A Method for Generating Uniformly Scattered Points on the L_p-norm Unit Sphere and Its Applications

Jiajuan Liang

University of New Haven, U.S.A.

(Joint work with Kai Wang Ng, The University of Hong Kong)

• Some Regular Regions in the s-dimensional ($s \geq 2$) Euclidean Space R^s and the Corresponding Uniform Distributions

The unit hypercube:

$$C^s = [0,1]^s = \{\boldsymbol{z} = (z_1, \cdots, z_s)' \in R^s,\ 0 \leq z_i \leq 1,\ i = 1, \ldots, s\} \tag{1}$$

The uniform distribution on C^s is denoted by $U(C^s)$.

The (surface of) L_p-norm unit sphere:

$$S_s^p = \{\boldsymbol{x} = (x_1, \cdots, x_s)' \in R^s,\ \|\boldsymbol{x}\|_p = (|x_1|^p + \cdots + |x_s|^p)^{1/p} = 1,\quad p > 0,\ s \geq 2\}. \tag{2}$$

When $p = 1$, $S_s^p = S_s^1$ reduces to a unit hypercube;

When $p = 2$, $S_s^p = S_s^2$ reduces to the (surface of) a unit sphere in the usual meaning.

Definition. A random vector $\boldsymbol{u}_s = (U_1, \cdots, U_s)'$ is said to have an L_p-norm uniform distribution ($p > 0$), denoted by $\boldsymbol{u}_s \sim \mathcal{U}(s,p)$, if $\Sigma_{i=1}^{s} |U_i|^p = 1$ and the joint p.d.f. of $U_1, \ldots, U_{s-1}$ is given by

$$g(u_1, \ldots, u_{s-1}) = \frac{p^{s-1}\Gamma(s/p)}{2^{s-1}\Gamma^s(1/p)} \left(1 - \sum_{i=1}^{s-1} |u_i|^p\right)^{(1-p)/p},$$

$$-1 < u_i < 1, \quad i = 1, \ldots, s-1, \quad \Sigma_{i=1}^{s-1} |u_i|^p < 1.$$

Sets of points in S_s^p ($p > 0,\ s \geq 2$) with certain uniformity are usually obtained by using the Inverse Transformation Method to project sets of points in C^s with certain optimal uniformity property onto S_s^p.

Major reference:

Fang, K.T. & Wang, Y. (1994). *Number-theoretic Methods in Statistics.* Chapman and Hall, London.

The purpose of this paper is to find a suitable Inverse Transformation Method.

• Measures of Uniformity

There are several ways of measuring uniformity by discrepancy.

<u>The usual discrepancy</u>: Let $\mathcal{P} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\} \subset R^s$.

$$D(n, \mathcal{P}) = \sup_{\boldsymbol{\gamma} \in C^s} \left| \frac{N(\boldsymbol{\gamma}, \mathcal{P})}{n} - v([\mathbf{0}, \boldsymbol{\gamma}]) \right|, \qquad (3)$$

is called the (usual) discrepancy.

$v([\mathbf{0}, \boldsymbol{\gamma}])$: the volume of the rectangle $[\mathbf{0}, \boldsymbol{\gamma}]$

$N(\boldsymbol{\gamma}, \mathcal{P})$: the number of points in $\mathcal{P}$ satisfying $\boldsymbol{z}_i \leq \boldsymbol{\gamma}$ (componentwise)

<u>The F-discrepancy</u>: Let $F(\boldsymbol{x})$ be a c.d.f and

$$F_n(\boldsymbol{x}) = \frac{1}{n} \sum_{i=1}^{n} I\{\boldsymbol{x}_i \leq \boldsymbol{x}\},$$

the e.c.f. based on $\mathcal{P} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$. Then

$$D_F(n, \mathcal{P}) = \sup_{\boldsymbol{x} \in R^s} |F_n(\boldsymbol{x}) - F(\boldsymbol{x})| \qquad (4)$$

is called the F-discrepancy of $\mathcal{P}$ with respect to $F(\boldsymbol{x})$.

