf{w}}_1^{-1}\tilde{\mathbf{w}}_2^{-1}\,(\lambda\,\mathbf{u}_1\tilde{\mathbf{u}}_1 - 1)\,(\lambda\,\mathbf{u}_2\tilde{\mathbf{u}}_2 - 1) = \langle\,\chi|\cdot L_2(\lambda)\,L_1(\lambda)\,.$$

4 BS-S-operator from matrix conjugation operators $\mathbf{R}_{123}$

We consider two $N^3 \times N^3$ matrices $\mathbf{R}$ and $\widetilde{\mathbf{R}}$ of the 3-D model:

$$\mathbf{R}_{i_1 i_2 i_3}^{j_1 j_2 j_3} = \delta_{i_2+i_3, j_2+j_3} \omega^{(j_1-i_1)j_3} \frac{w_{p_1}(i_2-i_1) w_{p_2}(j_2-j_1)}{w_{p_3}(j_2-i_1) w_{p_4}(i_2-j_1)} .$$

with Fermat parameters p_i resp. $\tilde{p}_i$ $(i=1,2,3,4)$.

In this Section the Sergeev functional transformation $\mathcal{R}_{123}^{(f)}$ is chosen to be trivial.

Proposition:

The BS-intertwining $N^2 \times N^2$-matrix $\mathbf{S}$ can be written as

$$\mathbf{S}_{\alpha_1\alpha_2}^{\beta_1\beta_2} = \sum_{m,n\in\mathbb{Z}_N} \mathbf{R}_{\alpha_1,\alpha_2,m}^{\beta_1\beta_2,n} \widetilde{\mathbf{R}}_{-\alpha_1,-\alpha_2,n}^{-\beta_1,-\beta_2,m}$$

if the Fermat parameters are related to the CP-parameters by:

$$x_1 = \frac{y_{q'}}{\omega x_p}\,; \quad x_2 = \frac{x_q}{y_{p'}}\,; \quad x_3 = \frac{x_q}{\omega x_p}\,; \quad x_4 = \frac{y_{q'}}{\omega y_{p'}}\,;$$
$$\tilde{x}_1 = \frac{x_{q'}}{y_p}\,; \quad \tilde{x}_2 = \frac{y_q}{\omega x_{p'}}\,; \quad \tilde{x}_3 = \frac{y_q}{\omega y_p}\,; \quad \tilde{x}_4 = \frac{x_{q'}}{\omega x_{p'}}\,.$$

$$\frac{\tilde{y}_1}{y_1} = \omega^{1/2}\frac{\mu_p x_p}{\mu_{q'} y_p};\ \frac{\tilde{y}_2}{y_2} = \frac{1}{\omega^{1/2}}\frac{\mu_q y_{p'}}{\mu_{p'} x_{p'}};\ \frac{\tilde{y}_3}{y_3} = \omega^{1/2}\frac{\mu_p\mu_q x_p}{y_p};\ \frac{\tilde{y}_4}{y_4} = \frac{1}{\omega^{1/2}}\frac{y_{p'}}{\mu_{p'}\mu_{q'} x_{p'}}.$$

Proof: use property of $w_p(n)$ (q-analog of $\Gamma(x)\Gamma(1-x) = \pi/\sin\pi x$):

$$w_p(n) = \frac{(-1)^n}{w_{Op}(-n)\,\omega^{n^2/2}}, \qquad \text{where} \qquad Op = (\omega^{-1}x^{-1}, \omega^{-1/2}x^{-1}y).$$