The quasi F-discrepancy: Let $\boldsymbol{x}$ be an $s \times 1$ random vector with c.d.f. $F(\boldsymbol{x})$. $\boldsymbol{x}$ has a stochastic representation $\boldsymbol{x} = \boldsymbol{h}(\boldsymbol{z})$, where $\boldsymbol{z} \sim U(C^t)$ $(t \leq s)$. Let $\{\boldsymbol{c}_k : \ k = 1, \ldots, n\}$ be a set of points that are uniformly scattered in C^t with (usual) discrepancy d. Then the set of points $\mathcal{P}_F = \{\boldsymbol{h}(\boldsymbol{c}_k) : \ k = 1, \ldots, n\}$ is said to have a quasi F-discrepancy d with respect to $F(\boldsymbol{x})$.

In this paper we will provide an algorithm for generating a set of uniformly scattered points in S_s^p with a minimum quasi F-discrepancy by projecting the *glp* set in C^s.

Definition. Let $(n; h_1, \ldots, h_s)$ be a vector with integer components satisfying $1 \leq h_i \leq n$, $h_i \neq h_j$ $(i \neq j)$, $s < n$, and the greatest common divisors $(n, h_i) = 1$, $i = 1, \ldots, s$. Let

$$\begin{cases} q_{ki} = kh_i \pmod{n}, \\ x_{ki} = (2q_{ki} - 1)/2n, \end{cases} \quad k = 1, \ldots, n;\ i = 1, \ldots, s, \tag{5}$$

where $1 \leq q_{ki} < n$. Then the set $\mathcal{P}_n = \{\boldsymbol{x}_k = (x_{k1}, \ldots, x_{ks})',\ k = 1, \ldots, n\}$ is called the lattice point set with the generating vector $(n; h_1, \ldots, h_s)$. If the set $\mathcal{P}_n$ has the smallest discrepancy in the sense of (3) among all possible generating vectors, then the set $\mathcal{P}_n$ is called a *glp* set.

Fang & Wang (1994) provides the generating vectors for some *glp* set in C^s.

• Theoretical Results

Theorem 1. Let $\boldsymbol{u}_s = (U_1, \cdots, U_s)' \sim \mathcal{U}(s, p)$. Define the random variables B_i ($i = 1, \ldots, s-1$) by the following conditional distributions:

$$\begin{aligned}
B_1 &\stackrel{\mathrm{d}}{=} |U_1|^p, \\
B_2 &\stackrel{\mathrm{d}}{=} \{(1 - |U_1|^p)^{-1}|U_2|^p|U_1\}, \\
&\vdots \\
B_m &\stackrel{\mathrm{d}}{=} \{(1 - \Sigma_{i=1}^{m-1} |U_i|^p)^{-1}|U_m|^p|(U_1, \cdots, U_{m-1})\},
\end{aligned} \tag{6}$$

where $m = 2, \ldots, s-1$, the sign "$\stackrel{\mathrm{d}}{=}$" means that the two sides of the equality have the same probability distribution, and $\{\cdot|\cdot\}$ stands for the conditional distribution given the part on the right hand side of "$|$". Then $B_1, \ldots, B_{s-1}$ are mutually independent and $B_k \sim \mathrm{Beta}[1/p, (s-k)/p]$ (the beta distribution, $k = 1, \ldots, s-1$).

Theorem 2. Assume that $\boldsymbol{u} = (U_1, \cdots, U_s)' \sim \mathcal{U}(s,p)$. Let $V_1, \ldots, V_s$ be i.i.d. and $V_i \sim U(0,1)$, and $B_1, \ldots, B_{s-1}$ be independent such that $B_k \sim \text{Beta}[1/p, (s-k)/p]$ ($k = 1, \ldots, s-1$). Denote by $F_k(\cdot)$ the c.d.f. of B_k and $F_k^{-1}(\cdot)$ the inverse function of $F_k(\cdot)$. Then the random vector $\boldsymbol{u} = (U_1, \cdots, U_s)' \sim \mathcal{U}(s,p)$ has a stochastic representation

$$\boldsymbol{u} \overset{\mathrm{d}}{=} \boldsymbol{x} = (X_1, \cdots, X_s)', \tag{7}$$