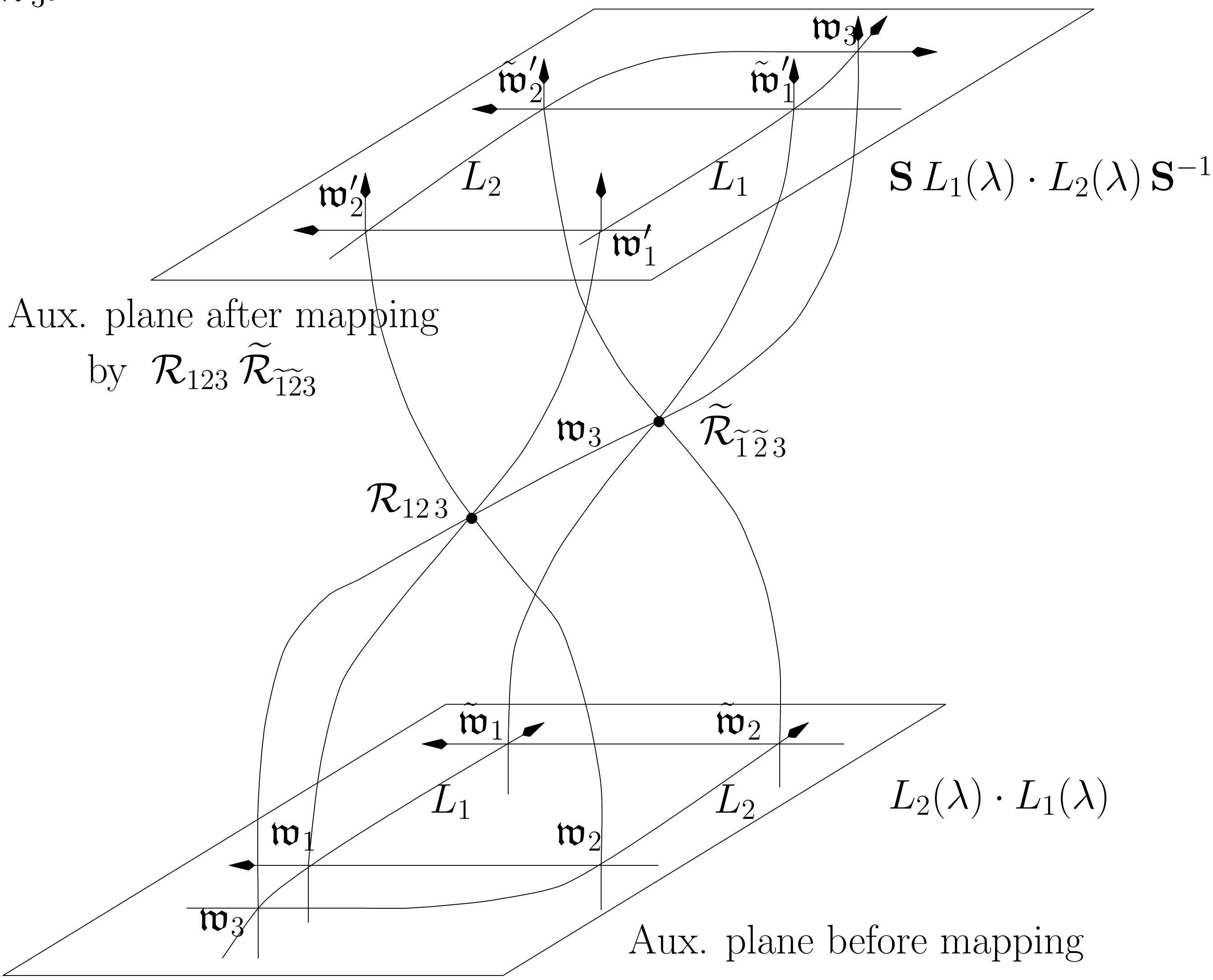
to get

$$\mathbf{S}_{\alpha_1\alpha_2}^{\beta_1\beta_2} = \frac{w_{\tilde{p}_1}(\alpha_1-\alpha_2)}{w_{Op_1}(\alpha_1-\alpha_2)}\,\frac{w_{p_2}(\beta_2-\beta_1)}{w_{O\tilde{p}_2}(\beta_2-\beta_1)}\,\frac{w_{O\tilde{p}_3}(\beta_2-\alpha_1)}{w_{p_3}(\beta_2-\alpha_1)}\,\frac{w_{Op_4}(\beta_1-\alpha_2)}{w_{\tilde{p}_4}(\beta_1-\alpha_2)}.$$

Now identify

$$W_{pq'}(\alpha_1-\alpha_2) \equiv \frac{w_{\tilde{p}_1}(\alpha_1-\alpha_2)}{w_{Op_1}(\alpha_1-\alpha_2)};\quad W_{p'q}(\beta_2-\beta_1) \equiv \frac{w_{p_2}(\beta_2-\beta_1)}{w_{O\tilde{p}_2}(\beta_2-\beta_1)}\ \ etc.$$

Intertwining of $L_1\, L_2$ visualized in 3D using auxiliary periodic variable $\mathfrak{w}_3$:



$$\mathcal{R}_{123}: \qquad (\,\mathfrak{w}_1,\ \mathfrak{w}_2,\ \mathfrak{w}_3) \;\mapsto\; (\,\mathfrak{w}_1',\ \mathfrak{w}_2',\ \mathfrak{w}_3);$$
$$\widetilde{\mathcal{R}}_{\tilde{1}\tilde{2}3}: \qquad (\,\tilde{\mathfrak{w}}_1,\ \tilde{\mathfrak{w}}_2,\ \mathfrak{w}_3) \;\mapsto\; (\,\tilde{\mathfrak{w}}_1',\ \tilde{\mathfrak{w}}_2',\ \mathfrak{w}_3);$$

From $\mathbf{S} = \mathbf{R}\,\tilde{\mathbf{R}}$ we get new parametrizations of $\mathbf{S}$: (5 contin. param.):

1. Chiral Potts: $\quad q,\ q',\ p,\ p',\ k\,.$
2. Fermat: $x_1,\ x_2,\ x_3,\ \tilde{x}_1,\ \tilde{x}_2,\ \tilde{x}_3$ with $\frac{y_1 y_2}{y_3 y_4} = \frac{\tilde{y}_1\tilde{y}_2}{\tilde{y}_3\tilde{y}_4}$.
3. Weyl: $\kappa_1,\ \tilde{\kappa}_1,\ \kappa_2,\ \tilde{\kappa}_2,\ w_1/\tilde{w}_1,\ w_2/\tilde{w}_2,$ $+$ constraint.
4. *New*: Cross-ratios of 8 points in $\mathbb{C}$: 5 independent

Trivial $\mathcal{R}^{(f)}_{123}$ means

$$(K_2U_2 - K_1U_1)\,W_2 \;=\; (K_1U_1 + U_2)\,K_3W_3\,;$$

$$(W_1 - K_3U_3)\,W_3 \;=\; (W_1 + U_3)\,W_2\,;\;\; (U_1 - U_2)\,U_3 \;=\; (K_1U_1 + U_2)\,W_1$$

The solution in terms of 6 points $X',\ X,\ \ Y',\ Y;\ \ Z_0',\ Z_0\ \in\mathbb{C}$ is:

$$U_1 = -\varepsilon\frac{Y-Z_0'}{Y-Z_0};\qquad U_2 = -\varepsilon\frac{X-Z_0'}{X-Z_0};\qquad U_3 = -\varepsilon\frac{X-Y'}{X-Y};$$

$$W_1 = \varepsilon\frac{Y'-Z_0}{Y-Z_0};\qquad W_2 = \varepsilon\frac{X'-Z_0}{X-Z_0};\qquad W_3 = \varepsilon\frac{X'-Y}{X-Y};$$

$$K_1 = -\left[\begin{array}{cc} Y' & Y \\ Z_0' & Z_0\end{array}\right];\;\; K_2 = -\left[\begin{array}{cc} X' & X \\ Z_0' & Z_0\end{array}\right];\;\; K_3 = -\left[\begin{array}{cc} X' & X \\ Y' & Y\end{array}\right],$$

where $\varepsilon \;=\; (-1)^N$ and for cross-ratios we write

$$\left[\begin{array}{cc} A & B \\ C & D\end{array}\right] \;\equiv\; \frac{(A-C)\,(B-D)}{(A-D)\,(B-C)}\,.$$

Same eqs. with tilde-variables: replace $Z_0',\ Z_0$ by $Z_1',\ Z_1$: then 8 points.

Nice result: Nth powers of the Fermat points are simple cross-ratios:

$$x_1^N = \left[\begin{array}{cc} X & Y' \\ Z_0' & Z_0\end{array}\right];\;\; x_2^N = \left[\begin{array}{cc} Y & X' \\ Z_0 & Z_0'\end{array}\right];\;\; x_3^N = \left[\begin{array}{cc} Y & X \\ Z_0 & Z_0'\end{array}\right];\;\; x_4^N = \left[\begin{array}{cc} X' & Y' \\ Z_0' & Z_0\end{array}\right];$$

$$y_1^N = \left[\begin{array}{cc} X & Z_0' \\ Y' & Z_0\end{array}\right];\;\; y_2^N = \left[\begin{array}{cc} Y & Z_0 \\ X' & Z_0'\end{array}\right];\;\; y_3^N = \left[\begin{array}{cc} Y & Z_0 \\ X & Z_0'\end{array}\right];\;\; y_4^N = \left[\begin{array}{cc} X' & Z_0' \\ Y' & Z_0\end{array}\right],$$

Also Nth powers of chiral Potts variables are cross-ratios:

$$k\,x_p^N = \left[\begin{array}{cc} X & Z_1' \\ Z_0 & Z_0'\end{array}\right];\quad k\,y_p^N = \left[\begin{array}{cc} X & Z_0' \\ Z_1 & Z_1'\end{array}\right];\quad k'\mu_p^N = \left[\begin{array}{cc} X & Z_1 \\ Z_0' & Z_1'\end{array}\right];$$

$$k\,x_{p'}^N = \left[\begin{array}{cc} X' & Z_0' \\ Z_1 & Z_1'\end{array}\right];\quad k\,y_{p'}^N = \left[\begin{array}{cc} X' & Z_1' \\ Z_0 & Z_0'\end{array}\right];\quad k'\mu_{p'}^N = \left[\begin{array}{cc} X' & Z_0 \\ Z_1' & Z_0'\end{array}\right];$$

$$k\,x_q^N = \left[\begin{array}{cc} Y & Z_1' \\ Z_0 & Z_0'\end{array}\right];\quad k\,y_q^N = \left[\begin{array}{cc} Y & Z_0' \\ Z_1 & Z_1'\end{array}\right];\quad k'\mu_q^N = \left[\begin{array}{cc} Y & Z_1 \\ Z_0' & Z_1'\end{array}\right];$$

$$k\,x_{q'}^N = \left[\begin{array}{cc} Y' & Z_0' \\ Z_1 & Z_1'\end{array}\right];\quad k\,y_{q'}^N = \left[\begin{array}{cc} Y' & Z_1' \\ Z_0 & Z_0'\end{array}\right];\quad k'\mu_{q'}^N = \left[\begin{array}{cc} Y' & Z_0 \\ Z_1' & Z_0'\end{array}\right].$$

with

$$k^2 = \left[\begin{array}{cc} Z_0 & Z_0' \\ Z_1 & Z_1'\end{array}\right] \qquad \text{or} \qquad k'^2 = \left[\begin{array}{cc} Z_0' & Z_1' \\ Z_0 & Z_1\end{array}\right].$$

In this cross-ratio parametriz. all constraints automatically fulfilled.

5 Intertwining of classical BS-L-operators from $\mathcal{R}_{123}^{(f)}$

Classical Linear Problem:

$$\begin{aligned} 0 &= \Psi_0 - \Psi_1 \Lambda \tilde{U}_1 + \Phi_0 \tilde{W}_1 + \Phi_1 \Lambda \tilde{K}_1 \tilde{U}_1 \tilde{W}_1 \\ 0 &= \Psi_1 - \Psi_0 U_1 \ + \ \Phi_1 W_1 \ + \ \Phi_0 K_1 U_1 W_1 \end{aligned}$$

in matrix form:

$$\Psi \, (\Lambda \, U_1 \, \tilde{U}_1 \, - \, 1) = \Phi \cdot \mathcal{L}_1(\Lambda)$$

defining the classical BS-$\mathcal{L}$-operator

$$\mathcal{L}_1(\Lambda) \ = \ \begin{pmatrix} \tilde{W}_1 + \Lambda U_1 \tilde{U}_1 K_1 W_1 & U_1 \left(\tilde{W}_1 + K_1 W_1 \right) \\ \Lambda \tilde{U}_1 \left(W_1 + \tilde{K}_1 \tilde{W}_1 \right) & W_1 + \Lambda \tilde{K}_1 U_1 \tilde{U}_1 \tilde{W}_1 \end{pmatrix}$$

acting in $\Psi = (\Psi_0, \Psi_1)$ and $\Phi = (\Phi_0, \Phi_1)$.

We take $\mathcal{L}_1(\Lambda)$ to define a discrete classical Bazhanov-Stroganov model.

Goal: describe solitonic solutions and soliton creation.

Define $\mathcal{L}_2(\Lambda)$ as above, but with $U_2, \, \tilde{U}_2, \, W_2, \, \tilde{W}_2, \, K_2, \, \tilde{K}_2$,

$\mathcal{L}_1^\star(\Lambda)$ with $U_1^\star, \, \tilde{U}_1, \, W_1^\star, \, \tilde{W}_1^\star, \, K_1, \, \tilde{K}_1$, analogously $\mathcal{L}_2^\star(\Lambda)$.

We are looking for $(U_1, \tilde{U}_1, W_1, \ldots, \tilde{W}_2) \ \mapsto \ (U_1^\star, \, \tilde{U}_1^\star, W_1^\star, \ldots, \tilde{W}_2^\star)$

which solves the intertwining relation

$$\mathcal{L}_2(\Lambda) \, \mathcal{L}_1(\Lambda) = \mathcal{L}_1^\star(\Lambda) \, \mathcal{L}_2^\star(\Lambda) \, .$$

Direct brute force solution is quite hopeless.

However, the 3D Sergeev functional mapping $\mathcal{R}_{123}^{(f)}$

$$\frac{U_1'}{U_1} = \frac{W_3'}{W_3} \ = \ \frac{K_2 U_2 W_2}{K_1 U_1 W_2 + K_3 U_2 W_3 + K_1 K_3 U_1 W_3} \, ;$$

$$\frac{W_1}{W_1'} = \frac{W_2'}{W_2} \ = \ \frac{W_1 W_3}{W_1 W_2 + U_3 W_2 + K_3 U_3 W_3} \, ; \quad \frac{U_2'}{U_2} \ = \ \frac{U_3}{U_3'} \ = \ \frac{U_1 U_3}{U_2 U_3 + U_2 W_1 + K_1 U_1 W_1} \, .$$

suggests the solution: Introduce additional variables $U_3, \ W_3$

and constant K_3, demand periodicity.