where the components $X_1, \ldots, X_s$ are given by

$$
\begin{aligned}
X_1 &= \text{sign}(2V_1 - 1)\left\{F_1^{-1}[(2V_1 - 1)\text{sign}(2V_1 - 1)]\right\}^{1/p}, \\
X_2 &= \text{sign}(2V_2 - 1)\left\{(1 - |X_1|^p) F_2^{-1}[(2V_2 - 1)\text{sign}(2V_2 - 1)]\right\}^{1/p}, \\
&\vdots \\
X_{s-1} &= \text{sign}(2V_{s-1} - 1)\left\{(1 - \textstyle\sum_{i=1}^{s-2} |X_i|^p)\cdot\right. \\
&\qquad \left. F_{s-1}^{-1}[(2V_{s-1} - 1)\text{sign}(2V_{s-1} - 1)]\right\}^{1/p}, \\
X_s &= \text{sign}(2V_s - 1)\left(1 - \textstyle\sum_{i=1}^{s-1} |X_i|^p\right)^{1/p},
\end{aligned}
\tag{7}
$$

here sign$(\cdot)$ stands for the sign function.

• The Algorithm for Generating Uniformly Scattered Points on S_s^p

Step 1. For the given number of points n and the dimension $s \geq 2$, find the *glp* set $\mathcal{P}_z = \{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_n\} \subset C^s$ from Appendix A in Fang & Wang (1994);

Step 2. Obtain the set of points $\mathcal{P}_x = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\} \subset S_s^p$ by projection in the following way: denote by $\boldsymbol{x}_i = (x_{i1}, \ldots, x_{is})'$ and $\boldsymbol{z}_i = (z_{i1}, \ldots, z_{is})'$ $(i = 1, \ldots, n)$, let

$$
\begin{aligned}
x_{i1} &= \mathrm{sign}(2z_{i1}-1)\left\{F_1^{-1}[(2z_{i1}-1)\mathrm{sign}(2z_{i1}-1)]\right\}^{1/p}, \\
x_{i2} &= \mathrm{sign}(2z_{i2}-1)\left\{(1-|x_{i1}|^p)F_2^{-1}[(2z_{i2}-1)\mathrm{sign}(2z_{i2}-1)]\right\}^{1/p}, \\
&\vdots \\
x_{i,s-1} &= \mathrm{sign}(2z_{i,s-1}-1)\left\{(1-\sum_{j=1}^{s-2}|x_{ij}|^p)F_{s-1}^{-1}[(2z_{i,s-1}-1)\mathrm{sign}(2z_{i,s-1}-1)]\right\}^{1/p} \\
x_{is} &= \begin{cases} \mathrm{sign}(2z_{is}-1)(1-\sum_{j=1}^{s-1}|x_{ij}|^p)^{1/p}, & \text{if } z_{is} \neq 0.5, \\ (1-\sum_{j=1}^{s-1}|x_{ij}|^p)^{1/p}, & \text{if } z_{is} = 0.5. \end{cases}
\end{aligned}
\tag{8}
$$

Then the set of points $\mathcal{P}_x = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\} \subset S_s^p$ has a quasi F-discrepancy d (d is the usual discrepancy of $\mathcal{P}_z \subset C^s$) with respect to the c.d.f. of $\boldsymbol{x} \sim \mathcal{U}(s, p)$, or $\mathcal{P}_x$ has an F-discrepancy d with respect to the c.d.f. $H(\boldsymbol{v}) = \Pi_{i=1}^s v_i$ of the random vector $\boldsymbol{V} = (V_1, \ldots, V_s)'$ with independent components $V_i \sim U(0,1)$ ($\boldsymbol{v} = (v_1, \ldots, v_s) \in C^s$).

It can be proved that d is equal to:

$$d = \sup_{\boldsymbol{r} \in C^s} \left| \frac{N(\mathcal{P}_x, G_{\boldsymbol{r}})}{n} - H(r) \right|, \tag{9}$$

where $\boldsymbol{r} = (r_1, \ldots, r_s)' \in C^s$, $H(\boldsymbol{r}) = \Pi_{i=1}^s r_i$, $N(\mathcal{P}_x, G_{\boldsymbol{r}})$ stands for number of points in $\mathcal{P}_x$ that fall in the set $G_{\boldsymbol{r}}$ defined by

$$G_{\boldsymbol{r}} = \{\boldsymbol{x} : \ \boldsymbol{x} = \boldsymbol{h}(\boldsymbol{v}), \ \boldsymbol{v} \leq r\}. \tag{10}$$

where $\boldsymbol{h}(\boldsymbol{v}) = (h_1(\boldsymbol{v}), \ldots, h_s(\boldsymbol{v}))$ ($\boldsymbol{v} = (v_1, \ldots, v_s) \in C^s$)