We write 3D Sergeev functional mapping $\mathcal{R}^{(f)}_{123}$ as

$$\mathcal{R}^{(f)}_{123} \;:\quad U_1, W_1, U_2, W_2, U_3, W_3 \;\mapsto\; U'_1, W'_1, U'_2, W'_2, U'_3, W'_3$$

Consider the *composition* of two of these rational transformations

$$\mathcal{R}^{(f)}_{123} \;:\quad U_1, W_1, U_2, W_2, U_3, W_3 \;\mapsto\; U^\star_1, W^\star_1, U^\star_2, W^\star_2, U'_3, W'_3\,,$$

$$\mathcal{R}^{(f)}_{\tilde{1}\tilde{2}3} \;:\quad \tilde{U}_1, \tilde{W}_1, \tilde{U}_2, \tilde{W}_2, U'_3, W'_3 \;\mapsto\; \tilde{U}^\star_1, \tilde{W}^\star_1, \tilde{U}^\star_2, \tilde{W}^\star_2, U^\star_3, W^\star_3\,,$$

together with a periodic condition

$$U^\star_3 = U_3\,, \quad W^\star_3 = W_3$$

and denote this composition by

$$S^{(f)}_{12} :\; U_1, W_1, U_2, W_2, \tilde{U}_1, \tilde{W}_1, \tilde{U}_2, \tilde{W}_2 \mapsto U^\star_1, W^\star_1, U^\star_2, W^\star_2, \tilde{U}^\star_1, \tilde{W}^\star_1, \tilde{U}^\star_2, \tilde{W}^\star_2\,.$$

With these definitions we have

Proposition: *The rational transformation* $S^{(f)}_{12}$ *solves the intertwining relations* $\mathcal{L}_2(\Lambda)\,\mathcal{L}_1(\Lambda) = \mathcal{L}^\star_1(\Lambda)\,\mathcal{L}^\star_2(\Lambda)\,.$

Proof: Solve first using the periodic b.c. for U_3, W_3, which gives

$$U_3 = \frac{U_1(\tilde{K}_1\tilde{U}_1\tilde{W}_1 + K_1\tilde{U}_2W_1) + \tilde{U}_2(U_2W_1 + U_1\tilde{W}_1)}{U_1\tilde{U}_1 - U_2\tilde{U}_2}$$

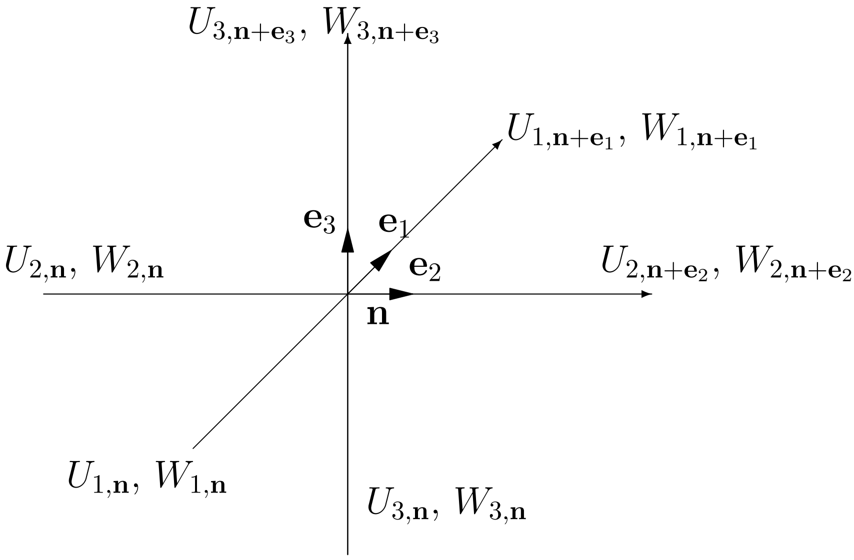
$$W_3 = \frac{W_2\tilde{W}_2(\tilde{K}_2K_2\tilde{U}_2U_2 - \tilde{K}_1K_1\tilde{U}_1U_1)}{K_3(U_2(\tilde{K}_1\tilde{U}_1\tilde{W}_2 + K_2\tilde{U}_2W_2) + \tilde{K}_1\tilde{U}_1(K_1U_1\tilde{W}_2 + K_2U_2W_2))},$$

then just insert.

6 Uniformization of the bulk functional mapping

Up to now we discussed *local* features of the models. Now bulk properties:
First we consider the classical part.

Functional mapping, written for a general vertex $\mathbf{n} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3$:



$$\frac{U_{1,\mathbf{n}+\mathbf{e}_1}}{U_{1,\mathbf{n}}} = \frac{W_{3,\mathbf{n}+\mathbf{e}_3}}{W_{3,\mathbf{n}}} = \frac{K_{2:n_1,n_3} U_{2,\mathbf{n}} W_{2,\mathbf{n}}}{K_{1:\ldots} U_{1,\mathbf{n}} W_{2,\mathbf{n}} + K_{3:\ldots} U_{2,\mathbf{n}} W_{3,\mathbf{n}} + K_{1:\ldots} K_{3:\ldots} U_{1,\mathbf{n}} W_{3,\mathbf{n}}},$$

$$\frac{W_{1,\mathbf{n}}}{W_{1,\mathbf{n}+\mathbf{e}_1}} = \frac{W_{2,\mathbf{n}+\mathbf{e}_2}}{W_{2,\mathbf{n}}} = \frac{W_{1,\mathbf{n}} W_{3,\mathbf{n}}}{W_{1,\mathbf{n}} W_{2,\mathbf{n}} + U_{3,\mathbf{n}} W_{2,\mathbf{n}} + K_{3:n_1,n_2} U_{3,\mathbf{n}} W_{3,\mathbf{n}}},$$

$$\frac{U_{2,\mathbf{n}+\mathbf{e}_2}}{U_{2,\mathbf{n}}} = \frac{U_{3,\mathbf{n}}}{U_{3,\mathbf{n}+\mathbf{e}_3}} = \frac{U_{1,\mathbf{n}} U_{3,\mathbf{n}}}{U_{2,\mathbf{n}} U_{3,\mathbf{n}} + U_{2,\mathbf{n}} W_{1,\mathbf{n}} + K_{1:n_2,n_3} U_{1,\mathbf{n}} W_{1,\mathbf{n}}}.$$

Legendre transf. from the 12 variables at each vertex $U_{1,\mathbf{n}+\mathbf{e}_1}, \ldots, U_{3,\mathbf{n}}$ to ratios of 9 fcts $\tau_{1,\mathbf{n}}, \ldots, \tau_{3,\mathbf{n}+\mathbf{e}_2}$ and 6 points $X_{\mathbf{n}_1}, X'_{\mathbf{n}_1}, Y_{\mathbf{n}_2}, Y'_{\mathbf{n}_2}, Z_{\mathbf{n}_3}, Z_{\mathbf{n}'_3}$:

$$U_{1,\mathbf{n}} = \frac{Y_{n_2} - Z'_{n_3}}{Y_{n_2} - Z_{n_3}} \frac{\tau_{2,\mathbf{n}}}{\tau_{2,\mathbf{n}+\mathbf{e}_3}}, \quad W_{1,\mathbf{n}} = -\frac{Z_{n_3} - Y'_{n_2}}{Z_{n_3} - Y_{n_2}} \frac{\tau_{3,\mathbf{n}+\mathbf{e}_2}}{\tau_{3,\mathbf{n}}}, \quad etc.$$

then $\mathcal{R}_{\mathbf{n}}^{(f)}$ takes the form of 3 trilinear Hirota equations: Pakuliak, Sergeev 2002

$$
\begin{aligned}
&(X_\alpha - X_\beta)(X'_\beta - X'_\gamma)(X_\gamma - X_\alpha)\tau_{\alpha,\mathbf{n}+\mathbf{e}_\beta+\mathbf{e}_\gamma}\tau_{\beta,\mathbf{n}}\tau_{\gamma,\mathbf{n}} \\
&\qquad\qquad + (X_\alpha - X'_\beta)(X_\beta - X_\gamma)(X'_\gamma - X_\alpha)\tau_{\alpha,\mathbf{n}}\tau_{\beta,\mathbf{n}+\mathbf{e}_\gamma}\tau_{\gamma,\mathbf{n}+\mathbf{e}_\beta} \\
&\quad = \ (X_\alpha - X_\beta)(X'_\beta - X_\gamma)(X'_\gamma - X_\alpha)\tau_{\alpha,\mathbf{n}+\mathbf{e}_\beta}\tau_{\beta,\mathbf{n}+\mathbf{e}_\gamma}\tau_{\gamma,\mathbf{n}} \\
&\qquad\qquad + (X_\alpha - X'_\beta)(X_\beta - X'_\gamma)(X_\gamma - X_\alpha)\tau_{\alpha,\mathbf{n}+\mathbf{e}_\gamma}\tau_{\beta,\mathbf{n}}\tau_{\gamma,\mathbf{n}+\mathbf{e}_\beta} \ ,
\end{aligned}
$$

$(\alpha,\beta,\gamma) = (1,2,3),(2,3,1),(3,1,2);\ X_1 = X_{n_1}, X'_1 = X'_{n_1}, X_2 = Y'_{n_2}, X'_2 = Y_{n_2}, X_3 = Z_{n_3}, X'_3 = Z'_{n_3}.$

Now well-known since 1978-84 : *any discrete integrable system can be solved by methods of algebraic geometry.* Kirchever, Shiota, Mulase

Trilinear Hirota eqs. identically solved by Double-Fay-Identity.

Practical managable solutions: "rational Θ-functions" H:

$$
H^{(g)}\left(\{P'_j, P_j, f_j\}_{k=0}^{g-1}\right) \ = \ \frac{\det|\, P_j^i \ - \ f_j {P'_j}^i \,|_{i,j=0}^{g-1}}{\prod_{i>j}(P_i - P_j)}
$$

$H(f_0) = 1 - f_0;\quad H(f_0,f_1) = 1 - \frac{P_1-P'_0}{P_1-P_0} f_0 - \frac{P_0-P'_1}{P_1-P_0} f_1 + \frac{P'_1-P'_0}{P_1-P_0}\, f_0 f_1;$

$$
\begin{aligned}
H(f_0,f_1,f_2) = 1 &- \tfrac{(P'_0-P_2)(P'_0-P_1)}{(P_0-P_1)(P_0-P_2)}\, f_0 + \ldots + \tfrac{(P'_1-P_2)(P'_0-P_2)(P'_1-P'_0)}{(P_2-P_1)(P_2-P_0)(P_1-P_0)}\, f_0 f_1 + \ldots \\
&+ \ldots + \tfrac{(P'_0-P'_1)(P'_1-P'_2)(P'_2-P_0)}{(P_2-P_1)(P_2-P_0)(P_1-P_0)}\, f_0 f_1 f_2.
\end{aligned}
$$

The solutions for the Weyl centers can be written in terms of the functions:

$$
U\left(\{f_k\}\,,\,\mathcal{A},\,\mathcal{B}\right) \ =: \ -\varepsilon \frac{A - B'}{A - B} \ \frac{H(\{P'_j, P_j,\ \mathbf{f}_j(A)\})}{H(\{P'_j, P_j,\ \mathbf{f}_j(A)\,\sigma_k(\mathcal{B})\})}
$$

$\mathcal{A} = (A', A),\ \ldots,\ \mathcal{Y} = (Y', Y),\ etc.$ stand for pairs of points, and

$$
\mathbf{f}_j(Y) \ = \ \frac{P_j - Y}{P'_j - Y}\ f_j; \qquad \sigma_j(\mathcal{Y}) \ = \ \begin{bmatrix} P'_j & P_j \\ Y' & Y \end{bmatrix}.
$$