with

$$
\begin{aligned}
h_1(\boldsymbol{v}) &= \operatorname{sign}(2v_1-1)\left\{F_1^{-1}[(2v_1-1)\operatorname{sign}(2v_1-1)]\right\}^{1/p},\\
h_2(\boldsymbol{v}) &= \operatorname{sign}(2v_2-1)\cdot\\
&\left\{[1-F_1^{-1}((2v_1-1)\operatorname{sign}(2v_1-1))]F_2^{-1}((2v_2-1)\operatorname{sign}(2v_2-1))\right\}^{1/p},\\
h_m(\boldsymbol{v}) &= \operatorname{sign}(2v_m-1)\left\{[1-\sum_{i=1}^{m-1}F_i^{-1}((2v_i-1)\operatorname{sign}(2v_i-1))]\cdot\right.\\
&\left.F_m^{-1}((2v_m-1)\operatorname{sign}(2v_m-1))\right\}^{1/p},\\
&\quad m=2,\ldots,s-1,\\
h_s(\boldsymbol{v}) &= \operatorname{sign}(2v_s-1)\left(1-\sum_{i=1}^{s-1}|h_i(\boldsymbol{v})|^p\right)^{1/p}.
\end{aligned}
\tag{11}
$$

The set of points $\mathcal{P}_x = \{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n\}$ obtained by the above algorithm has the smallest quasi F-discrepancy d with respect to the c.d.f. of the uniform distribution $\boldsymbol{x} \sim \mathcal{U}(s,p)$ on S_s^p, or $\mathcal{P}_x$ has the smallest F-discrepancy d with respect to the uniform c.d.f. $H(\boldsymbol{r}) = \Pi_{i=1}^s r_i$ with independent components, where $\boldsymbol{r} = (r_1,\ldots,r_s)' \in C^s$.

- Examples

(1) $s = 2$, $n = 8$, $p = 1/2,\ 1,\ 2,\ 3$, generating vector $(h_1, h_2) = (1, 5)$, the *glp* set in C^2 is $\left(\frac{k-0.5}{n}, \left\{\frac{h_2 k-0.5}{n}\right\}\right)$ for $n = 8$ and $1 \le k \le n$;

(2) $s = 2$, $n = 21$, $p = 1/2,\ 1,\ 2,\ 3$, $(h_1, h_2) = (1, 13)$, the *glp* set in C^2 is $\left(\frac{k-0.5}{n}, \left\{\frac{h_2 k-0.5}{n}\right\}\right)$ for $n = 21$ and $1 \le k \le n$;

(3) $s = 2$, $n = 55$, $p = 1/2,\ 1,\ 2,\ 3$, $(h_1, h_2) = (1, 34)$, the *glp* set in C^2 is $\left(\frac{k-0.5}{n}, \left\{\frac{h_2 k-0.5}{n}\right\}\right)$ for $n = 55$ and $1 \le k \le n$;

(4) $s = 2$, $n = 144$, $p = 1/2,\ 1,\ 2,\ 3$, $(h_1, h_2) = (1, 89)$, the *glp* set in C^2 is $\left(\frac{k-0.5}{n}, \left\{\frac{h_2 k-0.5}{n}\right\}\right)$ for $n = 144$ and $1 \le k \le n$.