Then

$$
U_{1,\mathbf{n}} \ = \ U\left(\{\mathbf{f}_j I_{j:\mathbf{n}}\}\,,\,\mathcal{Y}_{n_2},\ \mathcal{Z}_{n_3}\right);\quad U_{2,\mathbf{n}} \ = \ U\left(\{\mathbf{f}_j I_{j:\mathbf{n}}\sigma_j(\mathcal{Y}_{n_2})\}\,,\,\mathcal{Y}_{n_2},\ \mathcal{Z}_{n_3}\right);
$$

$$
W_{1,\mathbf{n}} = -U\left(\{\mathbf{f}_j I_{j:\mathbf{n}}\},\mathcal{Z}_{n_3},\mathcal{Y}_{n_2}\right);\quad W_{2,\mathbf{n}} = -U\left(\{\mathbf{f}_j I_{j:\mathbf{n}}\sigma_j(\mathcal{Z}_{n_3})\},\mathcal{Z}_{n_3},\mathcal{Y}_{n_2}\right);
$$

$$U_{3,\mathbf{n}} = U\left(\left\{\mathbf{f}_j I_{j:\mathbf{n}}\right\}, \mathcal{X}_{n_1}, \mathcal{Y}'_{n_2}\right); \quad W_{3,\mathbf{n}} = -U\left(\left\{\mathbf{f}_j I_{j:\mathbf{n}} \sigma_j(\mathcal{Y}'_{n_2})\right\}, \mathcal{Y}'_{n_2}, \mathcal{X}_{n_1}\right) \ .$$

where

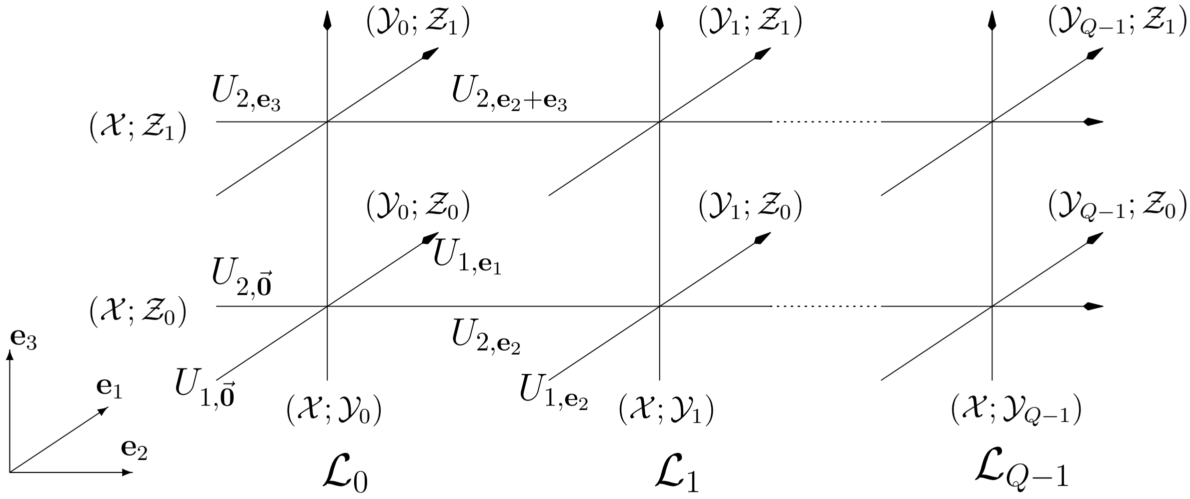
$$I_{j:\mathbf{n}} = \prod_{m_1=0}^{n_1-1} \frac{\sigma_j(\mathcal{X}'_{m_1})}{\sigma_j(\mathcal{X}_{m_1})} \prod_{m_2=0}^{n_2-1} \frac{\sigma_j(\mathcal{Y}_{m_2})}{\sigma_j(\mathcal{Y}'_{m_2})} \prod_{m_3=0}^{n_3-1} \frac{\sigma_j(\mathcal{Z}_{m_3})}{\sigma_j(\mathcal{Z}_{m_3})}.$$

7 The classical inhomogenous BS-Chain

We consider slice of the 3D-lattice:

Chain in $\vec{e_2}$-direction, single layer in $\vec{e_1}$, periodic after two steps in $\vec{e_3}$:

$$\tilde{U}_{i,\mathbf{n}} \equiv U_{i,\mathbf{n}+\mathbf{e}_3}, \quad \tilde{W}_{i,\mathbf{n}} \equiv W_{i,\mathbf{n}+\mathbf{e}_3}$$



$$\mathcal{L}_n(\Lambda) = \begin{pmatrix} \tilde{W}_{1,n\mathbf{e}_2} + \Lambda U_{1,n\mathbf{e}_2}\tilde{U}_{1,n\mathbf{e}_2} K_{1:n} W_{1,n\mathbf{e}_2} & U_{1,n\mathbf{e}_2}\left(\tilde{W}_{1,n\mathbf{e}_2} + K_{1:n} W_{1,n\mathbf{e}_2}\right) \\ \Lambda \tilde{U}_{1,n\mathbf{e}_2}\left(W_{1,n\mathbf{e}_2} + \tilde{K}_{1:n}\tilde{W}_{1,n\mathbf{e}_2}\right) & W_{1,n\mathbf{e}_2} + \Lambda \tilde{K}_{1:n} U_{1,n\mathbf{e}_2}\tilde{U}_{1,n\mathbf{e}_2}\tilde{W}_{1,n\mathbf{e}_2} \end{pmatrix}.$$

Classical monodromy matrix (chain length $\mathcal{N}$)

$$\mathcal{M}(\Lambda) = \left(\prod_{n'=0}^{\mathcal{N}-1} \tilde{W}^{-1}_{1,n'\mathbf{e}_2} \right) \mathcal{L}_0(\Lambda)\, \mathcal{L}_1(\Lambda) \cdots \mathcal{L}_n(\Lambda) \cdots \mathcal{L}_{\mathcal{N}-1}(\Lambda).$$