Insert Figure 1 around here

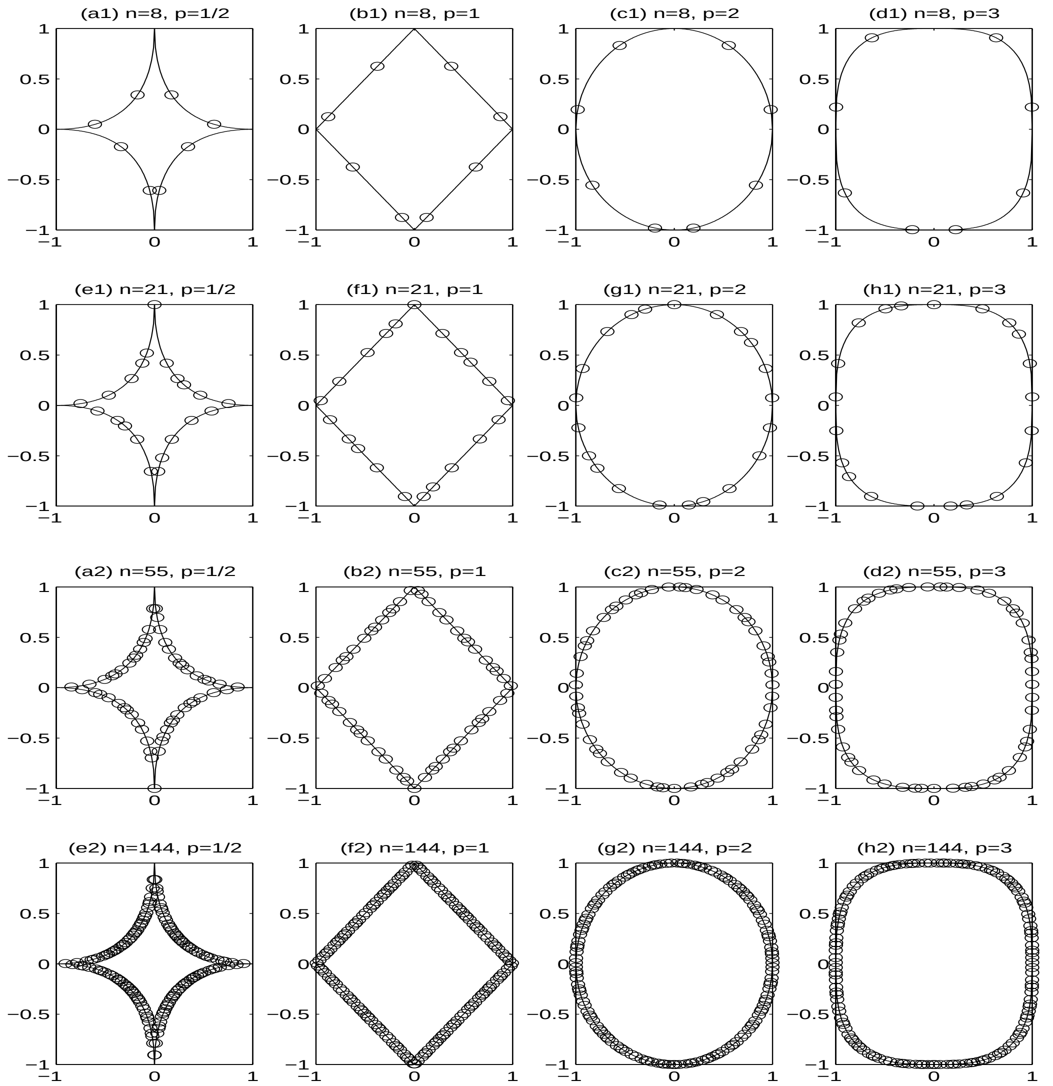

Figure 1: Illustration of projecting some *glp* sets in C^2 onto the L_p-norm unit sphere S_2^p by the algorithm given by (25) for some selected values of p.

• Applications

Definition. An s-variate random vector $\boldsymbol{x}$ is said to have an L_p-norm spherical distribution (denoted by $\boldsymbol{x} \sim SP(s,p)$) if

$$\boldsymbol{x} \stackrel{\mathrm{d}}{=} R\boldsymbol{u}, \tag{12}$$

where $\boldsymbol{u} \sim \mathcal{U}(s,p)$, R is a univariate nonnegative random variable that is independent of $\boldsymbol{u}$.

Example 1. Application in generating empirical samples from the class of p-generalized normal distributions. The p-generalized normal distribution was given by Goodman and Kotz (1973). Denote it by $N_s(\mathbf{0}, \boldsymbol{I}_s, p)$. $\boldsymbol{x} = (X_1, \ldots, X_s)' \sim N_s(\mathbf{0}, \boldsymbol{I}_s, p)$ has a p.d.f.

$$f(x_1, \ldots, x_s) = \frac{p^s r^{s/p}}{2^s \Gamma^s(1/p)} \cdot \exp\{-r \sum_{i=1}^{s} |x_i|^p\}, \quad (x_1, \cdots, x_s)' \in R^s,$$

where $r > 0$ is a parameter. It is easy to verify that

$\boldsymbol{x} \sim N_s(\mathbf{0}, \boldsymbol{I}_s, p) \stackrel{\mathrm{d}}{=} R\boldsymbol{u}$, $\boldsymbol{u} \sim \mathcal{U}(s,p)$, and R has a p.d.f.