Now we derive an *isospectrality transformation* of the transfer matrix $\mathrm{Tr}\,\mathcal{M}(\Lambda)$ by moving an auxiliary classical operator

$$\mathcal{L}_0^{aux}(\Lambda) = \begin{pmatrix} \tilde{W}_{2,\vec{\mathbf{0}}} + \Lambda U_{2,\vec{\mathbf{0}}}\tilde{U}_{2,\vec{\mathbf{0}}} K_{2:\vec{\mathbf{0}}} W_{2,\vec{\mathbf{0}}} & U_{2,\vec{\mathbf{0}}}\left(\tilde{W}_{2,\vec{\mathbf{0}}} + K_{2:\vec{\mathbf{0}}} W_{2,\vec{\mathbf{0}}}\right) \\ \Lambda \tilde{U}_{2,\vec{\mathbf{0}}}\left(W_{2,\vec{\mathbf{0}}} + \tilde{K}_{2:\vec{\mathbf{0}}}\tilde{W}_{2,\vec{\mathbf{0}}}\right) & W_{2,\vec{\mathbf{0}}} + \Lambda \tilde{K}_{2:\vec{\mathbf{0}}} U_{2,\vec{\mathbf{0}}}\tilde{U}_{2,\vec{\mathbf{0}}}\tilde{W}_{2,\vec{\mathbf{0}}} \end{pmatrix}$$

through the monodromy

$$\mathcal{L}_0^{aux}\mathcal{L}_0\, \mathcal{L}_1 \ldots \mathcal{L}_{\mathcal{N}-1} = \mathcal{L}_0^\star\, \mathcal{L}_1^{aux}\mathcal{L}_1 \ldots \mathcal{L}_{\mathcal{N}-1} = \ldots = \mathcal{L}_0^\star\, \mathcal{L}_1^\star \ldots \mathcal{L}_{\mathcal{N}-1}^\star\, \mathcal{L}_{\mathcal{N}}^{aux},$$

requiring $\mathcal{L}_0^{aux} = \mathcal{L}_{\mathcal{N}}^{aux}$ so that $\mathrm{Tr}\,\mathcal{M}(\Lambda) = \mathrm{Tr}\,\mathcal{M}^\star(\Lambda)$.

We have studied the intertwining of two operators $\mathcal{L}$:

$$\mathcal{L}(U_2, W_2, \tilde{U}_2, \tilde{W}_2, \Lambda) \cdot \mathcal{L}(U_1, W_1, \tilde{U}_1, \tilde{W}_1, \Lambda)$$
$$= \mathcal{L}(U_1^\star, W_1^\star, \tilde{U}_1^\star, \tilde{W}_1^\star, \Lambda) \cdot \mathcal{L}(U_2^\star, W_2^\star, \tilde{U}_2^\star, \tilde{W}_2^\star, \Lambda)$$

was described by the mapping

$$S_{12}^{(f)}: \; U_1, W_1, U_2, W_2, \tilde{U}_1, \tilde{W}_1, \tilde{U}_2, \tilde{W}_2 \mapsto U_1^\star, W_1^\star, U_2^\star, W_2^\star, \tilde{U}_1^\star, \tilde{W}_1^\star, \tilde{U}_2^\star, \tilde{W}_2^\star \,.$$

The mapping $S_{12}^{(f)}$ takes a simple form in terms of the functions U if the periodicity condition $U_3 = U_3^\star$, which here is $\sigma_j(\mathcal{Z}_0)\sigma_j(\mathcal{Z}_1) = 1$, is imposed.

Indicating the arguments of the variables U_j only, this reads:

$$\mathcal{L}_0^{aux}\left(\{f_j\sigma_j(\mathcal{Y}_0)\}, \mathcal{X}, \mathcal{Z}_0, \mathcal{Z}_1\right) \cdot \mathcal{L}_0\left(\{f_j\}, \mathcal{Y}_0, \mathcal{Z}_0, \mathcal{Z}_1\right)$$
$$= \mathcal{L}_0^\star\left(\{f_j\,\sigma_j(\mathcal{X})\}, \mathcal{Y}_0, \mathcal{Z}_0, \mathcal{Z}_1\right) \cdot \mathcal{L}_1^{aux}\left(\{f_j\}, \mathcal{X}, \mathcal{Z}_0, \mathcal{Z}_1\right):$$

In the U_2, W_2 arguments of operator $\mathcal{L}_0^{aux}$ the intertwining mapping is

$$f_j \rightarrow f_j\,\sigma^{-1}(\mathcal{Y}_0),$$

while the U_1, W_1-arguments of the $\mathcal{L}_0$ are mapped $f_j \rightarrow f_j\sigma_j(\mathcal{X})$.

Finally, periodicity

$$\mathcal{L}_0^{aux}\left(\{f_j\sigma_j(\mathcal{Y}_0)\}, \mathcal{X}, \mathcal{Z}_0, \mathcal{Z}_1\right) = \mathcal{L}_{\mathcal{N}}^{aux}\left(\left\{f_j\sigma_j(\mathcal{Y}_0)\prod_{i=0}^{\mathcal{N}-1}\sigma_j^{-1}(\mathcal{Y}_i)\right\}, \mathcal{X}, \mathcal{Z}_0, \mathcal{Z}_1\right)$$

requires

$$\prod_{i=0}^{\mathcal{N}-1}\sigma_j^{-1}(\mathcal{Y}_i) \;=\; 1\,.$$

If this is satisfied, the transfer matrices $\mathrm{Tr}\,\mathcal{M}(\Lambda)$ and $\mathrm{Tr}\,\mathcal{M}^\star(\Lambda)$ are *isospectral.* This is non-trivial if some $f_j \neq 0$.

8 Conclusions

- Well-known are the relations between the BS-model and the Chiral Potts model, and between Chiral Potts model and 3D ZBB-model.
- Here we studied in detail the direct relation between BS-model and 3D ZBB-model and its generalizations.
- BS-Lax-operator is obtained from the Sergeev linear problem.
- The BS quantum intertwiner matrix S is obtained as the product of two 3D mapping operators.
- 3d parameterization (Fermat points and Weyl centers) leads to a simple cross-ratio parameterization of BS operators which can be generalized to the inhomogenous case.
- Parameterization of inhomogenous BS model in terms of rational Theta-functions allows derivation of new BS isospectral transformations.

Proposition: Let M be length of the periodic BS-chain. Consider the system

$$\prod_{q=0}^{M-1} \frac{(P'-Y'_q)(P-Y_q)}{(P-Y'_q)(P'-Y_q)} \;=\; \prod_{j=0,1} \frac{(P'-Z'_j)(P-Z_j)}{(P-Z'_j)(P'-Z_j)} \;=\; 1\;.$$

where $Y_q, Y'_q, Z_0, \ldots$ are given by the CP-variables:

$$kx_q^N = \frac{(Y_q-Z_0)(Z'_1-Z'_0)}{(Y_q-Z'_0)(Z'_1-Z_0)}\;,\quad ky_q^N = \ldots\;;\quad k^2 = \frac{(Z_0-Z_1)(Z'_0-Z'_1)}{(Z'_0-Z_1)(Z_0-Z'_1)}.$$

This system has $g=M-1$ solutions $\{(P'_k,P_k)\}$. Define

$$I_k(q) \;=\; \prod_{j=0}^{q-1} \frac{\sigma_k(Y_j)}{\sigma_k(Y'_j)} \;\;:\quad I_k(0)=I_k(Q)=1\,.$$

Then, the spectrum of the BS transfer matrix with

$$\mathbf{X}_q^N \;=\; \frac{Y_q-Z'_0}{Y_q-Z_0}\;\frac{H(\{f_kI_k(q)\sigma_k^{-1}(Y'_q)\})}{H(\{f_kI_k(q)\sigma_k(Z'_0)\sigma_k^{-1}(Y'_q)\sigma_k^{-1}(Z_0)\}}\;,$$

$$\mathbf{Z}_q^N \;=\; \frac{Y'_q-Z_0}{Y_q-Z_0}\;\frac{Y_q-Z'_1}{Y'_q-Z_1}\;\frac{H(\{f_kI_k(q+1)\sigma_k^{-1}(Z_0)\})H(\{f_kI_k(q)\sigma_k^{-1}(Z_1)\})}{H(\{f_kI_k(q)\sigma_k^{-1}(Z_0)\})H(\{f_kI_k(q+1)\sigma_k^{-1}(Z_1)\})}$$

does not depend on the set of $\{f_k\}$*.*

Theta functions

Concepts: Γ_g: generic algebraic curve of genus g

$\boldsymbol{\omega}$ canonical g-dim. vector of the homomorphic differentials.

For $X, Y \in \Gamma_g$ let $\mathbf{I}_Y^X : \Gamma_g^2 \mapsto \mathrm{Jac}(\Gamma_g)$ be $\mathbf{I}_Y^X \stackrel{\text{def}}{=} \int_Y^X \boldsymbol{\omega}$

$E(X,Y) = -E(Y,X)$: prime form on Γ_g^2,

$\Theta(\mathbf{v})$: theta-function on $\mathrm{Jac}(\Gamma_g)$:

$$\Theta^{(g)}(\mathbf{z}) = \sum_{\mathbf{n}\in\mathbb{Z}^g} \exp\left(i\pi(\mathbf{n},\Omega\mathbf{n}) + 2i\pi(\mathbf{n},\mathbf{z})\right)$$

Fay identity for $A, B, C, D \in \Gamma_g$ and a $\mathbf{v} \in \mathrm{Jac}(\Gamma_g)$:

$$\begin{aligned} &\Theta(\mathbf{v})\,\Theta(\mathbf{v} + \mathbf{I}_B^A + \mathbf{I}_D^C) \\ &\quad = \Theta(\mathbf{v} + \mathbf{I}_D^A)\,\Theta(\mathbf{v} + \mathbf{I}_B^C)\,\frac{E(A,B)\,E(D,C)}{E(A,C)\,E(D,B)} \\ &\quad + \Theta(\mathbf{v} + \mathbf{I}_B^A)\,\Theta(\mathbf{v} + \mathbf{I}_D^C)\,\frac{E(A,D)\,E(C,B)}{E(A,C)\,E(D,B)}, \end{aligned}$$

In "physical" expressions prime forms only as cross ratios: expressible in thetas:

$$\begin{bmatrix} X & X' \\ Y & Y' \end{bmatrix}_E \stackrel{\text{def}}{=} \frac{E(X,Y)\,E(X',Y')}{E(X,Y')\,E(X',Y)} = \frac{\Theta_{\epsilon_{odd}}(\mathbf{I}_X^Y)\,\Theta_{\epsilon_{odd}}(\mathbf{I}_{X'}^{Y'})}{\Theta_{\epsilon_{odd}}(\mathbf{I}_X^{Y'})\,\Theta_{\epsilon_{odd}}(\mathbf{I}_{X'}^{Y})}.$$

More useful here: **Rational** limit of Θ : Sergeev 2000, Pakuliak, Sergeev 2002

$$\Theta^{(g)}(\mathbf{z}) \Rightarrow H^{(g)}\left(\{p_k, q_k, f_k\}_{k=0}^{g-1}\right) = \frac{\det|\, q_j^i - f_j\, p_j^i \,|_{i,j=0}^{g-1}}{\prod_{i>j}(q_i - q_j)}.$$

$f_0, f_1, \ldots, f_{g-1}$ remember $\mathbf{z}$ and $p_0, q_0, \ldots, p_{g-1}, q_{g-1}$ remember Ω.

$$\begin{bmatrix} X & X' \\ Y & Y' \end{bmatrix}_E \Rightarrow \frac{(X-Y)(X'-Y')}{(X-Y')(Y-X')};$$ analogous Fay-Identity also for $H^{(g)}$.

9 Conclusions

The intimate relation of the BS-model
to the 3D ZBB-model is no surprise:

Bazhanov and Stroganov found the relation of the BS-model to the integrable chiral Potts model,

Bazhanov and Baxter found that the $\mathbb{Z}_N$-generalization of the Zamolodchikov model for two layers leads to the integrable chiral Potts model.

Here, the relation of the Sergeev mapping and the Weyl Linear Problem, to the BS-model is made explicit.

The cross-ratio parametrization simplifies most functional variable changes.

It is related to the rational limit of the Theta-function parametrization of the functional mapping.

The introduction of a third pair of variables, which is afterwards eliminated by a periodicity condition, gives a transparent 3D-method to solve the intertwining relations of the BS-model.