$$g(t) = \frac{pr^{s/p}}{\Gamma(s/p)} \cdot t^{s-1} \exp(-rt^p), \quad t > 0.$$

Then the random variable $Y = rR^p$ has a gamma distribution with a p.d.f.

$$g(y) = \frac{1}{\Gamma(s/p)} y^{s/p-1} \exp(-y), \quad y > 0. \tag{13}$$

An i.i.d. sample $\{Y_1, \ldots, Y_n\}$ can be easily generated from the gamma distribution (13). Then an i.i.d. sample $\{R_1, \ldots, R_n\}$ can be obtained by

$$R_i = (Y_i/r)^{1/p}, \qquad i = 1, \ldots, n. \tag{14}$$

A random sample $\{\boldsymbol{x}_1, \ldots, x_n\}$ from $N_s(\mathbf{0}, \boldsymbol{I}_s, p)$ is obtained by

$$\boldsymbol{x}_i = R_i \boldsymbol{u}_i, \qquad i = 1, \ldots, n. \tag{15}$$

where $\{\boldsymbol{u}_i : \ i = 1, \ldots, n\}$ is a random sample from

$\mathcal{U}(s,p)$, which is obtained by generating a uniform sample $\boldsymbol{z}_i = (z_{i1}, \ldots, z_{is})' \in C^s$ with z_{ij} $(j = 1, \ldots, s)$ i.i.d. $U(0,1)$ and projecting this uniform sample onto S_s^p by the algorithm.

Example 2. Application in generating representative points (simply called rep-points) for the class of L_p-norm spherical distributions.

<u>Definition</u>. Let $F(\boldsymbol{x}) = F(x_1, \ldots, x_s)$ be a given s-dimensional continuous c.d.f. and $\mathcal{P} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\} \subset R^s$. The F-discrepancy $D_F(n, \mathcal{P})$ is a measure of the representation of $\mathcal{P}$ to $F(\boldsymbol{x})$. If we can find a set of points $\mathcal{P}^* = \{\boldsymbol{x}_1^*, \ldots, \boldsymbol{x}_n^*\}$ such that

$$D_F(n, \mathcal{P}^*) = \min_{\mathcal{P}} D_F(n, \mathcal{P}), \tag{16}$$

where $\mathcal{P}$ runs over all sets of n points in R^s, then $\mathcal{P}^*$ is called a set of cdf-rep-points of $F(\boldsymbol{x})$.

For the one-dimensional case $s = 1$, it is easy to find the set of cdf-rep-points of any given continuous c.d.f., for the high-dimensional case $s > 1$, it is usually difficult to find the set of cdf-rep-points $\mathcal{P}^*$ of any given c.d.f. $\boldsymbol{x} \sim F(\boldsymbol{x})$. We consider r.v. $\boldsymbol{x}$ has the stochastic representation of the type

$$\boldsymbol{x} \stackrel{\mathrm{d}}{=} R\boldsymbol{y}, \tag{17}$$

where $\boldsymbol{x} \sim F(\boldsymbol{x})$, $R > 0$ is a positive random variable, and $\boldsymbol{y} \sim \mathcal{U}(s, p)$. By using the algorithm in this paper to generate $\boldsymbol{y} \sim \mathcal{U}(s, p)$ and the NTSR algorithm in Fang & Wang (1994), we can generate the approximate cdf-rep-points of the r.v. in (17).

Example 3. Application in optimization problems. Let $f(\boldsymbol{x})$ ($\boldsymbol{x} = (x_1, \ldots, x_s)' \in R^s$) be a continuous function. Suppose that we want to find the maximal point $\boldsymbol{x}^* \in S_s^p$ such that

$$M = f(\boldsymbol{x}^*) = \max_{\boldsymbol{x} \in S_s^p} f(\boldsymbol{x}). \tag{18}$$

This is an optimization problem of $f(\boldsymbol{x}) = f(x_1, \ldots, x_s)$ subject to the restriction

$$|x_1|^p + \cdots + |x_s|^p = 1, \qquad p > 0.$$

By generating a set of uniformly scattered points in S_s^p, we can approximately obtain (18).

END OF TALK

THANK YOU FOR YOUR ATTENTION!